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INDEPENDENT SETS OF AXIOMS IN $L_{\kappa\alpha}$

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§0. Introduction. A set of sentences T is called *independent* if for every $\varphi \in T: T-\{\varphi\} \nvDash \varphi$. It is *countably independent* if every countable subset is independent. In finitary first order logic, $L_{\omega\omega}$, the two notions coincide because of compactness. This is not the case for infinitary logic. A theory is said to have an *independent* (respectively, countably independent) *axiomatization* if it is semantically equivalent to an independent (respectively, countably independent) set. Tarski [5] (1923) observed that any countable theory of $L_{\omega\omega}$ has an independent axiomatization. Reznikoff [3] (1965) extended the result to theories of any cardinality in $L_{\omega\omega}$.

Tarski's assertion may be easily generalized to $L_{\kappa\alpha}$, logic allowing conjunctions of less than κ formulas and homogeneous quantifier chains of length less than α in the form: Any theory of $L_{\kappa\alpha}$ with at most κ sentences has an independent axiomatization. In fact this holds for any logic related to the cardinal κ under very weak conditions.

Reznikoff's result appears more difficult to generalize to infinitary languages. The best we could get, following Reznikoff's ideas, is the next result for $L_{\omega_1\omega}$ which depends on the continuum hypothesis (CH): A theory of $L_{\omega_1\omega}$ of power at most $\aleph_\omega$ has a countably independent axiomatization. Once more this result depends only on certain general properties of $L_{\omega_1\omega}$, including Lopez-Escobar's interpolation theorem for this logic [2].

§1. Theories with at most κ sentences in $L_{\kappa\alpha}$. Our first assertion in the introduction follows from the result we prove now.

THEOREM 1. *Let κ be a cardinal and let L^* be a logic (in the sense of Barwise [1]) closed under conjunctions of less than κ sentences, and closed under "implication", this means that for each φ, ψ, sentences of L^*, there is a sentence $\varphi \to \psi$ such that $\mathcal{A} \vDash \varphi \to \psi$ iff $\mathcal{A} \nvDash \varphi$ or $\mathcal{A} \vDash \psi$. Then for any theory T in L^*, $|T| \leq \kappa$ implies that T has an independent axiomatization in L^*.*

Proof. Given $T = \{\varphi_\lambda : \lambda < \kappa\}$, assume it has non-valid sentences (otherwise, the empty set would be an independent axiomatization) and define inductively $T' = \{\sigma_\alpha : \alpha < \kappa_0\} \subseteq T$, with $\kappa_0 \leq \kappa$ as follows:

$$\sigma_\alpha = \text{first } \varphi_\lambda \in T \text{ such that } \bigwedge_{\beta<\alpha} \sigma_\beta \nvDash \varphi_\lambda.$$

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κ_0 is the first α such that the required φ_λ does not exist. An easy induction shows that for all $\lambda < \kappa$ there exists α such that $\bigwedge_{\beta<\alpha} \sigma_\beta \models \varphi_\lambda$; hence, $T' \equiv T$. Now construct a second theory

$$T'' = \{\sigma_0\} \cup \left\{ \bigwedge_{\beta<\alpha} \sigma_\beta \rightarrow \sigma_\alpha : 0 < \alpha < \kappa_0 \right\}.$$

Obviously, $T'' \equiv T'$ (induction on α and Modus Ponens). Moreover, T'' is independent. Let $\rho_\alpha = \bigwedge_{\beta<\alpha} \sigma_\beta \rightarrow \sigma_\alpha$, since the antecedent of this formula does not imply logically the consequent, there is a structure $\mathcal{A}$ such that $\mathcal{A} \not\models \rho_\alpha$. Then $\mathcal{A} \models \rho_\gamma$ for any other $\rho_\gamma \in T''$; if $\gamma > \alpha$ because $\mathcal{A}$ falsifies its antecedent; if $\gamma < \alpha$ because $\mathcal{A}$ satisfies its consequent. Q.E.D.

§2. Theories in $L_{\omega_1\omega}$ with more than ω_1 sentences. If T is a theory in $L_{\omega_1\omega}$, let M = smallest number of non-logical symbols appearing in any axiomatization of T. Note that if T has M symbols then $|T| \le M^{\aleph_0}$. Our second assertion in the introduction is just a corollary of the following result.

THEOREM 2. *If T has an axiomatization with at most M sentences, then T has a countably independent axiomatization.*

Proof. Let T' be an axiomatization of T with at most M sentences. We may assume $M > \aleph_0$, otherwise T' (and therefore T) would have an independent axiomatization by Theorem 1. Let $L(T')$ be the language of T', $L(\varphi)$ the language of the sentence φ. Define $T_0 = \{\sigma \in (L(T'))_{\omega_1\omega} : T' \models \sigma\}$, so $T_0 \equiv T' \equiv T$. Now we define a sequence of sentences from T_0, $\{\varphi_\alpha : \alpha < M\}$, by induction on α. Let $\alpha < M$, and assume φ_β has been defined for all $\beta < \alpha$. Then we may choose $\varphi_\alpha \in T_0$ such that

(1) $$T_0 \cap \left(\bigcup_{\beta<\alpha} L(\varphi_\beta) \right)_{\omega_1\omega} \not\models \varphi_\alpha$$

Such sentence exists because if it did not exist we would have an axiomatization of T with κ symbols, where $\kappa = |\bigcup_{\beta<\alpha} L(\varphi_\beta)| \le \sum_{\beta<\alpha} |L(\varphi_\beta)| \le |\alpha| \aleph_0 = \text{Max}(|\alpha|, \aleph_0) < M$, which is a contradiction. Define now:

(2) $$D_\alpha = L(\varphi_\alpha) \setminus \bigcup_{\beta<\alpha} L(\varphi_\beta)$$

Clearly, the φ_α's are all distinct and the D_α's are all mutually disjoint and non-empty.

CLAIM 1. If $\psi \in T_0$ and $D_\alpha \cap L(\psi) = \emptyset$, then $\psi \not\models \varphi_\alpha$.

Suppose $\psi \models \varphi_\alpha$; by the interpolation lemma for $L_{\omega_1\omega}$ there is a sentence σ such that $\psi \models \sigma$, $\sigma \models \varphi_\alpha$, and $L(\sigma) \subseteq L(\psi) \cap L(\varphi_\alpha)$. But $L(\varphi_\alpha) \subseteq D_\alpha \cup [\bigcup_{\beta<\alpha} L(\varphi_\beta)]$ and $L(\psi) \cap D_\alpha = \emptyset$; therefore $L(\sigma) \subseteq \bigcup_{\beta<\alpha} L(\varphi_\beta)$. We also

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have $\sigma \in T_0$ because $T_0 \models \psi \models \sigma$; then the fact that $\sigma \models \varphi_\alpha$ contradicts the definition of φ_α above. This proves the claim.

Now we associate to every $\psi \in T_0$ a sentence $\psi^* \in T_0$ in the following way:

(3) $$\varphi_0^* = \varphi_0$$

If $\psi \neq \varphi_0$, let $S(\psi) = \{\varphi_\beta : D_\beta \cap L(\psi) \neq \varnothing,\ \varphi_\beta \neq \psi\}$, and define:

(4) $$\psi^* = \varphi_0 \wedge \bigwedge S(\psi) \rightarrow \psi$$

Since the D_β's are disjoint, at most countably many meet $L(\psi)$ and so the above conjunction is countable. Since $\psi \models \psi^*$ trivially, we have $\psi^* \in T_0$. For each φ_β appearing in the antecedent of φ_α^* we must have $\beta < \alpha$, because $\varphi_\beta \neq \varphi_\alpha$ which means $\alpha \neq \beta$, and $\beta > \alpha$ would imply $D_\beta \cap L(\varphi_\alpha) = \varnothing$. Therefore:

Claim 2. (a) $\{\varphi_\alpha^* : \alpha < M\} \models \varphi_\alpha \quad (\alpha < M)$.
(b) $\{\varphi_\alpha^* : \alpha < M\} + \psi^* \models \psi \quad (\psi \in T_0)$.
(c) If A_α denotes the antecedent of φ_α^*, then $D_\alpha \cap L(A_\alpha) = \varnothing$.

(a) follows by induction on $\alpha < M$, the last remark, and Modus Ponens; (b) follows trivially from (a); (c) follows by definition of D_α from last remark.

Claim 3. ϕ_α^* is not implied by countably many other ψ^*'s, $\psi \in T_0 (0 < \alpha < M)$.

Let $\{\psi_i : i \in \omega\} \subseteq T_0$. Divide this set into two groups:

$$\{\psi_i : i \in I\} = \{\psi_i : L(\psi_i) \cap D_\alpha = \varnothing\}$$

$$\{\psi_j : j \in J\} = \{\psi_j : L(\psi_j) \cap D_\alpha \neq \varnothing\},$$

By Claim 2 (c) we have: $L((\bigwedge_{i \in I} \psi_i) \wedge A_\alpha) \cap D_\alpha = \varnothing$. Therefore, by Claim 1:

$$\left(\bigwedge_{i \in I} \psi_i\right) \wedge A_\alpha \not\models \varphi_\alpha.$$

Choose a structure $\mathscr{A}$ such that $\mathscr{A} \models \psi_i (i \in I)$, $\mathscr{A} \models A_\alpha$, but $\mathscr{A} \not\models \varphi_\alpha$. Then $\mathscr{A} \models \psi_i^*$ for all $i \in I$ since α satisfies the consequent ψ_i of ψ_i^*. Also $\mathscr{A} \not\models A_\alpha \rightarrow \varphi_\alpha = \varphi_\alpha^*$ For $j \in J$, we have $D_\alpha \cap L(\psi_j) \neq \varnothing$ and $\psi_j \neq \varphi_\alpha$, then φ_α must appear in the antecedent of ψ_j^*; therefore, $\mathscr{A}$ will falsify this antecedent and so $\mathscr{A} \models \psi_j^*$. In this way we have obtained: $\mathscr{A} \models \{\psi_k^* : k \in \omega\}$, $\mathscr{A} \not\models \varphi_\alpha$, proving the claim.

Now recall that $T' \subseteq T_0$. Let $C = \{\varphi_\alpha^* : 0 < \alpha < M\}$ and $D = \{\varphi_0\} \cup \{\psi^* : \psi \in T'\} \setminus C$, then $C \cup D \models T'$ by Claim 2 (b). But $T' \equiv T_0 \supseteq C \cup D$, so we get: $C \cup D \equiv T' \equiv T$. Finally, $|D| \leq |T'| + 1 \leq M = |C|$ and $C \cap D = \varnothing$, this means that the hypothesis of the following lemma applies to C and D, proving Theorem 2.

Lemma (essentially from Reznikoff [3]). Let C and D be disjoint sets of sentences with $|D| \leq |C|$. If every $\varphi \in C$ is not implied by (countably many) other sentences of $C \cup D$, then $C \cup D$ is equivalent to a (countably) independent set.

Proof. Let $f: D \to C$ be one to one, then the set

$$(C \setminus f(D)) \cup \{\varphi \wedge f(\varphi) : \varphi \in D\}$$

is countably independent and equivalent to $C \cup D$. Q.E.D.

COROLLARY 3. (C.H.) *If* $|T| \leq \aleph_\omega$, *then* T *has a countably independent axiomatization.*

Proof. If T is finite the result follows from Theorem 1. Assume T is infinite, and let M be as in Theorem 2, then

$$(*) \qquad M \leq |L(T)| \leq |T| \aleph_0 = |T| \leq \aleph_\omega.$$

There are three possibilities for M:

(i) $M \leq \aleph_1 = \omega_1$. In this case T has an axiomatization with at most $\aleph_1^{\aleph_0} = \aleph_1$ sentences (C.H.), and the result follows from Theorem 1.

(ii) $\aleph_1 < M < \aleph_\omega$. It follows from the C.H. that $M^{\aleph_0} = M$ because of Hausdorff's recursion formula: $\aleph_{\alpha+1}^\kappa = \aleph_{\alpha+1} \aleph_\alpha^\kappa$, [4]. Therefore, T has an axiomatization with at most M sentences and the result follows from Theorem 2.

(iii) $M = \aleph_\omega$. Then we have $M = |T|$ by (*) above, and we may apply Theorem 2. Q.E.D.

§3. Conclusion. If $\vdash$ denotes the proof relation in any standard formal system for $L_{\omega_1\omega}$, complete for validities [2], we may call T *syntactically independent* in case that $T - \{\varphi\} \nvdash \varphi$ for all $\varphi \in T$. Professor Charles Pinter pointed out to us that countable independence in the sense of this paper is equivalent to syntactical independence because every proof may be taken enumerably long. So Theorem 2 could be stated in terms of *full* syntactical independence. However, countable independence does not imply full semantical independence, as the next example shows. Take $L = \{R(-,-)\} \cup \{c_\alpha : \alpha < \omega_1\}$ and make T consist of the following sentences:

1. $\{\neg(c_\alpha = c_\beta) : \alpha < \beta < \omega_1\}$
2. "$R(-,-)$ is a function"
3. $\exists x \forall y \left(\bigvee_{n<\omega} y = c_n \to \neg R(y, x) \right)$

The last sentence says that the function does not map a fixed countable subset onto the universe. Since (3) follows from (1) and (2) by a cardinality argument, T is not independent. However, every countable subset is.

We finish with the obvious question. Does every theory in $L_{\omega_1\omega}$ have a (countably) independent axiomatization? Is there an example of theory in $L_{\omega_1\omega}$ which has a countably independent axiomatization but does not have a fully independent one?

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