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A LOCAL ERGODIC THEOREM ON L_p

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1. Introduction. Two general types of pointwise ergodic theorems have been studied: those as t approaches infinity, and those as t approaches zero. This paper deals with the latter case, which is referred to as the local case.

Let $(X, \mathscr{F}, \mu)$ be a complete, σ-finite measure space. Let $\{T_t\}$ be a strongly continuous one-parameter semi-group of contractions on $\mathscr{L}_1(X, \mathscr{F}, \mu)$, defined for $t \geqq 0$. For T_t positive, it was shown independently in [**2**] and [**5**] that

$$(1.1) \qquad \lim_{t\to 0} \frac{1}{t} \int_0^t T_s f(x) ds = f(x)$$

almost everywhere on X, for any $f \in \mathscr{L}_1$. The same result was obtained in [**1**], with the continuity assumption weakened to having it hold for $t > 0$.

It was also shown in [**5**] that (1.1) holds without the positivity assumption on T_t, provided that T_t is a contraction of $\mathscr{L}_\infty$. In [**3**] the $\mathscr{L}_\infty$ restriction is removed, so that (1.1) actually holds for any continuous one-parameter semi-group of contractions on $\mathscr{L}_1$.

If we let $\{T_t\}$ be a strongly continuous semi-group of contractions of $L_p(X, \mathscr{F}, \mu)$, for a fixed p, $1 \leqq p < +\infty$, and defined for $t \geqq 0$, then the limit (1.1) still holds for $f \in L_p$, provided that the semi-group is positive. This was shown in [**4**], where $\{T_t\}$ is not required to be a contraction, but merely a bounded operator. The question is raised in [**4**] of whether (1.1) remains true in the non-positive case. In this paper we prove:

THEOREM 1. *The limit* (1.1) *holds for* $f \in L_p$ *if* $\{T_t\}$ *is a strongly continuous semi-group of contractions of* $L_p(X, \mathscr{F}, \mu)$, *for a fixed* p, $1 \leqq p < +\infty$, *and defined for* $t \geqq 0$, *provided that* $\{T_t\}$ *is also simultaneously a semi-group of contractions of* $L_\infty(X, \mathscr{F}, \mu)$.

We also obtain a more general result:

THEOREM 2. *Let* $\{T_t\}$, $t \geqq 0$ *be a strongly continuous one-parameter semi-group of contractions on* $\mathscr{L}_p(X, \mathscr{F}, \mu)$, $1 \leqq p < \infty$. *Let there exist a measurable function* h *on* $[0, \infty) \times X$ *such that*

(i) $h > 0$ *everywhere, and*

(ii) $f \in \mathscr{L}_p$, $|f(x)| \leqq h(t,x)$ *for almost all* $x \in X$ *implies* $|T_s f(x)| \leqq h(t+s,x)$ *for almost all* $x \in X$, *for any* $t, s \geqq 0$. *Then* (1.1) *holds for all* f *in* $\mathscr{L}_p$.

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Theorem 2 yields Theorem 1 when $h \equiv 1$. More generally Theorem 2 gives convergence if $\|T_t\|_\infty \leqq e^{\alpha t}$.

In section 2 below, we prove a maximal lemma. Theorem 1 is proved in section 3, and Theorem 2 is obtained from Theorem 1 in section 4.

2. A maximal lemma. The main result of this section is Lemma 2. The following preliminary lemma is useful.

LEMMA 1. *Let T be a linear operator on a vector space V. Let f, h_k, g_k, $k = 0, 1, \ldots, n$, and d_k, $k = 1, \ldots, n$, be elements of V such that:*

(2.1) $f = h_0 + g_0$,

(2.2) $Tg_k = g_{k+1} + d_{k+1}, \; h_{k+1} = h_k + d_{k+1}, \; k = 0, 1, \ldots, n-1.$

Then

(2.3) $f + Tf + \ldots + T^n f = h_n + Th_{n-1} + \ldots + T^n h_0 + g_0 + \ldots + g_n,$

and

(2.4) $T^n f = T^n h_0 + d_n + Td_{n-1} + \ldots + T^{n-1} d_1 + g_n.$

The proof of Lemma 1 is immediate by induction.

We now define a truncation operation for complex numbers.

Definition 1. For any complex numbers a and b and any $\gamma > 0$ such that $|a| \leqq \gamma$, define

(2.5) $C_\gamma(a, b) = a + \lambda(b - a)$, where $0 \leqq \lambda \leqq 1$

and λ is the largest number between 0 and 1 such that $|a + \lambda(b - a)| \leqq \gamma$.

It is a straightforward matter to verify that for fixed γ, C_γ is a continuous function of the two variables a and b.

LEMMA 2. *Let $(X, \mathscr{F}, \mu)$ be a measure space. Let T be a linear contraction on $\mathscr{L}_p = \mathscr{L}_p(X, \mathscr{F}, \mu)$, such that*

(2.6) $\|Tf\|_\infty \leqq \|f\|_\infty$ *for each* $f \in \mathscr{L}_p \cap \mathscr{L}_\infty$.

Let f be in $\mathscr{L}_p$, H a set of positive, finite measure, and let $\beta > 0$ a number such that $\beta \geqq |f|$ on H. Let R be a measurable function on H, $|R| \geqq 3\beta$ on H, and let N be a positive integer such that for each $x \in H$ there exists an integer j, $0 \leqq j \leqq N$, such that

(2.7) $$R(x) = \frac{1}{j+1} \sum_{i=0}^{j} T^i f(x).$$

Then there exist functions $d_1, \ldots, d_N$, g on X such that:

(2.8) $d_k = 0$ *on* $X - H$ *and* $|d_1 + \ldots + d_k| \leqq 2\beta$

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on H, $k = 1, \ldots, N$;

(2.9) $\|g\|_p \leqq \|f - C_\beta(0, f)\|_p$,

(2.10) $T^N f = T^N C_\beta(0, f) + d_N + T d_{N-1} + \ldots + T^{N-1} d_1 + g$,

(2.11) $C_\beta(f, R) = f + d_1 + \ldots + d_N$ *on* H.

Proof. Let $h_0 = C_\beta(0, f)$ and let $g_0 = f - h_0$. Having defined h_i and g_i for $i = 0, 1, \ldots, k$, and having defined d_i for $i = 1, \ldots, k$, $0 \leqq k \leqq N - 1$, let a function U_{k+1} be defined as follows:

(2.12) $U_{k+1}(x)$ = the projection as a two-dimensional vector of $Tg_k(x)$ along $C_\beta(f, R)(x) - h_k(x)$ if $x \in H$ and $Tg_k(x) \cdot [C_\beta(f, R)(x) - h_k(x)] > 0$ (Here " $\cdot$ " denotes scalar product.); in all other cases let $U_{k+1} = 0$.

Let

(2.13) $h_{k+1} = C_\beta(h_k, h_k + U_{k+1})$.

(By an obvious induction we have $|h_k| \leqq \beta$ on X.) Let

(2.14) $d_{k+1} = h_{k+1} - h_k$,

(2.15) $g_{k+1} = Tg_k - d_{k+1}$.

This process defines $g_0, \ldots, g_N, h_0, \ldots, h_N$, and $d_1, \ldots, d_N$. Let $g = g_N$. From (2.13),

(2.16) $d_{k+1}(x) = \lambda_{k+1}(x)\, U_{k+1}(x)$, where $0 \leqq \lambda_{k+1}(x) \leqq 1$.

Since $U_{k+1}(x)$ is either 0 or a projection of $Tg_k(x)$ we see by (2.15) that

(2.17) $|g_{k+1}(x)| \leqq |Tg_k(x)|$.

Hence

(2.18) $\|g_{k+1}\|_p \leqq \|g_k\|_p$,

and (2.9) follows.

By (2.12) and (2.16) we have $d_k = 0$ on $X - H$. By (2.14), $d_1 + \ldots + d_k = h_k - h_0$, so by (2.13) we see that (2.8) holds.

Equation (2.10) is merely a rewritten version of (2.3). Thus only (2.11) remains to be proved. We can rewrite (2.11) as

(2.19) $C_\beta(f, R) = h_N$ on H.

Suppose for some point $x \in H$ and some $k \leqq N - 1$ that $h_k(x) = C_\beta(f,R)(x)$. It follows at once from (2.12) and (2.13) that $h_{k+1}(x) = C_\beta(f, R)(x)$ also. Thus, if at some point $x \in H$ we have $C_\beta(f, R)(x) \neq h_N(x)$, we also have

(2.20) $C_\beta(f, R)(x) \neq h_k(x) \qquad k = 0, 1, \ldots, N$.

Again, suppose for some point $x \in H$ and some $k \leqq N - 1$ that $d_{k+1}(x) \neq U_{k+1}(x)$. Then clearly $h_{k+1}(x) \neq h_k(x) + U_{k+1}(x)$. That is, $C_\beta(h_k(x), h_k(x) + U_{k+1}(x)) \neq h_k(x) + U_{k+1}(x)$, so that $|h_k(x) + U_{k+1}(x)| > \beta$. Also, since

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$U_{k+1}(x) \neq 0$, we must have $h_k(x) \neq C_\beta(f, R)(x)$, by (2.12). $U_{k+1}(x)$ has the same direction as $C_\beta(f, R)(x) - h_k(x)$, by (2.12). Thus $C_\beta(f, R)(x)$ is a point on the line joining $h_k(x)$ and $h_k(x) + U_{k+1}(x)$. We have $|h_k(x)| \leqq \beta$, $h_k(x) \neq C_\beta(f, R)(x)$, $|C_\beta(f, R)| = \beta$, and $|h_k(x) + U_{k+1}(x)| > \beta$. From these facts it follows by definition that $C_\beta(h_k(x), h_k(x) + U_{k+1}(x)) = C_\beta(f, R)(x)$. Thus we have shown that if for some $x \in H$ and some $k \leqq N - 1$ we have $d_{k+1}(x) \neq U_{k+1}(x)$, then

(2.21) $h_{k+1}(x) = C_\beta(f, R)(x)$.

Now let $x \in H$ be a point for which $C_\beta(f, R)(x) \neq h_N(x)$. We will obtain a contradiction. Using (2.20) and (2.21) we see that

(2.22) $d_{k+1}(x) = U_{k+1}(x) \quad \text{for } k = 0, 1, \ldots, N - 1.$

Hence, by (2.12),

(2.23) $g_{k+1}(x) \cdot [C_\beta(f, R)(x) - h_k(x)] \leqq 0, \qquad k = 0, 1, \ldots, N - 1.$

By induction, it is easy to show that $C_\beta(f, R)(x) - h_k(x)$ is a positive multiple of $R(x) - f(x)$, $k = 0, 1, \ldots, N$.
Thus

(2.24) $g_{k+1}(x) \cdot [R(x) - f(x)] \leqq 0, k = 0, \ldots, N - 1.$

Since $g_0(x) = 0$,

(2.25) $g_k(x) \cdot [R(x) - f(x)] \leqq 0, k = 0, \ldots, N.$

Choose j, $0 \leqq j \leqq N$, such that

(2.26) $$(j + 1)R(x) = \sum_{i=0}^{j} T^i f(x).$$

By (2.4),

(2.27) $$(j + 1)R(x) = h_j(x) + Th_{j-1}(x) + \ldots + T^j h_0(x) + g_0(x) + \ldots + g_j(x).$$

By (2.25),

(2.28) $$(j + 1)R(x) \cdot [R(x) - f(x)] \leqq (h_j(x) + Th_{j-1}(x) + \ldots + T^j h_0(x)) \cdot [R(x) - f(x)].$$

Since $|T^k h_{j-k}(x)| \leqq \beta$ for each k, by (2.6), we have

(2.29) $$(j + 1)(|R(x)|^2 - \beta|R(x)|) \leqq (j + 1)\beta[|R(x)| + \beta] \leqq (j + 1) \times \beta(4/3)|R(x)|,$$

or

(2.30) $|R(x)| \leqq (7/3)\beta$,

a contradiction. This completes the proof of Lemma 2.

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3. Proof of Theorem 1. We are given a complete, σ-finite measure space $(X, \mathscr{F}, \mu)$, and a set $\{T_t\}$, $t \geqq 0$, of bounded linear operators on $\mathscr{L}_p$ $(1 \leqq p < \infty)$ satisfying:

(3.1) $\quad T_{t+s} = T_t T_s \quad$ for all $t \geqq 0, s \geqq 0,$

(3.2) $\quad T_0 = I,$

(3.3) $\quad \|T_t\|_p \leqq 1, \quad$ for every $t \geqq 0,$

(3.4) $\quad \lim_{t \to s} T_t f = T_s f \quad$ for every $f \in \mathscr{L}_p$, $s \geqq 0$.

We also assume that

(3.5) $\quad \|T_t f\|_\infty \leqq \|f\|_\infty \quad$ for every $f \in \mathscr{L}_p \cap \mathscr{L}_\infty$, $t \geqq 0$.

We wish to prove that (1.1) holds for any $f \in \mathscr{L}_p$. Before proceeding, we establish some well-known facts about

$$\int_0^t T_s f(x) ds.$$

Fix $a \geqq 0$, $f \in \mathscr{L}_p(X, \mathscr{F}, \mu)$. Define

(3.6) $\quad g^n(t, x) = T_{ka/n} f(x) \quad$ for $\quad ka/n \leqq t < (k+1)a/n,$

$x \in X$, $k = 0, \ldots, n-1$. Thus g^n is defined on $[0, a) \times X$. Since the map $t \to T_t f$ is a continuous map into $\mathscr{L}_p(d\mu)$, it is easy to see that there exists a function g on $[0, a) \times X$ such that

(3.7) $\quad g^n \to g$ in $\mathscr{L}_p(dt \times d\mu)$ as $n \to \infty$.

For a subsequence n_j we have $g^{n_j}(t, x) \to g(t, x)$ as $j \to \infty$, for almost every (t, x). Thus for almost every t, $g^{n_j}(t, x) \to g(t, x)$ as $j \to \infty$, for almost every x. However, considered as a function in $\mathscr{L}_p(d\mu)$, it is clear that $g^{n_j}(t, \cdot) \to T_t f$ in $\mathscr{L}_p(d\mu)$ as $j \to \infty$, for every t. Hence

(3.8) $\quad g(t, x) = T_t f(x) \quad$ for almost every (t, x).

Thus g does not depend on our interval $[0, a)$, and we can define g on $[0, \infty) \times X$.

Returning to a finite interval $[0, a)$, we can choose our subsequence n_j such that $g^{n_j}(t, x) \to g(t, x)$ as $j \to \infty$ for almost every (t, x), and such that $\sum_j \|g^{n_j} - g\|_p < \infty$. Let $r = |g| + \sum_j |g^{n_j} - g|$. Then $\|r\|_p < \infty$ and $|r| \geqq |g^{n_j}|$ for all j. For almost every x, $r(\cdot, x)$ will have finite $\mathscr{L}_p(dt)$-norm, and hence finite $\mathscr{L}_1(dt)$-norm. Also, for almost every x, $g^{n_j}(t, x) \to g(t, x)$ as $j \to \infty$ for almost every t. Hence, by Lebesgue's dominated convergence theorem,

(3.9) $\quad \int_0^a g^{n_j}(t, x) dt \to \int_0^a g(t, x) dt \quad$ as $j \to \infty$,

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for almost every x. But clearly

$$(3.10) \quad \int_0^a g^{n_j}(t, \cdot)dt \to \int_0^a T_t f dt \quad \text{in } \mathscr{L}_p(d_\mu)$$

as $j \to \infty$. Thus

$$(3.11) \quad \int_0^a g(t, x)dt = \int_0^a T_t f dt(x)$$

for almost every x. We *define*

$$(3.12) \quad \int_0^a T_t f(x)dt = \int_0^a g(t, x)dt,$$

for any $a \geqq 0$. The point of this definition is that the left hand integral is a continuous function of t, for almost every x.

For future use we note that the sequence n_j appearing in (3.9) can clearly be chosen to be divisible by any fixed integer, and the proof of (3.9) shows that for almost every x

$$(3.13) \quad \int_0^b g^{n_j}(t, x)dt \to \int_0^b T_t f(x)dt$$

for all $b \leqq a$.

LEMMA 3. *For any* $f \in \mathscr{L}_p$,

$$(3.14) \quad \limsup_{t \to 0} \left| \frac{1}{t} \int_0^t T_s f(x)ds \right| \leqq 5|f(x)|$$

for almost every x.

Proof. Let E be a set of finite positive measure, and β a positive number such that

$$(3.15) \quad \limsup_{t \to 0} \left| \frac{1}{t} \int_0^t T_s f(x)ds \right| \geqq 5\beta \quad \text{on } E.$$

We must show that

$$(3.16) \quad |f(x)| \geqq \beta \quad \text{for almost all } x \in E.$$

Clearly we may assume that

$$(3.17) \quad |f(x)| \leqq \beta \quad \text{for all } x \text{ in } E.$$

Define

$$(3.18) \quad S_t(x) = C_{5\beta}\left(f(x), \frac{1}{t} \int_0^t T_s f(x)ds\right) \quad \text{for } x \in E.$$

By the continuity of $C_{5\beta}$ we see that $S_t(x)$ is measurable on $[0, \infty) \times E$ and $S_t(x)$ is a continuous function of t for almost every $x \in E$.

It is easy to see that we can find a measurable function S defined on E, such that

(3.19) $|S(x)| = 5\beta \quad \text{for } x \in E,$

and such that for almost every $x \in E$ there exists a sequence t_j, $t_j \to 0$ as $j \to \infty$, with

(3.20) $S(x) = \lim_{j\to\infty} S_{t_j}(x).$

Let $\epsilon > 0$ be given. We will now introduce some notation to clarify the remainder of the proof. Let h stand for a function in $\mathscr{L}_p$, appearing in this proof. In general there will be many ways of choosing the function for which h stands. It may, for example, depend on the choice of ϵ, or on subsequent choices. Let us regard all choices made in the proof *prior* to the choosing of ϵ as fixed. Then if h_1 and h_2 stand for functions appearing in this proof, we will write

(3.21) $h_1 \equiv h_2,$

if there exists one function $\sigma(\epsilon)$ such that

(3.22) $\|h_1 - h_2\|_p \leqq \sigma(\epsilon)$

for all possible choices of h_1, h_2, and ϵ, and

(3.23) $\sigma(\epsilon) \to 0 \quad \text{as } \epsilon \to 0.$

Let a positive integer l be chosen, with $l > 1/\epsilon$. Let a positive number δ be chosen, such that for every $t \leqq (l+1)\delta$ we have

(3.24) $\|(I - T_t)f\|_p < \epsilon, \qquad \|(I - T_t)C_\beta(0,f)\|_p < \epsilon,$

and $\|(I - T_t)h\|_p < \epsilon$, where $h(x) = C_\beta(f, S)(x) - f(x)$ for $x \in E$, $h(x) = 0$ for $x \notin E$. We note that (3.24) can be rewritten as

(3.25) $T_t f \equiv f, \qquad T_t C_\beta(0,f) \equiv C_\beta(0,f), \qquad T_t h \equiv h,$

for $t \leqq (l+1)\delta$.

Since $S_t(x)$ is a continuous function of t for almost every $x \in E$, we can find a set $E_1 \subseteq E$ with $\mu(E - E_1) < \epsilon$, and a positive integer n such that for any $x \in E_1$, an integer k exists, $1 \leqq k \leqq n$, with

(3.26) $|S_{k\delta/n}(x)| > 4\beta$

and

(3.27) $|C_\beta(f(x), S(x)) - C_\beta(f(x), S_{k\delta/n}(x))| < \epsilon.$

By (3.18) we see that for each $x \in E_1$, an integer k exists, $1 \leqq k \leqq n$, with

(3.28) $$\left| \frac{1}{(k\delta/n)} \int_0^{k/n\delta} T_t f(x)\,dt \right| > 4\beta$$

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and

(3.29) $$\left| C_\beta(f(x), S(x)) - C_\beta\left(f(x), \frac{1}{(k\delta/n)} \int_0^{k\delta/n} T_t f(x)dt\right)\right| < 2\epsilon.$$

By (3.13) we can choose a sequence $n_j \to \infty$ such that each n_j is a multiple of n, and such that for almost every $x \in E_1$

(3.30) $$\int_0^{k\delta/n} g^{n_j}(t, x)dt \to \int_0^{k\delta/n} T_t f(x)dt,$$

for $k = 1, \ldots, n$. Here $g^{n_j}(t, x)$ is defined to be $T_{i\delta/n_j}(x)$ for $i\delta/n_j \leqq t < (i + 1)\delta/n_j, i = 0, \ldots, n_j - 1$.

It follows that for some $n_j = N$ that there exists a set $H \subseteq E_1, \mu(E_1 - H) < \epsilon$, such that for each $x \in H$ an integer k exists, $1 \leqq k \leqq n$, with

(3.31) $$\left| \frac{1}{(k\delta/n)} \int_0^{k\delta/n} g^N(t, x)dt \right| > 3\beta$$

and

(3.32) $$\left| C_\beta(f(x), S(x)) - C_\beta(f(x)), \frac{1}{(k\delta/n)} \int_0^{k\delta/n} g^N(t, x)dt \right| < 3\epsilon.$$

For each $x \in H$, let

$$R(x) = \frac{1}{(k\delta/n)} \int_0^{k\delta/n} g^N(t, x)dt,$$

where k is chosen as in (3.31) and (3.32).

Let $T = T_{\delta/n}$. Since N is a multiple of n, it is easy to see that for each $x \in H$ there exists an integer j, $0 \leqq j \leqq N$, with

(3.33) $$R(x) = \frac{1}{j + 1} \sum_{i=0}^{j} T^i f(x).$$

We now apply Lemma 2 from section 2 to obtain functions $d_1, \ldots, d_N$, g on X such that (2.8)-(2.11) hold.

Define the operator W by

(3.34) $$W = \frac{1}{lN} \sum_{i=0}^{lN-1} T^i,$$

l as defined just before (3.24). Using (2.8), it follows easily that

(3.35) $$\left\| W\left(\sum_{i=1}^{N} T^{N-i} d_i - \sum_{i=1}^{N} d_i \right) \right\|_p \leqq \frac{2\beta}{l} m(E)^{1/p}.$$

Thus

(3.36) $$W \sum_{i=1}^{N} T^{N-i} d_i \equiv W \sum_{i=1}^{N} d_i.$$

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By (2.10)

$$W \sum_{i=1}^{N} d_i + Wg = W(T^N f - T^N C_\beta(0,f)) \equiv f - C_\beta(0,f) \quad \text{by (3.25).} \tag{3.37}$$

By (2.11)

$$\sum_{i=1}^{N} d_i = h_1, \tag{3.38}$$

where $h_1(x) = C_\beta(f(x), R(x)) - f(x)$ for $x \in H$ and $h_1(x) = 0$ for $x \notin H$.
By (3.32), $h_1 \equiv h$, where h is defined as in (3.25). Thus

$$W \sum_{i=1}^{N} d_i \equiv Wh \equiv h. \tag{3.39}$$

From (3.37) and (3.39) we obtain

$$Wg \equiv f - C_\beta(0,f) - h. \tag{3.40}$$

But $f - C_\beta(0,f)$ and h have disjoint supports by the definition of h, while $\|Wg\|_p \leqq \|f - C_\beta(0,f)\|_p$ by (2.10). Since h does not depend on ϵ, we must have $\|h\|_p = 0$. This proves (3.16) and completes the proof of Lemma 3.

Proof of Theorem 1. Let V be the collection of elements f in $\mathscr{L}_p$ such that (1.1) holds. V is obviously a linear space. It follows easily from Lemma 3 that V is closed in the norm topology. For any $f \in \mathscr{L}_p$, we can find a sequence $t_n \to 0$ such that

$$\frac{1}{t_n} \int_0^{t_n} T_s f(x) ds \in V$$

for each n. (This is Lemma 1 in [**4**], a generalization of Lemma 2 in [**2**].) Hence V is dense in $\mathscr{L}_p$, and hence $V = \mathscr{L}_p$. This proves Theorem 1.

4. Proof of Theorem 2. Let $Y = [0, \infty) \times X$. Let $\mathscr{G}$ be the usual product σ-algebra, and let $d\nu = dt \times d\mu$. For any bounded operator T on $\mathscr{L}_p(X, \mathscr{F}, \mu)$, define $\tilde{T}$ on $\mathscr{L}_p(Y, \mathscr{G}, \nu)$ for any f in $\mathscr{L}_p(Y, \mathscr{G}, \nu)$ by

$$\tilde{T}f(t, x) = Tf_t(x), \tag{4.1}$$

where $f_t(x)$ is the function on X defined by $f_t(x) = f(t, x)$. This defines $\tilde{T}f(t, x)$ for almost every (t, x). It is a straightforward matter to show that $\tilde{T}f \in \mathscr{L}_p(Y, \mathscr{G}, \nu)$, and that $\|\tilde{T}\|_p \leqq \|T\|_p$. Given a strongly continuous one-parameter semi-group $\{T_t\}$ (i.e., $\{T_t\}$ satisfying (3.1)-(3.4)) it is easy to see that $\{\tilde{T}_t\}$ is also a strongly continuous one-parameter semi-group of contractions on $\mathscr{L}_p(Y, \mathscr{G}, \nu)$.

Define the shift operator A_t for f in $\mathscr{L}_p(Y, \mathscr{G}, \nu)$ by

$$\begin{aligned} A_t f(s, x) &= f(s - t, x) \quad \text{for } s \geqq t, \\ A_t f(s, x) &= 0 \quad \text{for } s < t. \end{aligned} \tag{4.2}$$

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$\{A_t\}$ is clearly a strongly continuous one-parameter semi-group. Clearly $A_t\tilde{T}_s = \tilde{T}_s A_t$ for all t, s.

Now let h be a measurable function on Y satisfying (i) and (ii) of Theorem 2. Define the operator U_t on $\mathscr{L}_p(Y, \mathscr{G}, h^p d\nu)$ as follows:

$$U_t g = (1/h)A_t\tilde{T}_t hg \quad \text{for any } g \in \mathscr{L}_p(Y, \mathscr{G}, h^p d\nu). \tag{4.3}$$

It is easy to verify that $\{U_t\}$ is a semi-group, since A_t and $\tilde{T}_t$ commute. Let θ be the map from $\mathscr{L}_p(Y, \mathscr{G}, \nu)$ onto $\mathscr{L}_p(Y, \mathscr{G}, h^p d\nu)$ defined by $\theta f = f/h$. Since θ is an isometry, and $U_t = \theta A_t\tilde{T}_t\theta^{-1}$, it follows that $\{U_t\}$ is a strongly continuous one-parameter semi-group of contractions on $\mathscr{L}_p(Y, \mathscr{G}, h^p d\nu)$.

Finally we see that because of condition (ii) of Theorem 2 we have

$$\|U_t g\|_\infty \leqq \|g\|_\infty \quad \text{for each } g \text{ in } \mathscr{L}_p(Y, \mathscr{G}, h^p d\nu) \cap \mathscr{L}_\infty(Y, \mathscr{G}, h^p d\nu). \tag{4.4}$$

By Theorem 1, we have

$$\lim_{\alpha \to 0} \frac{1}{\alpha} \int_0^\alpha U_s g(t, x) ds = g(t, x) \tag{4.5}$$

for almost all (t, x) in Y.

Fix f in $\mathscr{L}_p(X, \mathscr{F}, \mu)$. Define g in $\mathscr{L}_p(Y, \mathscr{G}, h^p d\nu)$ by the equation

$$g(t, x) = \frac{1}{h(t, x)} f(x), \qquad 0 \leqq t < b, \tag{4.6}$$

$g(t, x) = 0$, $t \geqq b$, for some $b > 0$.

Then for a fixed (t, x), $0 \leqq t < b$, and any $s \leqq t$, we have

$$\begin{aligned} U_s g(t, x) &= \frac{1}{h(t, x)} A_s\tilde{T}_s hg(t, x) \\ &= \frac{1}{h(t, x)} \tilde{T}_s hg(t - s, x) \\ &= \frac{T_s f(x)}{h(t, x)}. \end{aligned} \tag{4.7}$$

Hence

$$\int_0^\alpha U_s g(t, x) ds = \frac{1}{h(t, x)} \int_0^\alpha T_s f(x) ds \quad \text{for } 0 \leqq t < b, \alpha \leqq t. \tag{4.8}$$

For almost every x, (4.5) holds for almost every t. For such an x and such a t, $0 < t < b$, we have

$$\lim_{\alpha \to 0} \frac{1}{h(t, x)} \frac{1}{\alpha} \int_0^\alpha T_s f(x) ds = g(t, x) = \frac{f(x)}{h(t, x)}. \tag{4.9}$$

This is (1.1), so Theorem 2 is proved.

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