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AN UNCERTAINTY PRINCIPLE LIKE HARDY'S THEOREM FOR NILPOTENT LIE GROUPS

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Dedicated to Eberhard Kaniuth on his 65th birthday

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Abstract

We extend an uncertainty principle due to Cowling and Price to threadlike nilpotent Lie groups. This uncertainty principle is a generalization of a classical result due to Hardy. We are thus extending earlier work on $\mathbb{R}^n$ and Heisenberg groups.

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Introduction

A classical theorem of Hardy [6] on Fourier transform pairs says that a non zero function f on the real line $\mathbb{R}$ and its Fourier transform $\hat{f}$ cannot both be very rapidly decreasing. More precisely, let the Fourier transform be defined by

$$\hat{f}(y) = \int_{\mathbb{R}} f(x) e^{-2\pi i x y}\, dx, \quad y \in \mathbb{R}.$$

Hardy's theorem says that if $|f(x)| \leq Ce^{-\alpha\pi x^2}$ for all $x \in \mathbb{R}$ and $|\hat{f}(y)| \leq Ce^{-\beta\pi y^2}$ for all $y \in \mathbb{R}$ with $\alpha\beta > 1$ then $f = 0$ a.e. For a proof see [6] or [4, Theorem 3.2]. The following is a generalization of this theorem due to Cowling and Price [3].

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THEOREM (Cowling and Price). *Let $f : \mathbb{R} \to \mathbb{C}$ be measurable and*

(i) $\|e_a f\|_{L^p(\mathbb{R})} < \infty$,
(ii) $\|e_b \hat{f}\|_{L^q(\mathbb{R})} < \infty$,

where $a, b > 0$, $e_k(x) = e^{k\pi x^2}$ and $1 \leq \min(p, q) < \infty$. If $ab \geq 1$, then $f = 0$ almost everywhere. If $ab < 1$, then there exist infinitely many linearly independent functions satisfying (i) *and* (ii).

An analogue of the Cowling-Price Theorem has been proved in [1] for Euclidean spaces, the Heisenberg group $\mathbb{H}_n$ and the Euclidean motion group of the plane. In this paper we concern ourselves with results of this kind on certain nilpotent Lie groups, thereby considerably extending the results for $\mathbb{R}^n$ and $\mathbb{H}_n$.

Threadlike nilpotent Lie groups

For $n \geq 3$, let $\mathfrak{g}_n$ be the n-dimensional real nilpotent Lie algebra with basis $X_1, X_2, \ldots, X_n$ and non trivial Lie brackets $[X_n, X_{n-1}] = X_{n-2}, \ldots, [X_n, X_2] = X_1$. Here $\mathfrak{g}_n$ is a $(n-1)$-step nilpotent and is a semi-direct product of $\mathbb{R}X_n$ and the abelian ideal $\sum_{j=1}^{n-1} \mathbb{R}X_j$. Note that $\mathfrak{g}_3$ is the Heisenberg Lie algebra. Let $G_n = \exp \mathfrak{g}_n$.

For $\xi = \sum_{j=1}^{n-1} \xi_j X_j^* \in \mathfrak{g}_n^*$, the coadjoint action of G_n is given by

$$Ad^*(e^{tX_n})\xi = \sum_{j=1}^{n-1} P_j(\xi, t) X_j^*,$$

where, for $1 \leq j \leq n-1$, $P_j(\xi, t)$ is the polynomial in t defined by

$$P_j(\xi, t) = \sum_{k=1}^{j-1} (1/k!)(-1)^k t^k \xi_{j-k}.$$

The orbit of ξ is generic with respect to the basis $\{X_1^*, X_2^*, \ldots, X_n^*\}$ if and only if $\xi_1 \neq 0$, and the jumping indices are 2 to n; see [2] for details. The cross section X_{ξ_1} for the set of generic orbits is given by

$$X_{\xi_1} = \left\{\xi = (\xi_1, 0, \xi_3, \ldots, \xi_{n-1}, 0) : \xi_i \in \mathbb{R}, \xi_1 \neq 0\right\}.$$

For $\xi \in \mathfrak{g}_n^*$, let π_ξ denote the irreducible representation of G_n associated with ξ. Then the mapping $\xi \to \pi_\xi$ is bijection of X_{ξ_1} and the set of all generic irreducible representations. Plancherel measure on $\widehat{G}_n$ is supported by these π_ξ.

Denoting by $\mathscr{F}$ the Fourier transform on $\mathbb{R}^{n-1}$, it follows that the Hilbert-Schmidt norm of the operator $\pi_\xi(f), f \in L^1 \cap L^2(G_n)$ is given by

$$\|\pi_\xi(f)\|_{HS}^2 = \int_{\mathbb{R}^2} \left|\mathscr{F}f\,(P_1(\xi, t), \ldots, P_{n-1}(\xi, t), t - s)\right|^2 ds\, dt$$

(for details see [2] and [5]).

Given a function $f : G_n \to \mathbb{C}$ and $y = (y_2, \ldots, y_n) \in \mathbb{R}^{n-1}$, let $f_y, f_y^* : \mathbb{R} \to \mathbb{C}$ be defined by

$$f_y(x_1) = f\left(e^{x_1 X_1 + \sum_{j=2}^n y_j X_j}\right),$$

and $f_y^*(x_1) = \overline{f_y(-x_1)}$.

The following lemma is proved in [7, Section 2 and Section 3].

LEMMA 1. *Let $f : G_n \to \mathbb{C}$ be a measurable function such that $|f(x)| \le ce^{-a\pi\|x\|^2}$ for some $a, c > 0$ and all $x \in G_n$. Let $g : \mathbb{R} \to \mathbb{C}$ be defined by*

$$g(x_1) = \int_{\mathbb{R}^{n-1}} f_y * f_y^*(x_1)\, dy.$$

Then $|g(x_1)| \le Ce^{-a\pi x_1^2/2}$ for some $C > 0$ and all $x_1 \in \mathbb{R}$ and

$$(*) \qquad \hat{g}(\xi_1) = |\xi_1| \int_{\mathbb{R}^{n-3}} \|\pi_\xi(f)\|_{HS}^2\, d\xi_3 \cdots d\xi_{n-1}.$$

THEOREM 2. *Let a, b and q be real numbers such that $a, b > 0$ and $q \ge 2$. Let $f : G_n \to \mathbb{C}$ be a measurable function and suppose that f satisfies:*

(i) $|f(x)| \le Ce^{-a\pi\|x\|^2}$ *for some $C > 0$ and all $x \in G_n$.*
(ii) $\int_{\mathbb{R}^{n-2}} |\xi_1| e^{bq\pi\|\xi\|^2} \|\pi_\xi(f)\|_{HS}^q\, d\xi_1\, d\xi_3 \cdots d\xi_{n-1} < \infty$.

Then the following hold:

(1) *If $q = 2$ and $ab \ge 1$, then $f = 0$ a.e.*
(2) *If $q > 2$ and $ab > 1$, then $f = 0$ a.e.*

PROOF. For $\alpha \in \mathbb{R}$, let $e_\alpha : \mathbb{R} \to \mathbb{R}$ denote the function $e_\alpha(t) = e^{\alpha\pi t^2}$. Let $g : \mathbb{R} \to \mathbb{C}$ be defined as in Lemma 1. We apply the Cowling-Price Theorem [3] to conclude that $g = 0$. Then Lemma 1 shows that $\pi_\xi(f) = 0$ for almost all $\xi \in \mathbb{R}^{n-2}$, whence $f = 0$ a.e.

For $q = 2$ by hypothesis (ii),

$$\|e_{2b}\hat{g}\|_1 = \int_{\mathbb{R}} e_{2b}(\xi_1) \left(\int_{\mathbb{R}^{n-3}} |\xi_1| \|\pi_\xi(f)\|_{HS}^2\, d\xi_3 \cdots d\xi_{n-1}\right) d\xi_1 < \infty.$$

Since $|g(x_1)| \le Ce^{-a\pi x_1^2/2}$ by Lemma 1 and $ab \ge 1$ so the Cowling-Price Theorem yields $g = 0$.

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For $q > 2$ and $ab > 1$, choose $\epsilon > 0$ such that $ab' > 1$, $b' = b - \epsilon$. Then for $\xi' = (\xi_3, \dots, \xi_{n-1})$, we have

$$\begin{aligned}
\|e_{2b'}\hat{g}\|_{q/2}^{q/2} &= \int_{\mathbb{R}} e_{b'q}(\xi_1)|\hat{g}(\xi_1)|^{q/2} d\xi_1 \\
&= \int_{\mathbb{R}} e_{b'q}(\xi_1)\left(\int_{\mathbb{R}^{n-3}} |\xi_1| \|\pi_\xi(f)\|_{HS}^2 \, d\xi'\right)^{q/2} d\xi_1 \\
&\leq \int_{\mathbb{R}} e_{b'q}(\xi_1)|\xi_1|^{q/2}\left(\int_{\mathbb{R}^{n-3}} e_{2b'}(\|\xi'\|)\|\pi_\xi(f)\|_{HS}^2 \, d\xi'\right)^{q/2} d\xi_1 \\
&= \int_{\mathbb{R}} e_{b'q}(\xi_1)|\xi_1|^{q/2}\left(\int_{\mathbb{R}^{n-3}} e_{2b}(\|\xi'\|)\|\pi_\xi(f)\|_{HS}^2 e_{-2\epsilon}(\|\xi'\|) d\xi'\right)^{q/2} d\xi_1 .
\end{aligned}$$

Applying Hölder's inequality with $q/2$ and $q/(q-2)$ we obtain

$$\begin{aligned}
\|e_{2b'}\hat{g}\|_{q/2}^{q/2} &\leq \int_{\mathbb{R}} \left(e_{b'q}(\xi_1)|\xi_1|^{q/2}\left(\int_{\mathbb{R}^{n-3}} e_{-(2\epsilon q)/(q-2)}(\|\xi'\|) d\xi'\right)^{(q/2)-1}\right. \\
&\qquad \left. \times \int_{\mathbb{R}^{n-3}} e_{bq}(\|\xi'\|)\|\pi_\xi(f)\|_{HS}^q \, d\xi'\right) d\xi_1 \\
&\leq K_1 \int_{\mathbb{R}} \left(e_{qb}(\xi_1)(e_{-q\epsilon}(\xi_1)|\xi_1|^{(q/2)-1}\right) |\xi_1| \\
&\qquad \times \left(\int_{\mathbb{R}^{n-3}} e_{bq}(\|\xi'\|)\|\pi_\xi(f)\|_{HS}^q \, d\xi'\right) d\xi_1 \\
&\leq K \int_{\mathbb{R}^{n-2}} e_{bq}(\|\xi\|)\|\pi_\xi(f)\|_{HS}^q |\xi_1| \, d\xi < \infty,
\end{aligned}$$

for certain positive constants K_1 and K. Thus $g = 0$ by the Cowling-Price Theorem. □

REMARK 3. If the formula $(*)$ in Lemma 1 reduces to $\hat{g}(\xi_1) = |\xi_1|\|\pi_\xi(f)\|_{HS}^2$ for some G_n, then for $1 \leq q < 2$ and $ab \geq 2$ along with the hypothesis in Theorem 2 implies that $f = 0$ a.e. The proof can be given as in [1, Theorem 2.1]. The above condition is satisfied if $G_n = G_3$, $G_{5,1}$, $G_{5,3}$ and $G_{5,6}$; see [9] for the definitions and structure of these groups.

THEOREM 4. *Let a and b be positive real numbers and $1 \leq \min(p, q) < \infty$. Suppose that $f \in L^1(G_n) \cap L^2(G_n)$ satisfies the following conditions*:

(i) $\int_{G_n} e^{pa\pi\|x\|^2}|f(x)|^p \, dx < \infty$,

(ii) $\int_{\mathbb{R}^{n-2}} |\xi_1| e^{b\pi q\|\xi\|^2} \|\pi_\xi(f)\|_{HS}^q \, d\xi < \infty$.

If $q \geq 2$ and $ab > 1$ then $f = 0$ a.e.

PROOF. Easy computations show that when, as before, identifying G_n as a set with $\mathbb{R}^n$, the product of two elements $y = (y_1, \dots, y_n)$ and $x = (x_1, \dots, x_n)$ of G_n is given by $yx = y + x + \sum_{j=1}^{n-2}(1/j!)y_n^j(x_{j+1}, \dots, x_{n-1}, 0, \dots, 0)$. For $\|x\| \geq 1$, this implies

$$\|yx\| \geq \|x\| - \|y\| - \|x\| \sum_{j=1}^{n-2} \frac{1}{j!}|y_n|^j \geq \|x\| \left(1 - \|y\| - \sum_{j=1}^{n-2} \frac{1}{j!}\|y\|^j\right).$$

Define $\varphi : (0, \infty) \to \mathbb{R}$ by $\varphi(\epsilon) = 1 - \epsilon - \sum_{j=1}^{n-2}(\epsilon^j/j!)$. Thus $\|yx\| \geq \|x\|\varphi(\epsilon)$, whenever $\|x\| \geq 1$ and $\|y\| \leq \epsilon$.

Let g be a continuous function on G_n such that $g(y) = g(y^{-1})$ for all $y \in G_n$ and $g(y) = 0$ for all y such that $\|y\| \geq \epsilon$. Since G_n is unimodular, for $x \in G_n$ such that $\|x\| \geq 1$,

$$\begin{aligned}(|g| * e_a|f|)(x) &= \int_{G_n} |g(y)|e_a(\|yx\|)|f(yx)|\,dy \\ &\geq \int_{G_n} |g(y)|e_a(\|x\|\varphi(\epsilon))|f(y^{-1}x)|\,dy \\ &= e_a(\|x\|\varphi(\epsilon))(|g| * |f|)(x).\end{aligned}$$

By (i) $e_a|f|$ is an L^p-function and $|g|$ is an $L^{p'}$ function $(1/p + 1/p' = 1)$, so $g * e_a|f|$ is an L^∞ function. Thus with $C = \||g| * e_a|f|\|_\infty < \infty$, we have

$$|g * f(x)| \leq |g| * |f|(x) \leq Ce_{-a}(\|x\|\varphi(\epsilon))$$

for all $x \in G_n$ such that $\|x\| \geq 1$. Since $g * f$ is continuous, it follows that for some constant $C > 0$, $|g * f(x)| \leq Ce_{-a}(\|x\|\varphi(\epsilon))$ for all $x \in G_n$. In addition,

$$\|\pi_\xi(g * f)\|_{HS} \leq \|\pi_\xi(g)\| \cdot \|\pi_\xi(f)\|_{HS} \leq \|g\|_1\|\pi_\xi(f)\|_{HS}$$

and hence, by hypothesis

$$\int_{\mathbb{R}^{n-2}} |\xi_1|e_{bq}(\|\xi\|)\|\pi_\xi(g * f)\|_{HS}^q\,d\xi \leq \|g\|_1^q \int_{\mathbb{R}^{n-2}} |\xi_1|e_{bq}(\|\xi\|)\|\pi_\xi(f)\|_{HS}^q\,d\xi < \infty.$$

Now for $\epsilon > 0$ sufficiently small, $ab\varphi(\epsilon) > 1$ so by Theorem 2 it follows that $g * f = 0$. Taking for g an approximate identity, we conclude that $f = 0$ a.e. □

The following result follows from Theorem 2, Remark 3 and Theorem 4.

THEOREM 5. *If $G_n = G_3, G_{5,1}, G_{5,3}$ or $G_{5,6}$ and $a, b > 0$. Suppose that p and q are such that $1 \leq \min(p, q) < \infty$ and $f \in L^1 \cap L^2(G_n)$ satisfies*

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(i) $\int_{\mathbb{R}^n} e^{pa\pi\|x\|^2}|f(x)|^p dx < \infty$ *if* $p < \infty$ *and* $|f(x)| \le Ce^{-a\pi\|x\|^2}$ *if* $p = \infty$,

(ii) $\int_{\mathbb{R}^{n-2}} |\xi_1| e^{b\pi q\|\xi\|^2} \|\pi_\xi(f)\|_{HS}^q d\xi < \infty$ *if* $q < \infty$ *and* $\|\pi_\xi(f)\|_{HS} \le Ce^{-b\pi\|\xi\|^2}$ *if* $q = \infty$.

Then the following hold:

(1) *If* $q \ge 2$ *and* $ab > 1$, *then* $f = 0$ *a.e.*

(2) *If* $1 \le q < 2$ *and* $ab > 2$, *then* $f = 0$ *a.e.*

Let $G = \exp \mathfrak{g}$ be a simply connected nilpotent Lie group. Let U denote the Zariski open subset of $\mathfrak{g}^*$ consisting of all elements in generic orbits with respect to the basis $\{X_1^*, \dots, X_n^*\}$ [2, Section 3.1, Theorem 3.1.9]. Let S be the set of jump indices, and set $T = \{1, 2, \dots, n\} \setminus S$ and $\mathfrak{g}_T^* = \sum_{j \in T} \mathbb{R} X_j^*$.

Then $X = U \cap \mathfrak{g}_T^*$ is a cross-section for the generic orbits and $\{\pi_\xi : \xi \in X\}$ supports the Plancherel measure on $\widehat{G}$.

The following is a generalization of Morgan's Theorem [8] which can be proved using [7, Lemma 2].

THEOREM 6. *Let* $G = \exp \mathfrak{g}$ *be a simply connected nilpotent Lie group. Let* α, β *and* C *be positive real numbers and suppose that* $f : G \to \mathbb{C}$ *is a measurable function such that*

(i) $|f(x)| \le Ce^{-\alpha\pi\|x\|^p}$,

(ii) $\|\pi_\xi(f)\|_{HS} \le Ce^{-\beta\pi\|\xi\|^q}$ *for all* $\xi = (\xi_1, \xi_2, \dots, \xi_n) \in X$,

where $p \ge 2$, $1/p + 1/q = 1$. *If* $(\alpha p)^{1/p}(\beta q)^{1/q} > 2$ *then* $f = 0$ *a.e.*

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