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# ON THE REAL-VALUED GENERAL SOLUTIONS OF THE D'ALEMBERT EQUATION WITH INVOLUTION

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**Abstract**

We find all real-valued general solutions $f : S \to \mathbb{R}$ of the d'Alembert functional equation with involution

$$f(x + y) + f(x + \sigma y) = 2f(x)f(y)$$

for all $x, y \in S$, where $S$ is a commutative semigroup and $\sigma : S \to S$ is an involution. Also, we find the Lebesgue measurable solutions $f : \mathbb{R}^n \to \mathbb{R}$ of the above functional equation, where $\sigma : \mathbb{R}^n \to \mathbb{R}^n$ is a Lebesgue measurable involution. As a direct consequence, we obtain the Lebesgue measurable solutions $f : \mathbb{R}^n \to \mathbb{R}$ of the classical d'Alembert functional equation

$$f(x + y) + f(x - y) = 2f(x)f(y)$$

for all $x, y \in \mathbb{R}^n$. We also exhibit the locally bounded solutions $f : \mathbb{R}^n \to \mathbb{R}$ of the above equations.

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## 1. Introduction

Throughout this paper we denote by $S$ a commutative semigroup, $G$ a commutative group, $\mathbb{F}$ a field and $\mathbb{R}$, $\mathbb{C}$ and $\mathbb{R}^n$ the sets of real numbers and complex numbers and the $n$-dimensional Euclidean space, respectively. A function $A : S \to \mathbb{F}$ is called *an additive function* provided that $A(x + y) = A(x) + A(y)$ for all $x, y \in S$, $m : S \to \mathbb{F}$ is called *an exponential function* provided that $m(x + y) = m(x)m(y)$ for all $x, y \in S$ and $\sigma : S \to S$ is called *an involution* provided that $\sigma(x + y) = \sigma(x) + \sigma(y)$ and $\sigma(\sigma(x)) = x$ for all $x, y \in S$. For simplicity, we denote $\sigma(x)$ by $\sigma x$.

The functional equation

$$f(x + y) + f(x - y) = 2f(x)f(y) \tag{1.1}$$

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is known as the *d'Alembert functional equation*. It has a long history going back to d'Alembert [6]. This functional equation was introduced by d'Alembert in connection with the composition of forces and plays a central role in determining the sum of two vectors in Euclidean and non-Euclidean geometries [7, 8]. As remarkable results on the d'Alembert functional equation, Cauchy [3] found all the continuous solutions $f : \mathbb{R} \to \mathbb{R}$ of the equation (1.1) (see also [1, page 103]) and Baker [2] found all general solutions $f : \mathbb{R} \to \mathbb{C}$ of the equation (see also [1, page 220]). The general solutions $f : \mathbb{R} \to \mathbb{R}$ of (1.1) are not yet known. Generalising the d'Alembert functional equation, several authors have studied the d'Alembert functional equation with involution

$$f(x + y) + f(x + \sigma y) = 2f(x)f(y) \tag{1.2}$$

for all $x, y \in S$. Sinopoulos [9] determined the general solutions $f : S \to \mathbb{F}$ of (1.2) when $S$ is a commutative semigroup and $\mathbb{F}$ is a quadratically closed commutative field of characteristic different from 2. Stetkær [10] studied (1.2) when $\mathbb{F} = \mathbb{C}$, $S$ is a commutative topological group and $f$ and $\sigma$ are continuous. Recently, Chung [5] found the locally integrable solutions $f : \mathbb{R}^n \to \mathbb{C}$ of the equation (1.2) defined in an almost everywhere sense. However, in all the previous results, the authors assumed that the target space of $f$ is a quadratically closed commutative field; therefore, it is not possible to exhibit the real-valued general solutions of the equations (1.1) and (1.2). The author is not aware of any results on the real-valued general solutions of the d'Alembert functional equation (1.1) and its generalisation (1.2). In this paper we exhibit all real-valued general solutions of the d'Alembert functional equation with involution (1.2) and obtain those of (1.1) as a direct consequence. Based on the result, we also prove that all Lebesgue measurable solutions $f : \mathbb{R}^n \to \mathbb{R}$ of the equation (1.2) with measurable involution $\sigma$ are given by $f(x) = \frac{1}{2}(e^{a\cdot x} + e^{a\cdot\sigma x})$ or $f(x) = e^{c\cdot x}\cos(b \cdot x)$ for some $a, b, c \in \mathbb{R}^n$ with $c\sigma = c$, $b\sigma = -b$. As a direct consequence, we obtain the Lebesgue measurable solutions $f : \mathbb{R}^n \to \mathbb{R}$ of the equation (1.1). This appears to be the first direct method to find the Lebesgue measurable solutions of the equations (1.1) and (1.2) (see, for example, [4] for the locally integrable solutions of the d'Alembert equation).

## 2. Main results

We first find the real-valued general solutions of the d'Alembert equation with involution (1.2).

THEOREM 2.1. *A nonzero function $f : S \to \mathbb{R}$ satisfies d'Alembert's functional equation with involution (1.2) for all $x, y$ in $S$ if and only if $f$ has one of the following forms:*

$$f(x) = \frac{m(x) + m(\sigma x)}{2}, \quad f(x) = E(x)\cos B(x) \tag{2.1}$$

*for all $x \in S$, where $m$, $E : S \to \mathbb{R}$ are exponential functions and $E$ and $B : S \to \mathbb{R}$ satisfy $E(\sigma x) = E(x)$ for all $x \in S$, $B(x + y) \equiv B(x) + B(y) \pmod{2\pi}$ for all $x, y \in S \setminus K$ and $B(\sigma x) \equiv -B(x) \pmod{2\pi}$ for all $x \in S \setminus K$ with $K = \{x \in S : E(x) = 0\}$.*

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Proof. Replacing $y$ by $\sigma y$ in (1.2) and equating the right-hand sides of the result and (1.2) yields

$$f(x) = f(\sigma x) \tag{2.2}$$

for all $x \in S$. Now we divide the equation into two cases.

*Case 1.* Suppose that $f(x + \sigma y) = f(x + y)$ for all $x, y \in S$. Then the equation (1.2) reduces to the exponential functional equation $f(x + y) = f(x)f(y)$. Let $f(x) = m(x)$ for some exponential function $m$. By (2.2), $m(\sigma x) = m(x)$ for all $x \in S$ and $f$ has the first form of (2.1).

*Case 2.* Suppose that $f(x_0 + \sigma y_0) \neq f(x_0 + y_0)$ for some $x_0, y_0 \in S$. Let

$$g(x) = f(x + y_0) - f(x + \sigma y_0) \tag{2.3}$$

for all $x \in S$. Then $g(x_0) \neq 0$ and, by (2.2),

$$g(\sigma x) = -g(x) \tag{2.4}$$

for all $x \in S$. From (1.2) and (2.3),

$$\begin{aligned} g(x + y) + g(x + \sigma y) &= f(x + y + y_0) - f(x + y + \sigma y_0) \\ &\quad + f(x + \sigma y + y_0) - f(x + \sigma y + \sigma y_0) \\ &= 2f(x + y_0)f(y) - 2f(x + \sigma y_0)f(y) \\ &= 2g(x)f(y) \end{aligned} \tag{2.5}$$

for all $x, y \in S$. Replacing $(x, y)$ by $(y, x)$ in (2.5), adding the result and (2.5) and using (2.4) yields

$$g(x + y) = g(x)f(y) + f(x)g(y) \tag{2.6}$$

for all $x, y \in S$. Replacing $y$ by $y + z$ in (2.6) and then using (2.6),

$$\begin{aligned} g(x + y + z) &= g(x)f(y + z) + f(x)g(y + z) \\ &= g(x)f(y + z) + f(x)g(y)f(z) + f(x)f(y)g(z) \end{aligned} \tag{2.7}$$

for all $x, y, z \in S$. Again replacing $(x, y)$ by $(x + y, z)$ in (2.6) and using (2.6),

$$\begin{aligned} g(x + y + z) &= g(x + y)f(z) + f(x + y)g(z) \\ &= g(x)f(y)f(z) + f(x)g(y)f(z) + f(x + y)g(z) \end{aligned} \tag{2.8}$$

for all $x, y, z \in S$. From (2.7) and (2.8),

$$g(x)\,[f(y + z) - f(y)f(z)] = g(z)[f(x + y) - f(y)f(x)] \tag{2.9}$$

for all $x, y, z \in S$. Inserting $z = x_0$ in (2.9) yields

$$g(x)\,[f(y + x_0) - f(y)f(x_0)] = g(x_0)[f(x + y) - f(y)f(x)],$$

which reduces to

$$f(x + y) - f(y)f(x) = g(x)h(y), \tag{2.10}$$

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where

$$h(y) = \frac{f(y + x_0) - f(y)f(x_0)}{g(x_0)} \tag{2.11}$$

for all $y \in S$. If we interchange $x$ and $y$ in (2.10),

$$f(y + x) - f(x)f(y) = g(y)h(x)$$

and, by comparing this equation to (2.10),

$$g(x)\,h(y) = g(y)\,h(x)$$

for all $x, y \in S$. Therefore,

$$h(x) = \alpha^2 g(x) \tag{2.12}$$

for all $x \in S$ and for some constant

$$\alpha \in \mathbb{R}, \text{ or } \alpha = i\beta \quad \text{with } \beta \in \mathbb{R}.$$

Substituting (2.12) in (2.10) yields

$$f(x + y) = f(x)f(y) + \alpha^2 g(x)g(y) \tag{2.13}$$

for all $x, y \in S$. If $\alpha = 0$, then (2.13) becomes

$$f(x + y) = f(x)f(y)$$

for all $x, y \in S$ and we return to the first case. Thus, it remains to consider the case $\alpha \neq 0$. Multiplying (2.6) by $\alpha$ yields

$$\alpha g(x + y) = \alpha g(x)f(y) + \alpha f(x)g(y). \tag{2.14}$$

Adding (2.14) to (2.13) and simplifying the resulting equation gives

$$f(x + y) + \alpha g(x + y) = [f(x) + \alpha g(x)][f(y) + \alpha g(y)] \tag{2.15}$$

for all $x, y \in S$. Similarly, subtracting (2.14) from (2.13) yields

$$f(x + y) - \alpha g(x + y) = [f(x) - \alpha g(x)][f(y) - \alpha g(y)] \tag{2.16}$$

for all $x, y \in S$.

*Subcase 2.1.* Suppose that $\alpha \in \mathbb{R}$. From (2.15) and (2.16), both $m_1 := f + \alpha g$ and $m_2 := f - \alpha g$ are real-valued exponential functions. Since $f$ is $\sigma$-even and $g$ is $\sigma$-odd, $m_1(\sigma x) = f(\sigma x) + \alpha g(\sigma x) = f(x) - \alpha g(x) = m_2(x)$. Letting $m_1 := m$,

$$f(x) = \frac{m(x) + m(\sigma x)}{2} \tag{2.17}$$

for all $x \in S$. Thus, we get the first solution of (2.1).

*Subcase 2.2.* Suppose that $\alpha = i\beta$ with $\beta \in \mathbb{R}$. Let

$$m^*(x) = f(x) + i\beta g(x) \tag{2.18}$$

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for all $x \in S$. Then, from (2.15), $m^*$ is a complex-valued exponential function. Let $E(x) = |m^*(x)|$ for all $x \in S$ and $K = \{x \in S : E(x) = 0\}$. Then $E$ is a real-valued exponential function and $m^*$ can be written in the form

$$m^*(x) = E(x)e^{iB(x)}, \tag{2.19}$$

where $B : S \to \mathbb{R}$ takes a value of $\arg m^*(x)$ for each $x \notin K$ and $B(x)$ takes arbitrary values for all $x \in K$. Since $m^*$ and $E$ are exponential functions, it follows from (2.19) that $B(x + y) \equiv B(x) + B(y) \pmod{2\pi}$ for all $x, y \in S \setminus K$ (see [1, page 54] for $S = \mathbb{R}$). Then, from (2.18) and (2.19),

$$f(x) = \Re(m^*(x)) = E(x) \cos B(x) \tag{2.20}$$

for all $x \in S$. On the other hand, since

$$m^*(\sigma x) = f(\sigma x) + i\beta g(\sigma x) = f(x) - i\beta g(x) \tag{2.21}$$

for all $x \in S$, from (2.18), (2.19) and (2.21),

$$\begin{aligned} f(x) &= \frac{m^*(x) + m^*(\sigma x)}{2} = \frac{1}{2}(E(x)e^{iB(x)} + E(\sigma x)e^{iB(\sigma x)}) \\ &= \frac{1}{2}(E(x) \cos B(x) + E(\sigma x) \cos B(\sigma x)) \\ &\quad + \frac{i}{2}(E(x) \sin B(x) + E(\sigma x) \sin B(\sigma x)) \end{aligned} \tag{2.22}$$

for all $x \in S$. Equating (2.20) and (2.22),

$$E(x) \cos B(x) = E(\sigma x) \cos B(\sigma x), \tag{2.23}$$

$$E(x) \sin B(x) = -E(\sigma x) \sin B(\sigma x) \tag{2.24}$$

for all $x \in S$. Using (2.23) and (2.24),

$$\begin{aligned} E(x)^2 &= E(x)^2 \cos^2 B(x) + E(x)^2 \sin^2 B(x) \\ &= E(\sigma x)^2 \cos^2 B(\sigma x) + E(\sigma x)^2 \sin^2 B(\sigma x) = E(\sigma x)^2 \end{aligned}$$

for all $x \in S$, which implies that $E(x) = E(\sigma x)$ for all $x \in S$. Thus, from (2.23) and (2.24),

$$\cos B(x) = \cos B(\sigma x) = \cos(-B(\sigma x)), \tag{2.25}$$

$$\sin B(x) = -\sin B(\sigma x) = \sin(-B(\sigma x)) \tag{2.26}$$

for all $x \in S \setminus K$. From (2.25) and (2.26), it follows that $B(\sigma x) = -B(x) \pmod{2\pi}$ for all $x \in S \setminus K$. Thus, we get the second solution of (2.1). The proof is complete. □

Let $S = G$ in Theorem 2.1. The nonzero exponential functions $m, E : G \to \mathbb{R}$ in Theorem 2.1 can be written in the form $m(x) = e^{A(x)}$, $E(x) = e^{C(x)}$ for some additive functions $A, C : G \to \mathbb{R}$ and $K := \ker E = \emptyset$. Thus, as a direct consequence of Theorem 2.1, we obtain the following corollary.

Corollary 2.2. *A nonzero function $f : G \to \mathbb{R}$ satisfies the d'Alembert functional equation with involution (1.2) for all $x, y$ in $G$ if and only if $f$ has one of the following forms:*

$$f(x) = \frac{e^{A(x)} + e^{A(\sigma x)}}{2}, \quad f(x) = e^{C(x)} \cos B(x) \tag{2.27}$$

*for all $x \in G$, where $A$, $C : G \to \mathbb{R}$ are additive functions and $C$ and $B : G \to \mathbb{R}$ satisfy $C(\sigma x) = C(x)$ for all $x \in G$, $B(x + y) \equiv B(x) + B(y) \pmod{2\pi}$ for all $x, y \in G$ and $B(\sigma x) \equiv -B(x) \pmod{2\pi}$ for all $x \in G$.*

Remark 2.3. In general, if $S$ satisfies the property that for any $x, y \in S$, there exist a positive integer $k$ and $z \in S$ such that

$$x + z = ky, \tag{2.28}$$

then the solutions $f : S \to \mathbb{R}$ in (2.1) can be written in the form (2.27). Note that most well-known semigroups such as $S = \langle (0, 1), \times \rangle$ and $\langle (0, \infty), + \rangle$ satisfy the condition (2.28).

Corollary 2.4. *A nonzero function $f : G \to \mathbb{R}$ satisfies the d'Alembert functional equation (1.1) for all $x, y \in G$ if and only if $f$ has one of the following forms:*

$$f(x) = \cosh A(x), \quad f(x) = \cos B(x) \tag{2.29}$$

*for all $x \in G$, where $A : G \to \mathbb{R}$ is an additive function and $B : G \to \mathbb{R}$ satisfies $B(x + y) \equiv B(x) + B(y) \pmod{2\pi}$ for all $x, y \in G$.*

Proof. Let $\sigma(x) = -x$ for all $x \in G$ in Corollary 2.2. Then the condition $C(\sigma x) = C(x)$ for all $x \in G$ implies that $C = 0$, and the condition $B(x + y) \equiv B(x) + B(y) \pmod{2\pi}$ for all $x, y \in G$ implies that $B(-x) \equiv -B(x) \pmod{2\pi}$ for all $x \in G$. Thus, we get the solutions (2.29). The proof is complete. □

Now we find the Lebesgue measurable solutions of the equation (1.2).

Corollary 2.5. *Let $\sigma : \mathbb{R}^n \to \mathbb{R}^n$ be a Lebesgue measurable involution. Then a nonzero Lebesgue measurable function $f : \mathbb{R}^n \to \mathbb{R}$ satisfies the d'Alembert functional equation (1.2) for all $x, y \in \mathbb{R}^n$ if and only if $f$ has one of the following forms:*

$$f(x) = \frac{e^{a\cdot x} + e^{a\cdot \sigma x}}{2}, \quad f(x) = e^{c\cdot x} \cos(b \cdot x) \tag{2.30}$$

*for all $x \in \mathbb{R}^n$, where $a, b, c \in \mathbb{R}^n$ with $c \cdot \sigma x = c \cdot x$ and $b \cdot \sigma x = -b \cdot x$ for all $x \in \mathbb{R}^n$.*

Proof. Let $G = \mathbb{R}^n$ in Corollary 2.2 and consider the first solution of (2.27). If $A(x) = A(\sigma x)$ for all $x \in \mathbb{R}^n$, then we get $f(x) = e^{A(x)}$ and hence $A$ is Lebesgue measurable. If $A(x_0) \neq A(\sigma x_0)$ for some $x_0 \in \mathbb{R}^n$, then we can choose $\alpha \in \mathbb{R}$ such that $\alpha(e^{A(x_0)} - e^{A(\sigma x_0)}) = 1$. Then it follows from $f(x) = \frac{1}{2}(e^{A(x)} + e^{A(\sigma x)})$ that

$$f(x) + \alpha(f(x + x_0) - f(x + \sigma x_0)) = e^{A(x)} \tag{2.31}$$

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for all $x \in \mathbb{R}^n$ and hence $A$ is Lebesgue measurable. Thus, in both cases we have $A(x) = a \cdot x$ for some $a \in \mathbb{R}^n$. Thus, we obtain the first solution of (2.30).

Now we consider the second solution of (2.28). Let $g(x) = e^{C(x)+iB(x)}$. If $\sin B(x) = 0$ for all $x \in \mathbb{R}^n$, then $g(x) = e^{C(x)} \cos B(x) = f(x)$ and $g$ is Lebesgue measurable. If $\sin B(x_0) \neq 0$ for some $x_0 \in \mathbb{R}^n$, choose $\alpha \in \mathbb{R}$ such that $-2\alpha e^{C(x_0)} \sin B(x_0) = 1$. Then it follows from $f(x) = e^{C(x)} \cos B(x)$ that

$$f(x) + i\alpha(f(x + x_0) - f(x + \sigma x_0)) = e^{C(x)+iB(x)} \tag{2.32}$$

for all $x \in \mathbb{R}^n$. Thus, $g$ is Lebesgue measurable and $|g(x)| = e^{C(x)}$ is also Lebesgue measurable and hence $C(x) = c \cdot x$ for all $x \in \mathbb{R}^n$ and for some $c \in \mathbb{R}$ with $c \cdot x = c \cdot \sigma x$ for all $x \in \mathbb{R}^n$. Now let $h(x) := e^{-c \cdot x} g(x) = e^{iB(x)}$ for all $x \in \mathbb{R}^n$. Then $h$ is Lebesgue measurable, satisfying $|h(x)| = 1$ and

$$h(x + y) = h(x)h(y) \tag{2.33}$$

for all $x, y \in \mathbb{R}^n$. Choosing an infinitely differentiable function $\phi$ with compact support and convolving it on both sides of (2.33) as a function of $y$,

$$\begin{aligned}(h * \phi)(x + z) &= \int_{\mathbb{R}^n} h(x + z - y)\phi(y)\, dy \\ &= h(x) \int_{\mathbb{R}^n} h(z - y)\phi(y)\, dy = h(x)(h * \phi)(z)\end{aligned} \tag{2.34}$$

for all $x, z \in \mathbb{R}^n$. From the condition $|h(x)| = 1$ for all $x \in \mathbb{R}^n$, we can choose $\phi$ so that $h * \phi \neq 0$, otherwise we must have $h = 0$ almost everywhere. Since $h * \phi$ is infinitely differentiable, fixing $z = z_0$ with $(h * \phi)(z_0) \neq 0$ in (2.34), we see that $h$ is infinitely differentiable. Letting $y = (y_1, \ldots, y_n)$ and differentiating (2.33) with respect to $y_1$ and putting $y = 0$,

$$\partial_1 h(x) = c_1 h(x), \tag{2.35}$$

where $c_1 = \partial_1 h(0)$. The solutions of the differential equation (2.35) are given by

$$h(x_1, \ldots, x_n) = h_1(x_2, \ldots, x_n)e^{c_1 x_1}. \tag{2.36}$$

Putting (2.36) in (2.33),

$$h_1(x' + y') = h_1(x')h_1(y') \tag{2.37}$$

for all $x' = (x_2, \ldots, x_n), y' = (y_2, \ldots, x_n)$. Differentiating (2.37) with respect to $y_2$ and putting $y' = 0$, we get $\partial_2 h_1(x') = c_2 h_1(x')$ with $c_2 = \partial_2 h(0)$ and

$$h(x_1, \ldots, x_n) = h_1(x_2, \ldots, x_n)e^{c_1 x_1} = h_2(x_3, \ldots, x_n)e^{c_2 x_2 + c_1 x_1}.$$

Continuing the above process, we arrive at $h(x) = ke^{c_1 x_1 + \cdots + c_n x_n}$ for some $k \in \mathbb{C}$. Since $|h(x)| = 1$ for all $x \in \mathbb{R}^n$, we have $c_j = ib_j$ for some $b_j \in \mathbb{R}$, $j = 1, 2, \ldots, n$. Using $h(0) = 1$, we get $k = 1$. Thus, it follows from (2.32) that

$$f(x) = \Re g(x) = e^{c \cdot x} \Re h(x) = e^{c \cdot x} \cos(b \cdot x)$$

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with $b = (b_1, \dots, b_n)$. Finally, we prove that $b \cdot \sigma x = -b \cdot x$ for all $x \in \mathbb{R}^n$. Let $B(x) = (b \cdot \sigma x + b \cdot x)/(2\pi)$ for all $x \in \mathbb{R}^n$. Then, by Corollary 2.2, $B(x) \in \mathbb{Z}$, the set of integers, for all $x \in \mathbb{R}^n$. Thus, for each $x_0 \in \mathbb{R}^n$, we have $B(2^{-n}x_0) = 2^{-n}B(x_0) \in \mathbb{Z}$ for all positive integers $n$, which implies that $B(x_0) = 0$. Therefore, $B(x) = 0$ for all $x \in \mathbb{R}^n$. The proof is complete. □

Remark 2.6. Note that every measurable involution $\sigma : \mathbb{R}^n \to \mathbb{R}^n$ is given by an $n \times n$ matrix. It follows directly from Corollary 2.5 that all *continuous real-valued solutions* $f : \mathbb{R}^n \to \mathbb{R}$ of the equation (1.2) are given by (2.30). Note that all *locally bounded real-valued solutions* of the most well known functional equations are regular. For example, all locally bounded solutions $f, g : \mathbb{R}^n \to \mathbb{R}$, $h : \mathbb{R} \setminus \{0\} \to \mathbb{R}$ and $k : (0, \infty) \to \mathbb{R}$ of the Cauchy functional equation $f(x + y) = f(x) + f(y)$, the exponential functional equation $g(x + y) = g(x)g(y)$, the logarithmic functional equation $h(xy) = h(x) + h(y)$ and the multiplicative functional equation $k(xy) = k(x)k(y)$ are given by the regular functions,

$$f(x) = a \cdot x, \quad g(x) = e^{a \cdot x}, \quad h(x) = p \ln x, \quad k(x) = x^p$$

for some $a \in \mathbb{R}^n$ and $p \in \mathbb{R}$. However, not every *locally bounded real-valued solution* of the equation (1.2) is of the regular form (2.30). Indeed, we have the following result.

Corollary 2.7. *Let $\sigma : \mathbb{R}^n \to \mathbb{R}^n$ be a Lebesgue measurable involution. Then a nonzero locally bounded function $f : \mathbb{R}^n \to \mathbb{R}$ satisfies the d'Alembert functional equation (1.2) for all $x, y \in \mathbb{R}^n$ if and only if $f$ has one of the following forms:*

$$f(x) = \frac{e^{a \cdot x} + e^{a \cdot \sigma x}}{2}, \quad f(x) = e^{c \cdot x} \cos B(x) \tag{2.38}$$

*for all $x \in \mathbb{R}^n$, where $a, c \in \mathbb{R}^n$ with $c \cdot \sigma x = c \cdot x$, $B(x + y) \equiv B(x) + B(y) \pmod{2\pi}$ for all $x, y \in \mathbb{R}^n$ and $B(\sigma x) \equiv -B(x) \pmod{2\pi}$ for all $x \in \mathbb{R}^n$.*

Proof. It follows from (2.31) and (2.32) that $e^{A(x)}$ and $e^{C(x)}$ are locally bounded, which implies that $A$ and $C$ are bounded above. Thus, $A$, $C$ are of the form $A(x) = a \cdot x$, $C(x) = c \cdot x$ for some $a, c \in \mathbb{R}$ and we get the solutions (2.38). □

As a direct consequence of Corollaries 2.4 and 2.7, we obtain the following result.

Corollary 2.8. *A nonzero locally bounded function $f : \mathbb{R}^n \to \mathbb{R}$ satisfies the d'Alembert functional equation (1.1) for all $x, y \in \mathbb{R}^n$ if and only if $f$ has one of the following forms:*

$$f(x) = \cosh(a \cdot x), \quad f(x) = \cos B(x) \tag{2.39}$$

*for all $x \in \mathbb{R}^n$, where $a \in \mathbb{R}^n$ and $B(x + y) \equiv B(x) + B(y) \pmod{2\pi}$ for all $x, y \in \mathbb{R}^n$.*

As a direct consequence of Corollary 2.5, we obtain the following result.

Corollary 2.9. *A nonzero measurable function $f : \mathbb{R}^n \to \mathbb{R}$ satisfies the d'Alembert functional equation (1.1) for all $x, y \in \mathbb{R}^n$ if and only if $f$ has one of the following forms:*

$$f(x) = \cosh(a \cdot x), \quad f(x) = \cos(b \cdot x) \tag{2.40}$$

*for all $x \in \mathbb{R}^n$ and for some $a, b \in \mathbb{R}^n$.*

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Proof. Let $\sigma(x) = -x$ for all $x \in \mathbb{R}^n$ in Corollary 2.5. Then the condition $c \cdot \sigma x = c \cdot x$ for all $x \in \mathbb{R}^n$ implies that $c = 0$. Thus, we get the solution (2.40). □

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