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Hypergeometric Abelian Varieties

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Abstract. In this paper, we construct abelian varieties associated to Gauss' and Appell–Lauricella hypergeometric series. Abelian varieties of this kind and the algebraic curves we define to construct them were considered by several authors in settings ranging from monodromy groups (Deligne, Mostow), exceptional sets (Cohen, Wolfart, Wüstholz), modular embeddings (Cohen, Wolfart) to CM-type (Cohen, Shiga, Wolfart) and modularity (Darmon). Our contribution is to provide a complete, explicit and self-contained geometric construction.

Introduction

This paper provides an explicit construction of abelian varieties associated to Gauss' hypergeometric series (one variable) and, more generally, to Appell-Lauricella hypergeometric series (several variables). Roughly speaking, to one hypergeometric series F, one associates a family of nonsingular algebraic curves indexed by the variables of F, on which the numerator of the integral representation of F is a period. The associated abelian varieties are abelian subvarieties of the Jacobian varieties of these curves. They all have the same dimension depending on F only and multiplications by the same field.

Abelian varieties $T_{abc}(z)$ associated to Gauss' hypergeometric series $F(a, b, c\,; z)$ appear in different settings. Defined by Wolfart [21], [22], they were the support for his investigations about the size and nature of the so-called *exceptional set*, which is the set of algebraic points at which the series takes algebraic values. Using a consequence ([23] Satz 2) of Wüstholz's Analytic Subgroup Theorem ([24] Hauptsatz), Wolfart showed that, under some conditions on a, b, c, the points $z \in E(a, b, c)$ correspond to isogenous abelian varieties (of the same dimension) defined over $\bar{\mathbb{Q}}$ and having complex multiplication (see also Cohen and Wolfart [7]). The conditions on a, b, c make the monodromy group $\Delta(a, b, c)$ a triangle subgroup of $SL_2(\mathbb{R})$. Wolfart showed that the arithmeticity of the monodromy group implies the infinity of the exceptional set. Cohen and Wüstholz [10] proved the converse assertion (under the same conditions). For this they proposed and used a special case of a weak version of André-Oort's conjecture. This case of the conjecture was proved recently by Edixhoven and Yafaev [14]. Explicit determinations of exceptional sets of hypergeometric series with monodromy group isomorphic to $SL_2(\mathbb{Z})$ can be found in [4] for two of them and in [3] for a wider family.

The geometric objects associated to hypergeometric series provide tools for the study of the monodromy groups of these series. For instance, the families of algebraic curves that we will consider appear in Deligne and Mostow [13], where the

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monodromy group of the Appell-Lauricella hypergeometric series is shown to be, under certain assumptions on the parameters, a lattice in a projective unitary group. Embeddings of monodromy groups of hypergeometric series into modular groups are constructed by Cohen and Wolfart in [7] for one variable and in [9] for several variables. The monodromy group Δ of a hypergeometric series F is embedded into a modular group acting on the universal covering space of some Shimura variety. The subgroup fixing the subvariety parametrizing the abelian varieties associated to F contains with finite index the image of Δ under the embedding. Using ideas developed with Cohen, Shiga and Wolfart [17] found a criteria for an abelian variety over $\bar{\mathbb{Q}}$ with generalized complex multiplication to be of CM-type. They showed that for hypergeometric abelian varieties, CM-type is equivalent to the algebraicity of all quotients of periods. They conjecture that this should be equivalent to the algebraicity of one quotient of periods, as Wolfart proved in the one-variable case.

In the context of the generalized Fermat equation, Darmon [11], [12] studied the modularity of hypergeometric abelian varieties in one variable.

This paper is structured in the following way. Section 1 recalls the definition of the hypergeometric series in one and several variables and shows how one can associate a family of curves to one hypergeometric series via its integral representation. Section 2 gives the precise definition of this family of curves and, in particular, the hypotheses on the parameters. The possibly singular points of the curve are also determined in there. In Section 3, we construct the nonsingular model of the curve by first desingularizing locally (Section 3.1) and then glueing the local desingularizations (Section 3.2). The compositions of the desingularization morphism with local parametrizations of the nonsingular curve at the points lying above the singular points are calculated in Section 3.3. One application is the computation of Euler characteristic and of the genus of the nonsingular model (Section 4), another application is the computation of the order of differential forms (Section 6), which are eigenforms for the action of some roots of unity (Section 5). The dimension of the eigenspaces for this action is given in Sections 6.2–6.4. So-called *new* eigenspaces are selected which define an abelian subvariety of the Jacobian variety, called the *New Jacobian*, whose dimension depends only on the family of curves (Section 7). In the one-variable case, this New Jacobian is isomorphic to the $\varphi\big(\operatorname{lcd}(a,b,c)\big)$-dimensional abelian variety $T_{abc}(z)$ defined by Wolfart, as shown in Section 8. Finally, Section 9 treats the "zero-variable" case by constructing an abelian variety on which the Beta-function lives as a period. This matches with Rohrlich's construction in the Appendix of [20]. The last two sections translate our general construction to Wolfart's language in the special case of Gauss' hypergeometric series, completing this way Wolfart's interpretation of the integral representation of $F(a,b,c;z)$ as a quotient of periods, the key for his study of the algebraic values of F via the consequence of Wüstholz Analytic Subgroup Theorem (*cf.* Remark 11).

We wish to thank the referee for pointing out that different methods and existing results could be used to shorten the proofs of Theorems 6.2 and 6.8 respectively. For instance, the statement of Theorem 6.8 could be proved by applying the result of [6], as Deligne and Mostow [13] and Wolfart [21], [22] did in the one variable case ($r = 2$ in our notations) and Cohen and Wolfart [9] in the two variable case ($r = 3$) (see also Note 2). Our proof of Theorem 6.8 is yet totally independent of

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[6] and relies on Theorem 6.2, which is itself based on the geometric construction of the nonsingular model of the curve (Section 3). Although a different and shorter method could be used to prove Theorem 6.2 (see Note 1), we hope that the explicit geometric construction of the nonsingular model given in this paper will serve as an example and have further applications.

Most of the material of this paper is contained in Chapter 1 of the author's PhD thesis [2].

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1 Hypergeometric Series

Gauss' hypergeometric series is defined to be

$$F(a,b,c;z)=\sum_{n=0}^{\infty}\frac{(a;n)(b;n)}{(c;n)(1;n)}z^n, \tag{1}$$

where $(x;n):=\prod_{j=1}^{n}(x+n-j)$ and $a,b,c\in\mathbb{C}$, $-c\notin\mathbb{N}$. It converges in the unit disc, where it enjoys the so-called Euler's integral representation

$$F(a,b,c;z)=\frac{1}{B(b,c-b)}\int_0^1 x^{b-1}(1-x)^{c-b-1}(1-zx)^{-a}\,dx \tag{2}$$

provided the integral converges, *i.e.* $\operatorname{Re}(c)>\operatorname{Re}(b)>0$. The denominator is the Beta-function $B(\alpha,\beta)=\int_0^1 x^{\alpha-1}(1-x)^{\beta-1}\,dx$. Using the change of variables $x\mapsto\frac{1}{u}$ and the symmetry of (1) in a and b, we can rewrite (2) as

$$F(a,b,c;z)=\frac{1}{B(a,c-a)}\int_1^{\infty}u^{-c+b}(u-1)^{c-a-1}(u-z)^{-b}\,du. \tag{3}$$

Gauss' hypergeometric series has been generalized in many ways to series in several variables. We will be interested in the series F_D named after Appell and Lauricella, which has an integral representation of Euler type (see [1], [15], [18], [8]). Indeed, consider the Appell-Lauricella hypergeometric series of the complex variables $z_1,\dots,z_d$

$$F_D(a,b_1,\dots,b_d,c;z_1,\dots,z_d)=\sum_{n_1,\dots,n_d=0}^{\infty}\frac{(a;\sum_j n_j)\prod_j(b_j;n_j)}{(c;\sum_j n_j)\prod_j(1;n_j)}\prod_{j=1}^{d}z_j^{n_j},$$

where $a,b_1,\dots,b_d,c\in\mathbb{C}$, $-c\notin\mathbb{N}$ and j runs from 1 to d. This series converges if $|z_j|<1$ for each $j=1,\dots,d$. If $\operatorname{Re}(c)>\operatorname{Re}(a)>0$, it has the following integral representation

$$\frac{1}{B(a,c-a)}\int_1^{\infty}u^{-c+\sum_j b_j}(u-1)^{c-a-1}\prod_{j=1}^{d}(u-z_j)^{-b_j}\,du. \tag{4}$$

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Remark 1 $F(a,b,c\,;z)$ satisfies the so-called *hypergeometric differential equation*

$$z(1-z)\frac{d^2u}{dz^2}+\big(c-(a+b+1)z\big)\frac{du}{dz}-abu=0.$$

Similarly, the function F_D satisfies a system of partial linear differential equations, as can be found in [18].

If $a, c-a\notin\mathbb{Z}$, the two integrals in the integral representation (4) can be replaced up to algebraic factors by periods on curves. For convenience of further notations, we now consider the function $F_D(a,b_2,\dots,b_r,c\ ;\ \lambda_2,\dots,\lambda_r)$ of the $r-1$ variables $\lambda_2,\dots,\lambda_r$ and write its integral representation as

$$\frac{1}{B(a,c-a)}\int_1^\infty\prod_{i=0}^r(u-\lambda_i)^{-\mu_i}\,du, \tag{5}$$

where we have set $\lambda_0=0$, $\lambda_1=1$ and $\mu_0=c-\sum_{j=2}^r b_j$, $\mu_1=1+a-c$, $\mu_j=b_j$ for $j=2,\dots,r$. Suppose now that the μ_i's are rational numbers and let N be their least common denominator. Consider the projective curve C_N defined by the affine equation

$$y^N=\prod_{i=0}^r(x-\lambda_i)^{N\mu_i}.$$

Then the integral in the numerator of (5) is equal, up to an algebraic factor, to a period $\int_\gamma\frac{dx}{y}$ on C_N, where γ is a loop on C_N whose image in $\mathbb{P}^1_{\mathbb{C}}$ under the projection $(x,y)\mapsto x$ is a double contour loop (also called *Pochhammer loop*) around 1 and ∞. Note that the condition $c-a\notin\mathbb{Z}$ implies $N\nmid\sum_{i=0}^r N\mu_i$.

2 The Family of Curves

As seen in Section 1, the projective curve associated to an Appell-Lauricella hypergeometric series $F_D(a,b_2,\dots,b_r,c\ ;\ \lambda_2,\dots,\lambda_r)$ is affinely defined by the equation $y^N=\prod_{i=0}^r(x-\lambda_i)^{A_i}$, where the exponents are integers defined by the parameters $a,b_2,\dots,b_r,c$ of the series. Letting the complex variables $\lambda_2,\dots,\lambda_r$ varying in $\mathbb{C}$, we can view the curves associated to F_D as forming a family over $\mathbb{C}^{r-1}$. If we let λ_0 and λ_1 vary and allow the λ_i's to be infinite, we get a family over $(\mathbb{P}^1_{\mathbb{C}})^{r+1}$. We will restrict ourselves to the study of the nondegenerated fibers (*i.e.* those for which $\forall i$, $\lambda_i\neq\infty$ and $\forall j\neq i$, $\lambda_j\neq\lambda_i$), because the construction in the degenerated case can be recovered from that in the generic case.

Concerning the exponents of the equation, we have from Section 1 the condition $N\nmid\sum_{i=0}^r A_i$. For technical calculations (Sections 6.3–6.4) leading to a nice formula for the dimension of the New Jacobian (Section 7), we will make the assumption $N\nmid A_0,\dots,A_r$. We also want the curves to be irreducible, what amounts to $(N,A_0,\dots,A_r)=1$. This is no big deal, as the results for reducible curves can be obtained from the results for its irreducible components. Furthermore, since our results will be relevant only up to isomorphism of the curve, we can suppose the exponents to be positive as showed in the following remark.

Remark 2 Let X'_N be the desingularization of the projective curve C'_N defined affinely by

$$y'^N = \prod_{i=0}^{r} (x' - \lambda_i)^{A_i},$$

where $N \in \mathbb{N}$, $A_0, \dots, A_r \in \mathbb{Z}$. For each $i \in \{0, \dots, r\}$, write $A_i = k_i N + r_i$ with $0 \le r_i \le N - 1$ and $k_i \in \mathbb{Z}$. Note that if $A_i < 0$ then $k_i < 0$. With these notations, the equation reads

$$y'^N \prod_{A_i<0} (x' - \lambda_i)^{-k_i N} = \prod_{A_i<0} (x' - \lambda_i)^{r_i} \prod_{A_i>0} (x' - \lambda_i)^{A_i}.$$

Let C_N denote the projective curve defined by the affine equation

$$y^N = \prod_{A_i<0} (x - \lambda_i)^{r_i} \prod_{A_i>0} (x - \lambda_i)^{A_i}.$$

Then we have a map $\rho\colon C'_N \to C_N$ given by

$$(x', y') \mapsto \Big(x', y' \prod_{A_i<0} (x' - \lambda_i)^{-k_i}\Big) =: (x, y)$$

$$\infty \mapsto \infty.$$

It is well-defined, because for $(x', y') \in C'_N$, we have

$$y^N = \Big(y' \prod_{A_i<0} (x' - \lambda_i)^{-k_i}\Big)^N = \prod_{A_i<0} (x' - \lambda_i)^{r_i} \prod_{A_i>0} (x' - \lambda_i)^{A_i}$$

and $(x, y) \in C_N$. Since $A_i < 0$ implies $-k_i > 0$, ρ is a morphism. It has a rational inverse given by $(x, y) \mapsto \big(x, y \prod_{A_i<0} (x - \lambda_i)^{k_i}\big)$. The desingularization maps $\pi'\colon X'_N \to C'_N$ and $\pi\colon X_N \to C_N$ are birational morphisms. Then $\rho \circ \pi'$ is a birational morphism from X'_N to C_N and there exists a unique isomorphism $\tilde{\rho}\colon X'_N \to X_N$ such that the following diagram commutes

$$\begin{array}{ccc} X'_N & \xrightarrow{\tilde{\rho}} & X_N \\ {\scriptstyle \pi'}\big\downarrow & & \big\downarrow{\scriptstyle \pi} \\ C'_N & \xrightarrow[\rho]{} & C_N. \end{array}$$

Let's now define the family of curves we will be working on.

Definition 1 For $r \in \mathbb{Z}_{\ge 0}$ and $N, A_0, \dots, A_r \in \mathbb{N}$ such that $N \nmid A_0, \dots, A_r$, $\sum_{i=0}^{r} A_i$, $(N, A_0, \dots, A_r) = 1$ and $\lambda_0, \dots, \lambda_r \in \mathbb{C}$ such that $\lambda_i \ne \lambda_j$ if $i \ne j$, we denote by C_N the projective algebraic curve defined by the affine equation

$$y^N = \prod_{i=0}^{r} (x - \lambda_i)^{A_i}$$

and X_N for its desingularization.

Both curves are irreducible projective algebraic curves defined over $\mathbb{C}$. The projective equations of C_N read

Case 1: $N - \sum_{k=0}^{r} A_k > 0$. $\quad C_N : x_2^N = x_0^{N-\sum_{k=0}^{r} A_k} \prod_{i=0}^{r}(x_1 - \lambda_i x_0)^{A_i}$

Case 2: $N - \sum_{k=0}^{r} A_k < 0$. $\quad C_N : x_2^N x_0^{-N+\sum_{k=0}^{r} A_k} = \prod_{i=0}^{r}(x_1 - \lambda_i x_0)^{A_i}$.

The point at infinity is respectively $(0:1:0)$ in the first case and $(0:0:1)$ in the second one. It will be denoted by ∞ when we do not wish to specify the case. Note that the case $N - \sum_{k=0}^{r} A_k = 0$ is excluded by our hypotheses.

Recall that the singular points are those whose coordinates annihilate all the partial derivatives of the polynomial defining the curve. One can verify that the only possibly singular points are the point at infinity together with the affine points $(1:\lambda_i:0)$ for $i = 0, \ldots, r$. More precisely, we have for $i \in \{0, \ldots, r\}$:

$$(1:\lambda_i:0) \text{ is singular } \Leftrightarrow A_i > 1$$

and

$$\infty \text{ is singular } \Leftrightarrow \Big|N - \sum_{j=0}^{r} A_j\Big| > 1.$$

3 Construction of the Desingularization

This section gives the explicit construction of the desingularization X_N of C_N. We first desingularize locally above the possibly singular points of C_N (Section 3.1). These local desingularizations are glued together in Section 3.2 to build the nonsingular model X_N of C_N. Finally, Section 3.3 gives the desingularization map in local coordinates above the possibly singular points. This will be used to calculate the genus of X_N (Section 4) and a basis of regular differential forms on X_N (Section 6).

3.1 Local Desingularizations

Let P be a possibly singular point of C_N. We will work locally on an affine open neighbourhood of P.

3.1.1 Above $P_j := (1:\lambda_j:0)$

In the neighbourhood of an affine point, we have the classical isomorphism

$$C_N - \{\infty\} \xrightarrow{\sim} C_{\text{aff}}$$
$$(x_0:x_1:x_2) \mapsto \left(\frac{x_1}{x_0}, \frac{x_1}{x_0}\right).$$

It induces a morphism $\kappa_0 \colon C_{\text{aff}} \to C_N$ given by $(x, y) \mapsto (1:x:y)$. Setting $x := \frac{x_1}{x_0}$, $y := \frac{x_1}{x_0}$, we recover the affine equation of C_N

$$C_{\text{aff}} : y^N = \prod_{i=0}^{r}(x - \lambda_i)^{A_i}.$$

Let's fix $j \in \{0, \dots, r\}$ and work locally in a neighbourhood of $(\lambda_j, 0)$ on which $g_j(x) := \prod_{i \neq j}(x - \lambda_i)^{A_i} \neq 0$. Set

$$N' := \frac{N}{(N, A_j)} \quad \text{and} \quad A' := \frac{A_j}{(N, A_j)}.$$

Then there exist $n, m \in \mathbb{Z}$ such that

$$nN' + mA' = 1$$

and we have

$$(x, y) \in C_{\text{aff}} \Rightarrow \begin{cases} y^{N'} = (x - \lambda_j)^{A'} u & \text{and} \\ u^{(N, A_j)} = g_j(x). \end{cases}$$

Remark that

$$y^{mN'}(x - \lambda_j)^{nN'} = (x - \lambda_j)u^m$$

and that $u = y^{N'}(x - \lambda_j)^{-A'}$. Hence, if we set $z := y^m(x - \lambda_j)^n$ and define

$$X_j := \{(x, u, z) \in \mathbb{C}^3 \,;\, z^{N'} = (x - \lambda_j)u^m, u^{(N, A_j)} = g_j(x), g_j(x) \neq 0\}$$

and $C_{\text{aff},j} := C_{\text{aff}} - \{(x, y) \in C_{\text{aff}} \,;\, g_j(x) = 0\}$. Then the rational map

$$\begin{aligned} \nu_j \colon C_{\text{aff},j} &\to X_j \\ (x, y) &\mapsto \bigl(x, y^{N'}(x - \lambda_j)^{-A'}, y^m(x - \lambda_j)^n\bigr) \end{aligned}$$

becomes a morphism on the open dense subset $\hat{C}_{\text{aff}} := C_{\text{aff},j} - \{(\lambda_j, 0)\}$ of $C_{\text{aff},j}$. This morphism has an inverse given by the morphism

$$\begin{aligned} \tau_j \colon X_j &\to C_{\text{aff},j} \\ (x, u, z) &\mapsto (x, u^n z^{A'}). \end{aligned}$$

In particular, τ_j is a birational morphism, which restricts to an isomorphism from $\hat{X}_j := X_j - \bigl(\tau_j^{-1}\{(\lambda_j, 0)\}\bigr)$ to $\hat{C}_{\text{aff}} = C_{\text{aff}} - \{(\lambda_k, 0) \,;\, k = 0, \dots, r\}$. Moreover, since C_{aff} is isomorphic to $C_N - \{\infty\}$ under κ_0, $\hat{C}_{\text{aff}}$ is isomorphic to $C_N - \{P_0, \dots, P_r, \infty\}$ and

$$\hat{X}_j \xrightarrow{\sim} C_N - \{P_0, \dots, P_r, \infty\}.$$

In particular, X_j is birationally equivalent to C_N under the morphism $\pi_j := \kappa_0 \circ \tau_j$.

Remark 3 The point $(\lambda_j, 0) \in C_{\text{aff},j}$ has exactly (N, A_j) preimages under τ_j, which are the points $(\lambda_j, u, 0)$, where u runs among the (N, A_j)-th roots of $g_j(\lambda_j)$. (They are distinct, because $g_j(\lambda_j) \neq 0$.) $P_j \in C_N$ has then also (N, A_j) π_j-preimages on X_j.

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Remark 4 X_j is nonsingular. Indeed, calculating the Jacobian matrix of X_j, we get

$$\begin{pmatrix} u^m & g_j'(x) \\ m(x-\lambda_j)u^{m-1} & -(N,A_j)u^{(N,A_j)-1} \\ -N'z^{N'-1} & 0 \end{pmatrix}$$

Remember that $u \neq 0$ on X_j. If $x = \lambda_j$, then the upper square looks like $\left(\begin{smallmatrix} \neq 0 & * \\ 0 & \neq 0 \end{smallmatrix}\right)$ and has rank 2. If $x \neq \lambda_j$, then $z \neq 0$ and the lower square looks like $\left(\begin{smallmatrix} \neq 0 & \neq 0 \\ \neq 0 & 0 \end{smallmatrix}\right)$ and the matrix has rank 2.

3.1.2 Above Infinity

Case 1: $N - \sum_{k=0}^{r} A_k > 0$. In this case, the projective equation of C_N is

$$x_2^N = x_0^{N-\sum A_k} \prod_{i=0}^{r} (x_1 - \lambda_i x_0)^{A_i}$$

and the point at infinity has coordinates $(0:1:0)$. We choose the neighbourhood of $(0:1:0)$ on C_N, on which $x_1 \neq 0$, and have the isomorphism

$$C_N \cap \{x_1 \neq 0\} \xrightarrow{\sim} C_{\infty 1}$$
$$(x_0 : x_1 : x_2) \mapsto \left(\frac{x_0}{x_1}, \frac{x_2}{x_1}\right) =: (x, y).$$

Its inverse is given by $\kappa_1 \colon (x, y) \mapsto (x:1:y)$. Remark that the affine possibly singular points with coordinate $x_1 \neq 0$ also lie on $C_N \cap \{x_1 \neq 0\}$. In the coordinates (x, y), the equation of $C_{\infty 1}$ is

$$C_{\infty 1} : y^N = x^{N-\sum A_k} \prod_{i=0}^{r} (1 - \lambda_i x)^{A_i},$$

the point at infinity is $(x, y) = (0, 0)$ and the affine possibly singular points are the points $(\frac{1}{\lambda_k}, 0)$, for each $k \in \{0, \dots, r\}$ such that $\lambda_k \neq 0$. Let's set

$$h(x) := \prod_{i=0}^{r} (1 - \lambda_i x)^{A_i}, \quad N' := \frac{N}{(N, N - \sum A_k)} \quad \text{and} \quad A' := \frac{N - \sum A_k}{(N, N - \sum A_k)}.$$

Then there exist $n, m \in \mathbb{Z}$ such that

$$nN' + mA' = 1$$

and we have

$$(x, y) \in C_{\infty 1} \Rightarrow \begin{cases} y^{N'} = x^{A'} u & \text{and} \\ u^{(N, N - \sum A_k)} = h(x). \end{cases}$$

Note that

$$y^{mN'}x^{nN'} = xu^m \quad \text{and} \quad u = y^{N'}x^{-A'}.$$

Set further $z := y^m x^n$, then

$$X_{\infty 1} := \{(x,u,z) \in \mathbb{C}^3 \,;\, z^{N'} = xu^m, u^{(N,N-\sum A_k)} = h(x), h(x) \neq 0\}$$

and $C'_{\infty 1} := C_{\infty 1} - \{(x,y)\,;\, h(x) = 0\}$. Hence we have a rational map

$$\begin{aligned} \nu_{\infty 1} \colon C'_{\infty 1} &\to X_{\infty 1} \\ (x,y) &\mapsto (x, y^{N'}x^{-A'}, y^m x^n), \end{aligned}$$

which restricts to a morphism on the open dense subset $\hat{C}_{\infty 1} := C'_{\infty 1} - \{(0,0)\}$ of $C'_{\infty 1}$. This morphism has an inverse given by the morphism

$$\begin{aligned} \tau_{\infty 1} \colon X_{\infty 1} &\to C'_{\infty 1} \\ (x,u,z) &\mapsto (x, z^{A'}u^n). \end{aligned}$$

$\tau_{\infty 1}$ is then a birational morphism and restricts to an isomorphism of $\hat{X}_{\infty 1} := X_{\infty 1} - \tau_{\infty 1}^{-1}\{(0,0)\}$ to $\hat{C}_{\infty 1}$. Since $C_{\infty 1}$ is isomorphic to $C_N \cap \{x_1 \neq 0\}$ under κ_1, $\hat{C}_{\infty 1}$ is isomorphic to $C_N - \{P_0, \dots, P_r, \infty\}$ and so is $\hat{X}_{\infty 1}$, *i.e.*

$$\hat{X}_{\infty 1} \xrightarrow{\sim} C_N - \{P_0, \dots, P_r, \infty\}.$$

In particular, $X_{\infty 1}$ is birationally equivalent to C_N under the morphism $\pi_{\infty 1} := \kappa_1 \circ \tau_{\infty 1}$.

Remark 5 The point at infinity, which has coordinates $(x,y) = (0,0)$ on $C_{\infty 1}$, has $(N, N - \sum A_k)$ preimages under $\tau_{\infty 1}$. They are $(0,u,0)$, where u is a $(N, N-\sum A_k)$-th root of $h(0)$. Since $h(0) \neq 0$, this implies that the point at infinity on C_N has also $(N, N - \sum A_k)$ preimages on $X_{\infty 1}$. One verifies that $X_{\infty 1}$ is nonsingular.

Case 2: $N - \sum_{k=0}^{r} A_k < 0$. In this case, the projective equation of C_N reads

$$x_2^N x_0^{-N+\sum A_k} = \prod_{i=0}^{r} (x_1 - \lambda_i x_0)^{A_i}$$

and the point at infinity $(0\!:\!0\!:\!1)$. We choose a neighbourhood of $(0\!:\!0\!:\!1)$ on C_N, on which $x_2 \neq 0$. On this neighbourhood, there is no other possibly singular point than $(0\!:\!0\!:\!1)$, because they all have coordinate $x_2 = 0$. We have the isomorphism

$$\begin{aligned} C_N \cap \{x_2 \neq 0\} &\to C_{\infty 2} \\ (x_0\!:\!x_1\!:\!x_2) &\mapsto \left(\frac{x_0}{x_2}, \frac{x_1}{x_2}\right) =: (x,y), \end{aligned}$$

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with inverse $\kappa_2\colon (x,y)\mapsto (x{:}y{:}1)$. In these coordinates, the equation of $C_{\infty 2}$ is

$$C_{\infty 2} : x^{-N+\sum A_k} = \prod_{i=0}^{r} (y-\lambda_i x)^{A_i}$$

and the point at infinity $(x,y)=(0,0)$. But the equation in this shape is not easy to handle. We will see that after having blown up the point $(0,0)$ on $C_{\infty 2}$, everything becomes easier. In order to do this, we will use the expressions of the blow-up map in local coordinates.

In the first coordinates' set, the point $(0,0)$ has no preimage on the preimage of $C_{\infty 2}$ bereft of the exceptional divisor. In the second coordinates' set, the preimage of $C_{\infty 2}$ under the map $\varphi\colon (u,v)\mapsto (uv,v)$ is given by

$$u^{-N+\sum A_k} v^{-N+\sum A_k} = v^{\sum A_k} \prod_{i=0}^{r} (1-\lambda_i u)^{A_i}$$
$$\iff \begin{cases} v=0 \ \text{(exceptional divisor)} \quad \text{or} \\ C'_{\infty 2} : u^{-N+\sum A_k} = v^N \prod_{i=0}^{r} (1-\lambda_i u)^{A_i} \end{cases}$$

and the preimage of $(x,y)=(0,0)$ is $(u,v)=(0,0)$.

We can apply to $C'_{\infty 2}$ the same procedure as in the other cases, though it will be slightly more technical. As usual, we begin by setting

$$h(u) := \prod_{i=0}^{r} (1-\lambda_i u)^{A_i}, \quad N' := \frac{N}{(N, -N+\sum A_k)} \quad \text{and} \quad A' := \frac{-N+\sum A_k}{(N, -N+\sum A_k)}$$

and letting $n, m \in \mathbb{Z}$ be such that $nN' + mA' = 1$. Then $(u,v) \in C'_{\infty 2}$ implies

$$\begin{cases} u^{A'} = v^{N'} w, \\ w^{(N,-N+\sum A_k)} = h(u), \\ h(u) \neq 0. \end{cases}$$

Note that $u^{nA'} v^{mA'} = v w^n$ and $w = u^{A'} v^{-N'}$. Set $z := u^n v^m$ and define

$$X_{\infty 2} := \{(u,v,w,z) \in \mathbb{C}^4 \,;\, z^{A'} = v w^n, w^{(N,-N+\sum A_k)} = h(u), h(u) \neq 0\}.$$

Then we have a rational map

$$\nu_{\infty 2}\colon C'_{\infty 2} \to X_{\infty 2}$$
$$(u,v) \mapsto (u, v, u^{A'} v^{-N'}, u^n v^m),$$

which restricts to a morphism on the open dense subset $C'_{\infty 2} - \{(0,0)\}$ of $C'_{\infty 2}$. This morphism has an inverse given by the morphism

$$\tau'_{\infty 2}\colon X_{\infty 2} \to C'_{\infty 2}$$
$$(u,v,w,z) \mapsto (w^m z^{N'}, v).$$

Hence, $\tau'_{\infty 2}$ is a birational morphism and restricts to an isomorphism of $\hat{X}_{\infty 2} := X_{\infty 2} - \{(\tau'_{\infty 2})^{-1}(0,0)\}$ to $C'_{\infty 2} - \{(0,0)\}$. Remembering that the blow-up map $\varphi\colon C'_{\infty 2} \to C_{\infty 2}$ is a birational morphism and restricts to an isomorphism on $C'_{\infty 2} - \{\varphi^{-1}\{(0,0)\}\}$ and that $\varphi^{-1}\{(0,0)\} = \{(0,0)\}$, we get a birational morphism

$$\tau_{\infty 2} := \varphi \circ \tau'_{\infty 2}\colon X_{\infty 2} \to C_{\infty 2},$$

which induces an isomorphism from $\hat{X}_{\infty 2}$ to $\hat{C}_{\infty 2} := C_{\infty 2} - \{(0,0)\}$. Now, since $C_{\infty 2}$ is isomorphic to $C_N - \{P_0, \dots, P_r\}$ under κ_2, $\hat{C}_{\infty 2}$ is isomorphic to $C_N - \{P_0, \dots, P_r, \infty\}$ and so is $\hat{X}_{\infty 2}$, *i.e.*

$$\hat{X}_{\infty 2} \xrightarrow{\sim} C_N - \{P_0, \dots, P_r, \infty\}.$$

The birational morphism from $X_{\infty 2}$ to C_N is given by $\pi_{\infty 2} := \kappa_2 \circ \tau_{\infty 2}$.

Remark 6 $X_{\infty 2}$ is nonsingular and the $\tau_{\infty 2}$-preimages of $(0,0) \in C_{\infty 2}$ are $(0,0,w,0)$, where w runs trough the $(N, -N+\sum A_k)$-th roots of $h(0)$. Since $h(0) \neq 0$, their number is $(N, -N + \sum A_k) = (N, N - \sum A_k)$.

3.2 Construction of X_N by Glueing

We refer here to the construction described in [16], Volume 1, V.3.2. Let X_∞ resp. π_∞ denote $X_{\infty 1}$ resp. $\pi_{\infty 1}$ in the case $N - \sum A_k > 0$ and $X_{\infty 2}$ resp. $\pi_{\infty 2}$ in the case $N - \sum A_k < 0$.

Remember that, for each $j \in \{0, \dots, r, \infty\}$, the morphism $\pi_j\colon X_j \to C_N$ restricts to an isomorphism of the open dense subset $\hat{X}_j$ of X_j to $C_N - \{P_0, \dots, P_r, \infty\}$ and that X_j and C_N are birationally equivalent. Then one can define an equivalence relation on the disjoint union $\coprod_{j \in \{0,\dots,r,\infty\}} X_j$ by setting, for $Q_j \in \hat{X}_j, Q_k \in \hat{X}_k$ with $j, k \in \{0, \dots, r, \infty\}$ and $j \neq k$,

$$Q_j \sim Q_k \quad \Leftrightarrow \quad \pi_j(Q_j) = \pi_k(Q_k).$$

Moreover, the functions π_j, $j \in \{0, \dots, r, \infty\}$, induce a well-defined function π on the quotient $X := \coprod_{j \in \{0,\dots,r,\infty\}} X_j/_\sim$ by setting, for $\mathcal{C} \in X$ and $Q_j \in X_j$ with $[Q_j] = \mathcal{C}$,

$$\pi(\mathcal{C}) := \pi_j(Q_j).$$

By definition of the equivalence relation, this is independent of the choice of the representative of the class $\mathcal{C}$.

On the set X, we have the quotient topology and can define a sheaf induced from the sheaf of regular functions on each X_j, for which π is again a birational and finite morphism. This implies that X is again a projective curve. Since X is moreover nonsingular, because so is each X_j, and, as we have seen, birationally equivalent to C_N, it provides a model of the desingularization of C_N, hence is isomorphic to X_N. The desingularization map is given by the map π such that $\pi|_{X_j} = \pi_j$ for each j. Another isomorphic construction is given in [16] Volume 2, II.5.3, Theorem 6, Theorem 7.

Since, to our purpose, we only need to know the desingularization up to isomorphism, X will be identified with X_N in the following.

3.3 Compositions of π with Local Parametrizations of X_N

Because the restriction of π to $X_N - \left\{\pi^{-1}(\{P_0, \dots, P_r, \infty\})\right\}$ is an isomorphism to $C_N - \{P_0, \dots, P_r, \infty\}$, we only need to know the compositions of π with local parametrizations at the points of $\pi^{-1}(\{P_0, \dots, P_r, \infty\})$, which are isolated on X_N.

3.3.1 Above $P_j = (1 : \lambda_j : 0)$

Fix $j \in \{0, \dots, r\}$ and remember that the local desingularization above P_j is

$$X_j = \{(x, u, z) \in \mathbb{C}^3 \,;\, z^{N'} = (x - \lambda_j)u^m, u^{(N,A_j)} = g_j(x), g_j(x) \neq 0\}.$$

We have then the composition

$$\pi_j = \kappa_0 \circ \tau_j \colon X_j \to C_N$$
$$(x, u, z) \mapsto (1 : x : u^n z^{A'_j}).$$

Recall that X_j is nonsingular and open in an affine variety, hence each point has a neighbourhood for the complex topology which is isomorphic to an open neighbourhood on $\mathbb{C}$. Choose a complex open neighbourhood U_j of $s = 0$ in $\mathbb{C}$ on which $g_j(s^{N'} + \lambda_j) \neq 0$. Then the image of U_j under $s \mapsto g_j(s^{N'} + \lambda_j)$ is included in $\mathbb{C}$ bereft of a half-line through 0. Thus, branches of roots of $g_j(s^{N'} + \lambda_j)$ can be well-defined as holomorphic functions of s on U_j. Then choosing fixed branches, we have a well-defined holomorphic function

$$\varphi_j \colon s \mapsto \left(s^{N'} + \lambda_j, g_j(s^{N'} + \lambda_j)^{\frac{1}{(N,A_j)}}, s g_j(s^{N'} + \lambda_j)^{\frac{m}{N}}\right)$$

from U_j to X_j. Indeed, $u^{(N,A_j)} = g_j(s^{N'} + \lambda_j) = g_j(x)$ and

$$z^{N'} = s^{N'} g_j(s^{N'} + \lambda_j)^{\frac{m}{N} N'} = (s^{N'} + \lambda_j - \lambda_j) g_j(s^{N'} + \lambda_j)^{\frac{m}{(N,A_j)}} = (x - \lambda_j)u^m.$$

On the image of φ_j, we have a well-defined holomorphic inverse map

$$(x, u, z) \mapsto z g_j(x)^{-\frac{m}{N}}.$$

Hence, φ_j is an analytic isomorphism and a local parametrization of X_j at the point $\varphi(0) = \left(\lambda_j, g_j(\lambda_j)^{\frac{1}{(N,A_j)}}, 0\right)$, which is one of the π-preimages of $P_j \in C_N$. Remark again that the choices of branches for the (N, A_j)-th root of $g_j(s^{N'} + \lambda_j)$ are in bijective correspondence with the π-preimages on X_N of P_j.

The expression of π in this local parameter s at each π-preimage of P_j is given by the composition $\pi_j \circ \varphi_j \colon U_j \to C_N$

$$s \mapsto \left(1 : s^{\frac{N}{(N,A_j)}} + \lambda_j : s^{\frac{A_j}{(N,A_j)}} g_j(s^{\frac{N}{(N,A_j)}} + \lambda_j)^{\frac{1}{N}}\right). \tag{6}$$

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3.3.2 Above Infinity

Case 1: $N - \sum_{k=0}^{r} A_k > 0$. We are looking for an analytic parametrization of $X_{\infty 1}$ at the points $(0, u, 0)$, where u satisfies $u^{(N, N - \sum A_k)} = h(0)$. Let's choose a complex neighbourhood U_∞ of $s = 0$ in $\mathbb{C}$ on which $h(s^{N'}) \neq 0$, N' being here $\frac{N}{(N, N - \sum A_k)}$. On such a neighbourhood, we can define roots of $h(s^{N'})$ as analytic functions of s. Fix an N-th root $h(s^{N'})^{\frac{1}{N}}$. Then, for each branch of the $(N, N - \sum A_k)$-th root of $h(s^{N'})$, the map

$$\begin{aligned} \varphi_{\infty 1} \colon U_\infty &\to X_{\infty 1} \\ s &\mapsto \left(s^{N'}, h(s^{N'})^{\frac{1}{(N, N - \sum A_k)}}, s h(s^{N'})^{\frac{m}{N}} \right) \end{aligned}$$

is a well-defined analytic map such that $\varphi_{\infty 1}(0) = (0, u, 0)$, where u is the corresponding $(N, N - \sum A_k)$-th root of $h(0)$. $\varphi_{\infty 1}$ has an analytic inverse on its image, which is given by

$$(x, u, z) \mapsto z h(x)^{-\frac{m}{N}}.$$

Thus $\varphi_{\infty 1}$ is a local parametrization of $X_{\infty 1}$ at the preimage $\left(0, h(0)^{\frac{1}{(N, N - \sum A_k)}}, 0\right)$ of ∞. Since $\pi_{\infty 1} = \kappa_1 \circ \tau_{\infty 1} \colon X_{\infty 1} \to C_N$ is given by $(x, u, z) \mapsto (x : 1 : z^{A'} u^n)$, $\pi_{\infty 1} \circ \varphi_{\infty 1} \colon U_\infty \to C_N$ is given by

$$s \mapsto \left(s^{\frac{N}{(N, N - \sum A_k)}} : 1 : s^{\frac{N - \sum A_k}{(N, N - \sum A_k)}} h\big(s^{\frac{N}{(N, N - \sum A_k)}}\big)^{\frac{1}{N}} \right) \tag{7}$$

and is a local expression of π at each point on X_N lying above $(0 : 1 : 0)$.

Case 2: $N - \sum_{k=0}^{r} A_k < 0$. On U_∞ defined as in Case 1, we have a well-defined holomorphic map

$$\begin{aligned} \varphi_{\infty 2} \colon U_\infty &\to X_{\infty 2} \\ s &\mapsto \left(s^{N'}, s^{\frac{-N + \sum A_k}{(N, -N + \sum A_k)}} h(s^{N'})^{-\frac{1}{N}}, h(s^{N'})^{\frac{1}{(N, -N + \sum A_k)}}, s h(s^{N'})^{\frac{nN' - 1}{NA'}} \right), \end{aligned}$$

for a fixed choice of the branch of the roots of $h(s^{N'})$. On its image, this map has an analytic inverse given by

$$(u, v, w, z) \mapsto z h(x)^{\frac{1 - nN'}{NA'}}.$$

Each choice of the branch of the $(N, -N + \sum A_k)$-th root of $h(s^{N'})$ corresponds to a π-preimage of $(0 : 0 : 1)$ at which $\varphi_{\infty 2}$ is a local parametrization. The composition of $\varphi_{\infty 2}$ with π equals $\kappa_2 \circ \varphi \circ \tau'_{\infty 2} \circ \varphi_{\infty 2} \colon U_\infty \to C_N$ and is given by

$$s \mapsto \left(s^{\frac{\sum A_k}{(N, -N + \sum A_k)}} : s^{\frac{-N + \sum A_k}{(N, -N + \sum A_k)}} : h\big(s^{\frac{N}{(N, -N + \sum A_k)}}\big)^{\frac{1}{N}} \right). \tag{8}$$

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4 Genus of X_N

In this section, we calculate the algebraic genus $g[X_N] = \dim_{\mathbb{C}} \Omega^1[X_N]$ as the topological genus of the compact Riemann surface $X_N(\mathbb{C})$ of complex points on X_N. In order to do this, we apply Hurwitz's formula to the covering map $\nu\colon X_N(\mathbb{C}) \to \mathbb{P}^1_{\mathbb{C}}$ defined below.

Consider the projection $p\colon C_N \to \mathbb{P}^1_{\mathbb{C}}$ given by $(x_0 : x_1 : x_2) \mapsto (x_0 : x_1)$ and compose it with the desingularization map $\pi\colon X_N(\mathbb{C}) \to C_N$. The composition

$$\nu := p \circ \pi\colon X_N(\mathbb{C}) \to \mathbb{P}^1_{\mathbb{C}}$$

is nonconstant and regular (hence holomorphic). It is then a covering map between compact Riemann surfaces to which we can apply Hurwitz genus formula. The degree of ν is N, because each affine point $(x, y) \in C_N$ with $x \neq \lambda_0, \dots, \lambda_r$ has N distinct preimages on X_N corresponding to the N-th roots of $\prod_{i=0}^{r}(x - \lambda_i)^{A_i}$.

It remains to calculate the ramification indices.

4.1 Above Nonsingular Points

Each point $Q \in X_N(\mathbb{C})$ such that $\pi(Q) =: P$ is nonsingular is a regular point of the covering ν. Indeed, if $P \in C_N$ is nonsingular and $P = (x, y)$ (resp. ∞), then y (resp. x_1 and x_2) can be written as a function of x (resp. x_0) in a neighbourhood of P. π being locally an isomorphism at Q, x (resp. x_0) can also be taken as a local parameter of $X_N(\mathbb{C})$ at Q. With respect to this parameter, π is given by $x \mapsto \bigl(x, y(x)\bigr)$ (resp. by $x_0 \mapsto (x_0 : x_1(x_0) : x_2(x_0)))$ and ν by $x \mapsto x$ (resp. $x_0 \mapsto x_0$). Hence $r_\nu(Q) = 1$.

4.2 Above $P_j = (1 : \lambda_j : 0)$

For $j \in \{0, \dots, r\}$, let Q_j be one π-preimage of $(\lambda_j, 0) \in C_N$. The composition of π with a local parametrization of X_N at Q_j is given in (6) Section 3. Composing it with p, we get

$$s \mapsto (1 : s^{\frac{N}{(N,A_j)}} + \lambda_j).$$

If we now choose the chart $(x_0 : x_1) \mapsto \frac{x_1 - \lambda_j}{x_0}$ on $\{(x_0 : x_1) \in \mathbb{P}^1_{\mathbb{C}} \;;\; x_0 \neq 0\}$ and compose it with the above map, we get the expression of ν in local coordinates as

$$s \mapsto s^{\frac{N}{(N,A_j)}}.$$

This shows that each π-preimage of $(\lambda_j, 0)$ has ramification index equal to $\frac{N}{(N,A_j)}$.

4.3 Above Infinity

For the points lying above ∞, we will choose the chart (U_1, ψ) on $\mathbb{P}^1_{\mathbb{C}}$, where $U_1 := \{(x_0 : x_1) \in \mathbb{P}^1_{\mathbb{C}} \;;\; x_1 \neq 0\}$ and $\psi\colon (x_0 : x_1) \mapsto \frac{x_0}{x_1}$.

Case 1: $N - \sum_{k=0}^{r} A_k > 0$. The composition of p with the composition (7) of π with a local parametrization of X_N at each preimage of ∞ reads

$$s \mapsto (s^{\frac{N}{(N, N - \sum A_k)}} : 1).$$

Composing it with ψ, we get the expression of ν in local coordinates at each π-preimage of ∞ as

$$s \mapsto s^{\frac{N}{(N,N-\sum A_k)}}.$$

Case 2: $N - \sum_{k=0}^{r} A_k < 0$. In this case, we have to be a little more careful, because the map p is not defined at ∞. If only to consider momentarily the restriction of p to the punctured Riemann surface $X_N(\mathbb{C}) - \left\{\pi^{-1}\{(0{:}0{:}1)\}\right\}$, we can suppose $s \neq 0$ and consider the composition of (8) with p, which is

$$\nu \circ \varphi_{\infty 2} \colon s \mapsto \left(s^{\frac{\sum A_k}{(N,-N+\sum A_k)}} : s^{\frac{-N+\sum A_k}{(N,-N+\sum A_k)}}\right) = \left(s^{\frac{N}{(N,-N+\sum A_k)}} : 1\right).$$

If s tends to 0, $\nu \circ \varphi_{\infty 2}(s)$ tends to $(0 : 1)$, the point at infinity in $\mathbb{P}^1_{\mathbb{C}}$. Hence, we can extend the map continuously by setting $0 \mapsto (0 : 1)$. The composition with ψ of the extended map is

$$s \mapsto s^{\frac{N}{(N,-N+\sum A_k)}}.$$

This shows that the ramification index of each point lying above ∞ is equal to $N/(N, -N + \sum A_k)$. In both cases, the ramification index of each π-preimage of ∞ is equal to $(N, -N + \sum A_k) = (N, N - \sum A_k)$. The following table summaries all these data.

(9)

point P of C_N	nb of π-preimages Q	$r_\nu(Q)$
$(\lambda_j, 0)$, $j \in \{0, \dots, r\}$	(N, A_j)	$\frac{N}{(N,A_j)}$
∞	$(N, N - \sum_{k=0}^{r} A_k)$	$\frac{N}{(N,N-\sum_{k=0}^{r} A_k)}$
other points	1	1

Theorem 4.1 *Let X_N be the desingularization of the irreducible projective algebraic plane curve C_N defined over $\mathbb{C}$ by the affine equation*

$$y^N = \prod_{i=0}^{r} (x - \lambda_i)^{A_i},$$

where $\lambda_0, \dots, \lambda_r \in \mathbb{C}$ are such that, $\forall i, j \in \{0, \dots, r\}$ with $i \neq j$, $\lambda_i \neq \lambda_j$. Let further $N, A_0, \dots, A_r \in \mathbb{N}$ satisfy

$$N \neq \sum_{k=0}^{r} A_k \quad \textit{and} \quad (N, A_0, \dots, A_r) = 1.$$

Then the Euler characteristic of $X_N(\mathbb{C})$ is given by

$$\chi\big(X_N(\mathbb{C})\big) = -rN + \Big(N, N - \sum_{k=0}^{r} A_k\Big) + \sum_{j=0}^{r} (N, A_j)$$

and the genus of X_N by

$$g[X_N] = \big(X_N(\mathbb{C})\big) = 1 + \frac{1}{2}\Big(rN - \Big(N, N - \sum_{k=0}^{r} A_k\Big) - \sum_{j=0}^{r} (N, A_j)\Big).$$

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Proof We apply Hurwitz's formula to the covering map

$$\nu = p \circ \pi \colon X_N(\mathbb{C}) \to \mathbb{P}^1_{\mathbb{C}},$$

where $\pi \colon X_N \to C_N$ is the desingularization map and $p \colon C_N \to \mathbb{P}^1_{\mathbb{C}}$ the projection given by $(x_0 : x_1 : x_2) \mapsto (x_0 : x_1)$. As seen above, it has degree N and the only possible ramification points lie above the points $\infty \in \mathbb{P}^1_{\mathbb{C}}$ and $(1 : \lambda_j)$, $j \in \{0, \dots, r\}$. Using the ramification indices calculated above, the number of preimages calculated in Section 3 (all recalled in (9)) and the fact that $\chi(\mathbb{P}^1_{\mathbb{C}}) = 2$, we get, by Hurwitz's genus formula

$$\begin{aligned}\chi\big(X_N(\mathbb{C})\big) &= 2N - \Big(N, N - \sum A_k\Big)\Big(\frac{N}{(N, N - \sum A_k)} - 1\Big) \\ &\qquad - \sum_{j=0}^{r} (N, A_j)\Big(\frac{N}{(N, A_j)} - 1\Big) \\ &= -rN + \Big(N, N - \sum A_k\Big) + \sum_{j=0}^{r} (N, A_j).\end{aligned}$$

To get the second formula, we use $\chi\big(X_N(\mathbb{C})\big) = 2 - 2g\big(X_N(\mathbb{C})\big)$.

$$\begin{aligned}g[X_N] = g\big(X_N(\mathbb{C})\big) &= 1 - \frac{1}{2}\chi\big(X_N(\mathbb{C})\big) \\ &= 1 + \frac{1}{2}\Big(rN - \Big(N, N - \sum_{k=0}^{r} A_k\Big) - \sum_{j=0}^{r} (N, A_j)\Big). \qquad \blacksquare\end{aligned}$$

5 Actions of μ_N

Let μ_N be the group of complex N-th roots of unity. We will define an action of μ_N on X_N and show how it induces a linear action on the $\mathbb{C}$-vector space $\Omega^1[X_N]$ of regular differential 1-forms on X_N.

For $\zeta \in \mu_N$ and an affine point $(x, y) \in C_N$, define

$$\zeta \cdot (x, y) := (x, \zeta^{-1} y).$$

Further, set $\zeta \cdot \infty = \infty, \forall \zeta \in \mu_N$. As $(\zeta^{-1})^N = 1$, we have $\zeta \cdot (x, y) \in C_N$. This is an action, because μ_N is abelian and 1 acts as the identity. Moreover, since μ_N is included in the definition field $\mathbb{C}$ of C_N, for each $\zeta \in \mu_N$, the map

$$\begin{aligned}\varphi_\zeta \colon C_N &\to C_N \\ (x, y) &\mapsto \zeta \cdot (x, y) \\ \infty &\mapsto \infty\end{aligned}$$

is a morphism of algebraic varieties.

Now we want to extend this action to an action on X_N. Remember that the desingularization map $\pi\colon X_N \to C_N$ restricts to an isomorphism on the dense subset $\pi^{-1}(C_N^{\text{reg}})$ of X_N, where C_N^{reg} is the set of regular points in C_N. Let $P \in X_N$, if $\pi(P)$ is regular, set

$$\zeta \cdot P := \pi^{-1}\big(\zeta \cdot \pi(P)\big)\,.$$

If $P \in X_N$ is such that $\pi(P) = \infty$ or $\exists j \in \{0, \dots, r\}$ with $\pi(P) = (\lambda_j, 0)$, set

$$\zeta \cdot P := P.$$

Note that, if $(\lambda_j, 0)$ (respectively ∞) is regular on C_N, this last definition is coherent with the above one. This defines an action on X_N.

For $\zeta \in \mu_N$, set

$$\begin{aligned} \Phi_\zeta\colon X_N &\to X_N \\ P &\mapsto \zeta \cdot P. \end{aligned}$$

Then Φ_ζ makes the following diagram commute

$$\begin{array}{ccc} X_N & \xrightarrow{\Phi_\zeta} & X_N \\ {\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle\pi} \\ C_N & \xrightarrow[\varphi_\zeta]{} & C_N. \end{array} \tag{10}$$

Because φ_ζ and π are morphisms, Φ_ζ is also a morphism.

Let ω be a regular differential form on X_N. Since $\Phi_\zeta\colon X_N \to X_N$ is a morphism, the pull-back $\Phi_\zeta^*\omega$ is again regular on X_N. Hence, the following map is well-defined

$$\begin{aligned} \mu_N \times \Omega^1[X_N] &\to \Omega^1[X_N] \\ (\zeta, \omega) &\mapsto \Phi_\zeta^*\omega. \end{aligned}$$

It defines an action of μ_N on $\Omega^1[X_N]$, which is linear, because, for every $\zeta \in \mu_N$, the map $\Omega^1[X_N] \to \Omega^1[X_N]$, given by $\omega \mapsto \Phi_\zeta^*\omega$ is linear. Furthermore, the $\mathbb{C}$-vector space $\Omega^1[X_N]$ being finite dimensional, it furnishes a finite dimensional linear representation of μ_N.

Such a linear representation admits a decomposition in isotypical components (each isotypical component being the direct sum of all irreducible representations associated to a given character). For $n \in \{1, \dots, N\}$, V_n will denote the isotypical component associated to the character $\chi_n\colon \zeta \mapsto \zeta^n$. In these terms, we can write the canonical decomposition of $\Omega^1[X_N]$ as

$$\Omega^1[X_N] = \bigoplus V_n, \tag{11}$$

where the sum is taken over the indices n in $\{0, \dots, N-1\}$ for which $\dim V_n > 0$.

In the next paragraph, the dimension of V_n will be calculated for each n. It corresponds to the number of irreducible subrepresentations of $\Omega^1[X_N]$ having character χ_n.

6 Basis of Regular Differential Forms on X_N

In this section, we calculate a $\mathbb{C}$-basis of regular differential 1-forms on X_N. In view of the decomposition (11), it is sufficient to find a basis of V_n for each n in $\{0, \dots, N-1\}$. Once this being done, we will calculate $\dim_{\mathbb{C}} V_n$ by counting the basis elements and also the sum $\dim V_n + \dim V_{N-n}$ in the case $(n, N) = 1$.

Let's first make use of a result that goes back to Abel and Riemann and that is stated in Satz 1 Section 9.3 of [5] in the following way.

Proposition 6.1 *The nonvanishing holomorphic differential* 1*-forms on the Riemann surface C', which is the desingularization of an irreducible algebraic plane curve C with affine equation $f(x, y) = 0$, where the coordinates are chosen in such a way that $\frac{\partial f}{\partial y}$ is not identically zero, are given by* [the pull-backs under the desingularization map of]

$$\frac{\Phi(x, y)\,dx}{\frac{\partial f}{\partial y}(x, y)},$$

where $\Phi(x, y) = 0$ is the equation of an adjoint curve to C of degree $(\deg f) - 3$.

We do not want to introduce what an adjoint curve is, but this proposition allows us to choose a basis of regular differential 1-forms on X_N among the regular pull-backs under π of the differential forms

$$\frac{\Phi(x, y)dx}{y^{N-1}}, \tag{12}$$

on C_N, where $\Phi(x, y) \in \mathbb{C}[x, y]$.

If $(x, y) \in C_N$, each power y^{kN} with $k \in \mathbb{N}$, can be replaced by a polynomial expression in x. Hence, we can suppose that

$$\Phi(x, y) = \Phi_0(x) + \Phi_1(x)y + \cdots + \Phi_{N-1}(x)y^{N-1}.$$

That is

$$\frac{\Phi(x, y)\,dx}{y^{N-1}} = \frac{\Phi_0(x)\,dx}{y^{N-1}} + \frac{\Phi_1(x)\,dx}{y^{N-2}} + \cdots + \frac{\Phi_{N-2}(x)\,dx}{y} + \Phi_{N-1}(x)\,dx.$$

Hence, if the regular pull-backs of the differential forms (12) generate $\Omega^1[X_N]$, so do the regular pull-backs of the differential forms

$$\frac{\Psi(x)\,dx}{y^n},$$

where $n \in \{0, \dots, N-1\}$ and $\Psi(x) \in \mathbb{C}[x]$. Further, the polynomials $\Psi(x)$ will be replaced by polynomials that are fitter to reflect the topology of C_N (resp. X_N) and that also generate the ring $\mathbb{C}[x]$. Namely, polynomials of the form

$$\prod_{i=0}^{r}(x - \lambda_i)^{a_i} \in \mathbb{C}[x], a_i \in \mathbb{Z}.$$

This discussion may be summarized by saying that the regular pull-backs under π of the following differential forms on C_N

$$\omega_n(x,y) := \frac{\prod_{i=0}^{r}(x-\lambda_i)^{a_i}\,dx}{y^n},$$

with $a_i \in \mathbb{Z}$ and $n \in \{0,\dots,N-1\}$, generate $\Omega^1[X_N]$.

6.1 Regularity Conditions for $\pi^*\omega_n$

Let's now fix n in $\{0,\dots,N-1\}$. We are looking for conditions on $a_0,\dots,a_r$ for $\pi^*\omega_n$ to be regular on X_N.

6.1.1

On the dense subset $U := C_N - \{\infty,(\lambda_i,0)\,;\, j = 0,\dots,r\}$ of C_N, the differential form ω_n is obviously regular, because $(x,y) \mapsto x$ and $(x,y) \mapsto \frac{1}{y^n}$ are regular functions on U. Since the desingularization map $\pi\colon X_N \to C_N$ is a morphism, the pull-back $\pi^*\omega_n$ is regular on $\pi^{-1}(U)$.

6.1.2 Above $(1{:}\lambda_j{:}0)$

Let $j \in \{0,\dots,r\}$ and $Q_j \in X_N$ be such that $\pi(Q_j) = (\lambda_j,0)$. As calculated in Section 3.3.1, the composition of π with the local parametrization φ_j of X_N at Q_j is given in affine coordinates on C_N by (compare with (6))

$$s \mapsto \left(s^{\frac{N}{(N,A_j)}} + \lambda_j,\, s^{\frac{A_j}{(N,A_j)}} g_j(s^{\frac{N}{(N,A_j)}} + \lambda_j)^{\frac{1}{N}}\right),$$

where $g_j(x) := \prod_{i\neq j}(x-\lambda_i)^{A_i}$ and s takes values in a neighbourhood U_j of 0 in $\mathbb{C}$ on which $g_j(s^{\frac{N}{(N,A_j)}} + \lambda_j) \neq 0$. By definition, we have

$$\left((\pi_j\circ\varphi_j)^*(\omega_n)\right)(s) = \omega\left((\pi_j\circ\varphi_j)(s)\right)\circ d_s(\pi_j\circ\varphi_j)$$

and $d_{(\pi_j\circ\varphi_j)(s)}x \circ d_s(\pi_j\circ\varphi_j) = \frac{\partial(\pi_j\circ\varphi_j)_1}{\partial s}(s)\,d_s s$. Hence

$$\begin{aligned}\left((\pi_j\circ\varphi_j)^*(\omega_n)\right)(s) &= \frac{N}{(N,A_j)}\prod_{i=0}^{r}(s^{\frac{N}{(N,A_j)}} + \lambda_j - \lambda_i)^{a_i} s^{\frac{N}{(N,A_j)}-1-\frac{nA_j}{(N,A_j)}} \\ &\qquad g_j(s^{\frac{N}{(N,A_j)}} + \lambda_j)^{-\frac{n}{N}}\,ds \\ &= C(s)s^{a_j\frac{N}{(N,A_j)}+\frac{N-nA_j}{(N,A_j)}-1}\,ds,\end{aligned}$$

where $C(s) := \frac{N}{(N,A_j)}\prod_{i\neq j}(s^{\frac{N}{(N,A_j)}} + \lambda_j - \lambda_i)^{a_i} g_j(s^{\frac{N}{(N,A_j)}} + \lambda_j)^{-\frac{n}{N}}$ does not take the value zero on U_j and is regular (because $g_j(s^{\frac{N}{(N,A_j)}} + \lambda_j) \neq 0$ on U_j). Remark that this amounts to replacing x and y by their expressions in s and dx by $\frac{N}{(N,A_j)} s^{\frac{N}{(N,A_j)}-1}\, ds$ in $\omega_n(x,y)$. Since φ_j is an analytic isomorphism, it is an algebraic morphism and so is its inverse. This has the consequence that $\pi^*\omega_n$ is regular at $Q_j = \varphi_j(0)$ exactly when $(\pi_j \circ \varphi_j)^*(\omega_n)$ is regular at 0 (because $\pi_j^*\omega_n = (\varphi_j^{-1})^*\big((\pi_j \circ \varphi_j)^*(\omega_n)\big)$ and $(\pi_j \circ \varphi_j)^*(\omega_n) = (\varphi_j)^*(\pi_j^*\omega_n)$). Hence, we have

$$\pi^*\omega_n \text{ is regular at } Q_j \quad \Leftrightarrow \quad a_j \geq \frac{nA_j + (N, A_j)}{N} - 1.$$

Note that this condition ensures the regularity of $\pi^*\omega_n$ at *each* π-preimage of $(\lambda_j, 0)$.

6.1.3 Above Infinity

First of all, we have to write the differential form ω_n in projective coordinates. Setting $x := \frac{x_1}{x_0}$ and $y := \frac{x_2}{x_0}$, we get

$$dx = \frac{1}{x_0}\, dx_1 - \frac{x_1}{x_0^2}\, dx_0$$

and

$$\omega_n(x_0, x_1, x_2) = x_2^{-n} x_0^{n-2-\sum_{k=0}^r a_k} \prod_{i=0}^r (x_1 - \lambda_i x_0)^{a_i} (x_0 dx_1 - x_1 dx_0).$$

Case 1: $N - \sum_{k=0}^r A_k > 0$. The composition of π with the local parametrization $\varphi_{\infty 1}$ of X_N at each preimage Q of $(0{:}1{:}0)$ is given by (*cf.* (7) Section 3.3.2)

$$s \mapsto \Big(s^{\frac{N}{(N, N-\sum A_k)}} : 1 : s^{\frac{N-\sum A_k}{(N,N-\sum A_k)}} h\big(s^{\frac{N}{(N,N-\sum A_k)}}\big)^{\frac{1}{N}} \Big),$$

where $h(x) = \prod_{i=0}^r (1 - \lambda_i x)^{A_i}$ and s takes values in the complex neighbourhood U_∞ of $s = 0$ on which $h(s^{\frac{N}{(N,N-\sum A_k)}}) \neq 0$. Noting that $x_1 = 1 \Rightarrow dx_1 = 0$ and inserting the expressions for x_0, x_1, x_2, dx_0 into that of ω_n, we get

$$\big((\pi_{\infty 1} \circ \varphi_{\infty 1})^*\omega_n\big)(s) = C(s) s^{\frac{n\sum A_k - N - N\sum a_i}{(N,N-\sum A_k)} - 1}\, ds,$$

where C is regular on U_∞ and $C(s) \neq 0$ for $s \in U_\infty$. Therefore

$$\pi^*\omega_n \text{ is regular at } Q \quad \Leftrightarrow \quad \sum_{i=0}^r a_i \leq \frac{n\sum_{k=0}^r A_k - (N, N - \sum_{k=0}^r A_k)}{N} - 1.$$

Case 2: $N - \sum_{k=0}^r A_k < 0$. The composition of π with the local parametrization $\varphi_{\infty 2}$ of X_N at each preimage Q of $(0{:}0{:}1)$ is given, for $s \in U_\infty$, by (*cf.* (8) Section 3)

$$s \mapsto \Big(s^{\frac{\sum A_k}{(N, -N+\sum A_k)}} : s^{\frac{-N+\sum A_k}{(N,-N+\sum A_k)}} : h\big(s^{\frac{N}{(N,-N+\sum A_k)}}\big)^{\frac{1}{N}} \Big).$$

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Replacing $x_0, x_1, x_2, dx_0, dx_1$ by their expressions in s, we get

$$\left((\pi_{\infty 2} \circ \varphi_{\infty 2})^* \omega_n\right)(s) = C(s) s^{\frac{n \sum A_k - N \sum a_i - N}{(N, -N + \sum A_k)} - 1}\, ds,$$

where C is regular on U_∞ and $C(s) \neq 0$ for $s \in U_\infty$. Thus, we have

$$(\pi^* \omega_n)(s) \text{ is regular at } Q \quad \Leftrightarrow \quad \sum_{i=0}^{r} a_i \leq \frac{n \sum_{k=0}^{r} A_k - (N, -N + \sum_{k=0}^{r} A_k)}{N} - 1.$$

Summary 1 Since $(N, -N + \sum_{k=0}^{r} A_k) = (N, N - \sum A_k)$, we can summarize these conditions by saying that the pull-back under $\pi \colon X_N \to C_N$ of the differential form

$$\omega_n(x, y) = \frac{\prod_{i=0}^{r} (x - \lambda_i)^{a_i}\, dx}{y^n}$$

on C_N is regular on X_N if and only if

$$(13) \qquad \begin{cases} \displaystyle\sum_{i=0}^{r} a_i \leq \frac{n \sum_{k=0}^{r} A_k - (N, N - \sum_{k=0}^{r} A_k)}{N} - 1 \\[2ex] \displaystyle a_j \geq \frac{n A_j + (N, A_j)}{N} - 1, \quad \forall j \in \{0, \dots, r\}. \end{cases}$$

These conditions will be referred to as the *regularity conditions* for $\pi^* \omega_n$.

Note 1 As kindly noticed by the referee, there is no need to construct the non-singular model by glueing the local desingularizations for computing the order of differential forms on X_N. Indeed, it suffices to consider the local affine equation at each singular point and reduce it into irreducible factors. It will be of the form

$$(14) \qquad 0 = y^N - w^A = \prod_{\zeta_d} (y^{N'} - \zeta_d w^{A'}),$$

where $d = gcd(N, A)$, $N' = \frac{N}{d}$, $A' = \frac{A}{d}$ and the product runs over all d-th roots of unity. Let $n, m \in \mathbb{Z}$ be such that $nN' + mA' = 1$ and write $z := y^m w^n$, then the desingularization is locally given by

$$y = \zeta_d^n z^{A'} \quad \text{and} \quad w = \zeta_d^{-m} z^{N'}.$$

So that each factor in (14) corresponds exactly to one branch of the desingularization above the point $(0, 0)$. Substituting $w := x - \lambda_i$ at $(\lambda_i, 0)$ and $w := \frac{1}{x}$ at ∞, one can compute the order of differential forms on X_N.

Remark 7 We would like now to show that the pull-back $\pi^* \omega_n$ of a differential form $\omega_n(x, y) = y^{-n} \prod_{i=0}^{r} (x - \lambda_i)^{a_i}\, dx$ belongs to the isotypical component V_n of character

χ_n, if it satisfies the above conditions. If it is the case, $\pi^*\omega_n \in \Omega^1[X_N]$ and it remains to study the action of μ_N on $\pi^*\omega_n$, for a fixed $n \in \{0,\dots,N-1\}$. Let $\zeta \in \mu_N$, then

$$\begin{aligned}\zeta\cdot\pi^*\omega_n &= \Phi_\zeta^*(\pi^*\omega_n)\\ &= (\pi\circ\Phi_\zeta)^*\omega_n\\ &= (\varphi_\zeta\circ\pi)^*\omega_n \quad \text{by (10)}.\end{aligned}$$

Now, for $P \in X_N$, we have

$$\begin{aligned}\big((\varphi_\zeta\circ\pi)^*\omega_n)\big)(P) &= \omega_n\big((\varphi_\zeta\circ\pi)(P)\big)\circ d_P(\varphi_\zeta\circ\pi)\\ &= \omega_n\big(\varphi_\zeta\big(\pi(P)\big)\big)\circ d_{\pi(P)}\varphi_\zeta\circ d_P\pi\\ &= \zeta^n\omega_n\big(\pi(P)\big)\circ d_P\pi\\ &= \zeta^n(\pi^*\omega_n)(P).\end{aligned}$$

Hence, for every $\zeta \in \mu_N$, we have

$$\zeta\cdot(\pi^*\omega_n) = \chi_n(\zeta)\pi^*\omega_n.$$

This shows that $\pi^*\omega_n \in V_n$.

6.2 Dimension of V_n

Let n be fixed in $\{0,\dots,N-1\}$. In order to determine the dimension of V_n, we will count the number of elements in a maximal family of linearly independent differential forms of the form $y^{-n}\prod_{i=0}^r(x-\lambda_i)^{a_i}\,dx$, where $n, a_0,\dots,a_r \in \mathbb{Z}$ and satisfy the regularity conditions (13). Since $a_0,\dots,a_r$ are integers and according to the regularity conditions, the maximal possible value $(\sum_{i=0}^r a_i)_{\max}$ of $\sum_{i=0}^r a_i$ and the minimal possible value $(a_j)_{\min}$ of a_j, $j \in \{0,\dots,r\}$, are given by

$$\Big(\sum_{i=0}^r a_i\Big)_{\max} = \Big[\frac{n\sum A_k - (N, N-\sum A_k)}{N} - 1\Big] \quad \text{and}$$

$$(a_j)_{\min} = -\Big[1 - \frac{nA_j + (N, A_j)}{N}\Big], \quad j \in \{0,\dots,r\},$$

where $[x]$ denotes the integral part of x.

Definition 2 Let $x \in \mathbb{R}$, then x admits a unique decomposition as

$$x = [x] + \langle x\rangle,$$

where $[x] \in \mathbb{Z}$ and $\langle x\rangle \in [0,1)$ are respectively called the *integral part* and the *fractional part* of x.

Write further $(\sum_{i=0}^r a_i)_{\min} := \sum_{i=0}^r (a_i)_{\min}$ and $\ell := (\sum_{i=0}^r a_i)_{\max} - (\sum_{i=0}^r a_i)_{\min}$.

If $\ell \geq 0$, there is at least one solution. Write $\omega_{\min}$ for the solution where each a_i is minimal. Then $V_n = \langle x^k \omega_{\min} \rangle_{k=0,\dots,\ell}$ and $\dim V_n = \ell + 1$. Indeed, one verifies that each possible value for $\sum_{i=0}^r a_i$ brings exactly one element in the maximal family of linearly independent differential forms. For instance, if $\exists j \in \{0, \dots, r\}$ such that $(a_j)_{\min} + 1$ and $(\sum_{i=0}^r a_i)_{\min} + 1$ satisfy the regularity conditions, then

$$y^{-n}(x-\lambda_j)^{(a_j)_{\min}+1} \prod_{i \neq j} (x-\lambda_i)^{(a_i)_{\min}} = x\omega_{\min} - \lambda_j \omega_{\min} \in \langle \omega_{\min}, x\omega_{\min} \rangle.$$

Note that this is independent of j and conclude by induction on ℓ.

Theorem 6.2 *Let X_N be the curve defined in Theorem 4.1 and recall that the vector space $\Omega^1[X_N]$ of regular differential 1-forms on X_N furnishes a linear representation of μ_N (cf. Section 5). Then, for $n \in \{0, \dots, N-1\}$, the isotypical component V_n of character $\chi_n \colon \zeta \mapsto \zeta^n$ has dimension*

$$\dim V_n = \begin{cases} d_n & \text{if } d_n > 0 \\ 0 & \text{otherwise,} \end{cases}$$

where

$$d_n := \left[\frac{n \sum A_k - (N, N - \sum A_k)}{N} \right] + \sum_{i=0}^r \left[1 - \frac{nA_i + (N, A_i)}{N} \right].$$

Proof Use $[x-1] = [x] - 1$ to show that $d_n = \ell + 1$ and apply the above reasoning. ∎

Remark 8 If $\dim V_n = 0$, then V_n does not appears in the canonical decomposition (11) of $\Omega^1[X_N]$.

Remark 9 Since $g[X_N] = \dim_{\mathbb{C}}(\Omega^1[X_N])$, Theorem 4.1 and Theorem 6.2 together imply the following relation

$$1 + \frac{1}{2}\Big(rN - \big(N, N - \sum_{k=0}^r A_k\big) - \sum_{j=0}^r (N, A_j) \Big) = \sum_{\substack{n \in \{0,\dots,N-1\} \\ d_n > 0}} d_n.$$

6.3 Dimension of V_n, $(n, N) = 1$

Here will be used the conditions $N \nmid A_0, \dots, A_r, \sum_{k=0}^r A_k$. The goal here is to transform the formula for $\dim V_n$ of Theorem 6.2 in the case where $(n, N) = 1$ into a more treatable form. Some preparatory lemmata are given in order to prove Theorems 6.7 and 6.8.

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Lemma 6.3 *Let $x \in \mathbb{R}$, $\ell \in \mathbb{Z}$, $N, A \in \mathbb{N}$ and $n \in \{0, \dots, N-1\}$. Then we have*

1. $[x+\ell] = [x] + \ell$,
2. $[x] = x - \langle x \rangle$,
3. $\langle x+\ell \rangle = \langle x \rangle$.
4. *If $x \notin \mathbb{Z}$, then $\langle -x \rangle = 1 - \langle x \rangle$.*
5. *If $N \nmid A$ and $(n, N) = 1$, then $\langle \frac{nA-(N,A)}{N} \rangle = \langle \frac{nA}{N} \rangle - \frac{(N,A)}{N}$.*
6. *If $N \nmid A$ and $(n, N) = 1$, then $\big[-\frac{nA+(N,A)}{N} \big] = -\big[\frac{nA-(N,A)}{N} \big] - 1$.*

Proof The first four points follow directly from the definitions. For point 5., write $N' := \frac{N}{(N,A)}$, $A' := \frac{A}{(N,A)}$ and $nA' = kN' + r$, with $k \in \mathbb{Z}$ and $r \in \{1, \dots, N'-1\}$. Note that $r \neq 0$, because $N \nmid nA$. Then

$$\frac{nA'}{N'} = k + \frac{r}{N'} \quad \text{and} \quad \frac{nA'-1}{N'} = k + \frac{r-1}{N'}.$$

Since $r - 1 \in \{0, \dots, N'-2\}$ and $k \in \mathbb{Z}$, we have

$$\Big\langle \frac{nA'}{N'} \Big\rangle = \frac{r}{N'} \quad \text{and} \quad \Big\langle \frac{nA'-1}{N'} \Big\rangle = \frac{r}{N'} - \frac{1}{N'}.$$

This implies $\langle \frac{nA'-1}{N'} \rangle = \langle \frac{nA'}{N'} \rangle - \frac{1}{N'}$ or equivalently

$$\Big\langle \frac{nA-(N,A)}{N} \Big\rangle = \Big\langle \frac{nA}{N} \Big\rangle - \frac{(N,A)}{N}.$$

6. With the same notations and hypotheses as above, we have $[\frac{nA'-1}{N'}] = k$. Now,

$$\Big[\frac{-nA'-1}{N'} \Big] = \Big[-k - \frac{r+1}{N'} \Big] = -k + \Big[-\frac{r+1}{N'} \Big] = -k - 1,$$

because $-\frac{r+1}{N'} \in [-1, 0)$. Hence, $[-\frac{nA'+1}{N'}] = -[\frac{nA'-1}{N'}] - 1$ or equivalently

$$\Big[-\frac{nA+(N,A)}{N} \Big] = -\Big[\frac{nA-(N,A)}{N} \Big] - 1. \qquad \blacksquare$$

Lemma 6.4 *Let $n \in \{0, \dots, N-1\}$, $N, A_0, \dots, A_r \in \mathbb{N}$ and suppose $(n, N) = 1$ and $N \nmid A_0, \dots, A_r, \sum_{k=0}^{r} A_k$. Then, $\forall j \in \{0, \dots, N\}$, we have*

1. $\big[-\frac{nA_j+(N,A_j)}{N} \big] = \langle \frac{nA_j}{N} \rangle - \frac{nA_j}{N} - 1$ *and*
2. $\big[\frac{n\sum_{k=0}^{r} A_k - (N, N - \sum_{k=0}^{r} A_k)}{N} \big] = \frac{n\sum_{k=0}^{r} A_k}{N} - \big\langle \frac{n\sum_{k=0}^{r} A_k}{N} \big\rangle$.

Proof The reference number refers to Lemma 6.3.

1. Fix $j \in \{0, \dots, N\}$, then

$$\begin{aligned} \Big[-\frac{nA_j+(N,A_j)}{N} \Big] &= -\Big[\frac{nA_j-(N,A_j)}{N} \Big] - 1 \quad \text{by (5)} \\ &= -\frac{nA_j-(N,A_j)}{N} + \Big\langle \frac{nA_j-(N,A_j)}{N} \Big\rangle - 1 \quad \text{by (2)} \\ &= -\frac{nA_j}{N} + \frac{(N,A_j)}{N} + \Big\langle \frac{nA_j}{N} \Big\rangle - \frac{(N,A_j)}{N} - 1 \quad \text{by (5)} \\ &= \Big\langle \frac{nA_j}{N} \Big\rangle - \frac{nA_j}{N} - 1. \end{aligned}$$

2.

$$\begin{aligned}\left[\frac{n\sum A_k-(N,N-\sum A_k)}{N}\right] &= \frac{n\sum A_k-(N,N-\sum A_k)}{N}\\ &\quad -\left\langle\frac{n\sum A_k-(N,N-\sum A_k)}{N}\right\rangle \quad \text{by (2)}\\ &= \frac{n\sum A_k}{N}-\frac{(N,N-\sum A_k)}{N}\\ &\quad -\left\langle\frac{n\sum A_k}{N}\right\rangle+\frac{(N,N-\sum A_k)}{N} \quad \text{by (5)}\\ &= \frac{n\sum A_k}{N}-\left\langle\frac{n\sum A_k}{N}\right\rangle. \end{aligned}$$ ■

Proposition 6.5 *If $(n,N)=1$ and $N\nmid A_0,\dots,A_r,\sum_{k=0}^r A_k$, then the integer d_n defined in Theorem 6.2 is equal to*

$$d_n=-\left\langle\frac{n\sum_{k=0}^r A_k}{N}\right\rangle+\sum_{i=0}^r\left\langle\frac{nA_i}{N}\right\rangle.$$

Proof

$$\begin{aligned} d_n &= \left[\frac{n\sum A_k-(N,N-\sum A_k)}{N}\right]+\sum_{i=0}^r\left[1-\frac{nA_i+(N,A_i)}{N}\right]\\ &= \left[\frac{n\sum A_k-(N,N-\sum A_k)}{N}\right]+r+1+\sum\left[-\frac{nA_i+(N,A_i)}{N}\right]\\ &= \frac{n\sum A_k}{N}-\left\langle\frac{n\sum A_k}{N}\right\rangle+r+1+\sum\left\langle\frac{nA_i}{N}\right\rangle-\sum\frac{nA_i}{N}-(r+1)\\ &= -\left\langle\frac{n\sum A_k}{N}\right\rangle+\sum\left\langle\frac{nA_i}{N}\right\rangle. \end{aligned}$$

The second and third equalities are respectively obtained by applying (1) of Lemmas 6.3 and 6.4. ■

Under the hypotheses $(n,N)=1$ and $N\nmid A_0,\dots,A_r,\sum_{k=0}^r A_k$, we still can get a better result on $\dim V_n$. For this, we will use the following lemma.

Lemma 6.6 *Let $x_0,\dots,x_r$ be real numbers. Then we have*

$$-\left\langle\sum_{i=0}^r x_i\right\rangle+\sum_{i=0}^r\langle x_i\rangle\in\{0,\dots,r\}.$$

Proof First remark that

$$\left\langle\sum x_i\right\rangle=\left\langle\sum[x_i]+\sum\langle x_i\rangle\right\rangle=\left\langle\sum\langle x_i\rangle\right\rangle,$$

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because of $\sum [x_i] \in \mathbb{Z}$ applied to (3) of Lemma 6.3. Then we have

$$-\Big\langle \sum x_i \Big\rangle + \sum \langle x_i \rangle = -\Big\langle \sum \langle x_i \rangle \Big\rangle + \sum \langle x_i \rangle = \Big[\sum \langle x_i \rangle \Big] =: c,$$

by the above and by definition. Pay attention to the fact that c is an integer. Since $\sum \langle x_i \rangle \geq 0$ and $-\langle \sum x_i \rangle \in (-1, 0]$, the integer c cannot be negative, because -1 cannot be reached. Moreover, $c \leq r$, because $\sum \langle x_i \rangle < r + 1$. Hence c lies in $\{0, \dots, r\}$. ■

Theorem 6.7 *Let the notations be as in Theorem 6.2 and suppose $(n, N) = 1$ and $N \nmid A_0, \dots, A_r, \sum_{k=0}^r A_k$. Then we have*

$$\dim V_n = -\Big\langle \frac{n \sum_{k=0}^r A_k}{N} \Big\rangle + \sum_{i=0}^r \Big\langle \frac{nA_i}{N} \Big\rangle.$$

Proof By Proposition 6.5, we have

$$d_n = -\Big\langle \frac{n \sum_{k=0}^r A_k}{N} \Big\rangle + \sum_{i=0}^r \Big\langle \frac{nA_i}{N} \Big\rangle$$

and by Lemma 6.6, we know that $d_n \in \{0, \dots, r\}$. Finally, by Theorem 6.2, we get $\dim V_n = d_n$. ■

6.4 $\dim V_n + \dim V_{N-n}, (n, N) = 1$

Theorem 6.8 *Let the notations be as in Theorem 6.2. Suppose that $(n, N) = 1$ and $N \nmid A_0, \dots, A_r, \sum_{k=0}^r A_k$. Then we have*

$$\dim V_n + \dim V_{N-n} = r.$$

Proof

$$\begin{aligned}
\dim V_{N-n} &= -\Big\langle \frac{(N-n)\sum A_k}{N} \Big\rangle + \sum_{i=0}^r \Big\langle \frac{(N-n)A_i}{N} \Big\rangle \quad \text{by Theorem 6.7} \\
&= -\Big\langle \sum A_k - \frac{n \sum A_k}{N} \Big\rangle + \sum \Big\langle A_i - \frac{nA_i}{N} \Big\rangle \\
&= -\Big\langle -\frac{n \sum A_k}{N} \Big\rangle + \sum \Big\langle -\frac{nA_i}{N} \Big\rangle \quad \text{by (3) Lemma 6.3} \\
&= -1 + \Big\langle \frac{n \sum A_k}{N} \Big\rangle + r + 1 - \sum \Big\langle \frac{nA_i}{N} \Big\rangle \quad \text{by (4) Lemma 6.3} \\
&= r - \Big(-\Big\langle \frac{n \sum A_k}{N} \Big\rangle + \sum \Big\langle \frac{nA_i}{N} \Big\rangle \Big) \\
&= r - \dim V_n \quad \text{by Theorem 6.7.}
\end{aligned}$$

■

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Note 2 As the referee kindly pointed out, the result of Theorem 6.8 could be deduced directly from a result of Chevalley and Weil [6] giving a formula for the multiplicities of irreducible representations in the representation given by the action of a finite group on the space of holomorphic differential forms on a curve. This formula amounts basically to the holomorphic Lefschetz formula, a recent account of which can be found in the Appendix of [19]. This confirms our result based on a geometric viewpoint.

7 New Forms and New Jacobian

We are now approaching our goal of constructing an abelian variety on which $\int \pi^*\omega_1$ lives as a period. We could have taken the Jacobian variety of X_N, but its dimension (equal to $\dim \Omega^1[X_N] = g[X_N]$) would have depended not only on N but also on $A_0, \ldots, A_r$ (*cf.* Theorem 4.1). That is the reason why we will restrict ourselves to an abelian subvariety of $\mathrm{Jac}(X_N)$, whose dimension depends on N and on the number $r+1$ of factors in the equation, but not on the exponents.

In order to define this subvariety, we will select regular differential forms on X_N, which "do not come from under" and are therefore called *new*. This will be made more precise.

First of all, let's work at the level of the singular curve C_N, because it is here possible to work with explicit expressions for the differential forms, in coordinates that we choose to be affine.

Let $d \in \mathbb{N}$. If $d|N$, then we have a well-defined morphism

$$\begin{aligned} \psi_d \colon C_N &\to C_d \\ (x, y) &\mapsto (x, y^{\frac{N}{d}}) \\ \infty &\mapsto \infty. \end{aligned}$$

Let $(u, v) \in C_d$ be an affine point, then

$$\psi_d^{-1}\{(u, v)\} = \{(u, v_0), (u, \zeta_{\frac{N}{d}} v_0), \ldots, (u, \zeta_{\frac{N}{d}}^{\frac{N}{d}-1} v_0)\},$$

where v_0 is any fixed $\frac{N}{d}$-th root of v and $\zeta_{\frac{N}{d}} := e^{\frac{2\pi i}{N} d}$. We see that there is an open dense subset of C_d of points having $\frac{N}{d}$ preimages. The other points have exactly one preimage. ψ_d is a ramified topological covering. The set of preimages of a point $P \in C_d$ is called the *fiber* over P with respect to ψ_d.

As we have seen in Section 5, the group μ_N of N-th roots of unity acts on C_N

$$\begin{aligned} \mu_N \times C_N &\to C_N \\ (\zeta, P) &\mapsto \varphi_\zeta(P). \end{aligned}$$

Remark that the subgroup $I_d := \langle \zeta_N^d \rangle$, $\zeta_N := e^{\frac{2\pi i}{N}}$, of index d in μ_N acts transitively on each fiber by permutation. Hence, the covering is Galois.

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The action of μ_N on C_N induces an action of μ_N on the vector space $\Phi[C_N]$ of differential forms on C_N. This goes very similarly as for the definition of the induced action on $\Omega^1[X_N]$ (see Section 5). Indeed, we set

$$\begin{aligned}\mu_N \times \Phi[C_N] &\to \Phi[C_N]\\ (\zeta, \omega) &\mapsto \varphi_\zeta^* \omega.\end{aligned}$$

Now, suppose that you have a differential form η on C_d. It is clear that its pull-back $\psi_d^*\eta$ on C_N is invariant under the action of the subgroup I_d, because I_d preserves the fibers.

The converse is more subtle. Let $\omega \in \Phi[C_N]$ be invariant under the action of I_d. Does ω define a differential form $(\psi_d)_*\omega$ on C_d? The answer to this question is positive, because I_d acts transitively on each fiber. Hence, for $Q \in C_d$, we can define $\big((\psi_d)_*\omega\big)(Q)$ to be the unique linear form on $\theta_{C_d,Q}$ such that, for $P \in C_N$ with $\psi_d(P) = Q$, $\big((\psi_d)_*\omega\big)(Q) \circ d_P\psi_d = \omega(P)$. This is well-defined, because $\forall P' \in C_N$ with $\psi_d(P') = Q$, $\exists \xi \in I_d$ such that $\varphi_\xi(P') = P$ and then

$$\omega(P) = \omega\big(\varphi_\xi(P')\big) \circ d_{P'}\varphi_\xi = (\varphi_\xi^*\omega)(P') = \omega(P'),$$

by invariance of ω under I_d. Remark further that $\psi_d^*\big((\psi_d)_*\omega\big) = \omega$. Indeed, let $P \in C_N$, then

$$\psi_d^*\big((\psi_d)_*\omega\big)(P) = \big((\psi_d)_*\omega\big)\big(\psi_d(P)\big) \circ d_P\psi_d = \omega(P).$$

The so-defined differential form $(\psi_d)_*\omega \in \Phi[C_d]$ is called the *push-forward* of ω with respect to ψ_d.

For a differential form ω on C_N and $d|N$, we have shown

$$\omega \text{ is fixed under the action of } I_d \text{ on } \Phi[C_N] \quad \Leftrightarrow \quad \exists \eta \in \Phi[C_d] \text{ such that } \psi_d^*\eta = \omega.$$

Let's now consider the differential form $\omega_n(x, y) = y^{-n} \prod_{i=0}^r (x - \lambda_i)^{a_i}\, dx$ on C_N, where the a_i's are integer. Under which condition on n is ω_n fixed by the action of I_d? Well,

$$\begin{aligned}\forall \xi \in I_d, \quad \varphi_\xi^*\omega_n = \omega_n &\Leftrightarrow \forall \xi \in I_d, \quad \xi^n\omega_n = \omega_n\\ &\Leftrightarrow \forall k \in \Big\{0, \dots, \frac{N}{d} - 1\Big\}, \quad (\zeta_N^{dk})^n\omega_n = \omega_n\\ &\Leftrightarrow \exists \ell \in \mathbb{Z} \text{ s.t. } n = \ell\frac{N}{d}.\end{aligned}$$

The differential forms which satisfy this for a d dividing N and different from N are the ones we want to get rid of, because "they come from under". This is equivalent to the fact that $(N, n) \neq 1$. Indeed, if the above equivalent conditions hold, $\frac{N}{d}$ is $\neq 1$ and divides both N and n. Conversely, suppose that $(N, n) \neq 1$, then ω_n is fixed under the action of $I_{\frac{N}{(N,n)}}$.

Definition 3 A differential form ω_n (resp. $\pi^*\omega_n$) on C_N (resp. X_N) such that $(n, N) = 1$ and the linear combinations of such differential forms are said to be *new*. The vector subspace of $\Omega^1[X_N]$ consisting of all *new forms* on X_N which are holomorphic is

$$\Omega^1[X_N]_{\text{new}} := \bigoplus V_n,$$

where the sum is taken over the $n \in \{0, \dots, N-1\}$ such that $\dim V_n > 0$ and $(n, N) = 1$.

The Jacobian variety $\mathrm{Jac}(X_N)$ of X_N is the abelian variety defined by the following quotient:

$$\Omega^1[X_N]^* \;/\; \imath\big(H_1\big(X_N(\mathbb{C}), \mathbb{Z}\big)\big).$$

The vector subspace $\Omega^1[X_N]_{\text{new}}$ defines a subquotient of this quotient which corresponds to abelian subvariety of $\mathrm{Jac}(X_N)$. This abelian subvariety will be called the *New Jacobian* of X_N and denoted by $\mathrm{Jac}_{\text{new}}(X_N)$. Its dimension is equal to $\dim_{\mathbb{C}}(\Omega^1[X_N]_{\text{new}})$. By definition, we have

$$\dim_{\mathbb{C}}(\Omega^1[X_N]_{\text{new}}) = \sum_{\substack{(n,N)=1\\0<n<N}} \dim(V_n) = \frac{1}{2} \sum_{\substack{(n,N)=1\\0<n<N}} (\dim V_n + \dim V_{N-n}). \tag{15}$$

Under the assumptions $N \nmid A_0, \dots, A_r, \sum_{k=0}^{r} A_k$, we can apply Theorem 6.8 to get $\dim V_n + \dim V_{N-n} = r$. This implies

$$\dim \mathrm{Jac}_{\text{new}}(X_N) = \frac{r\varphi(N)}{2}, \tag{16}$$

where $\varphi(N) := \sum_{\substack{(n,N)=1\\0<n<N}} 1$ is Euler's function.

Remark 10 The endomorphism algebra of $\mathrm{Jac}_{\text{new}}(X_N)$ contains $\mathbb{Q}(\mu_N)$.

8 Abelian Varieties Associated to Gauss' Hypergeometric Series

The family of curves, which is often associated to Gauss' hypergeometric series, as in [21], [22], is isomorphic to but not equal to the one we defined in Section 2. Indeed, our construction is based on the second integral representation (3) of $F(a, b, c\,; z)$ and not on Euler's. Since the two families of curves are isomorphic, so are their Jacobian varieties. Hence, the New Jacobian defines an abelian subvariety of the Jacobian of the curve coming from Euler's integral representation, which is precisely the abelian variety $T_{abc}(z)$ used by Wolfart. On the way to show this, we will have all intermediate results about genus, order of differential forms, dimensions, some of which will slightly correct some of Wolfart's assertions (see Remark 12). Remark 11 gives some light about the motivation coming from Wolfart's work.

Let's consider a hypergeometric series $F(a, b, c\,; z)$ with rational parameters a, b, c and $-c \notin \mathbb{N}$. For $|z| < 1$ and $c > b > 0$, Euler's integral representation (2) can be written as

$$F(a, b, c\,; z) = \frac{\mathcal{P}(z)}{\mathcal{P}(0)}, \quad \text{where} \quad \mathcal{P}(z) = \int_0^1 x^{b-1}(1-x)^{c-b-1}(1-zx)^{-a}\, dx. \tag{17}$$

If $z \neq 0, 1$ and $b, c - b \notin \mathbb{Z}$, this integral can be replaced up to an algebraic factor by a period $\int_\gamma \frac{dx}{y}$ on the projective curve $C(N, z)$ defined affinely by the equation

$$y^N = x^A(1-x)^B(1-zx)^C,$$

where $N := \operatorname{lcd}(a, b, c)$, $A := N(1-b)$, $B := N(1+b-c)$, $C := Na$. If we choose γ to be a loop on $C(N, z)$ whose projection in $\mathbb{P}^1_{\mathbb{C}}$ under $(x, y) \mapsto x$ is a double contour loop around 0 and 1 with interior not containing $\frac{1}{z}$, we have the relation

$$\int_\gamma \frac{dx}{y} = (1-\zeta_N^{-A})(1-\zeta_N^{-B}) \int_0^1 x^{b-1}(1-x)^{c-b-1}(1-zx)^{-a}\,dx.$$

Let $X(N, z)$ denote the desingularization of $C(N, z)$ and π_z the desingularization morphism. We have

$$\int_\gamma \frac{dx}{y} = \int_{\pi_z^*\gamma} \pi_z^*\Big(\frac{dx}{y}\Big),$$

where $\pi_z^*(\frac{dx}{y})$ is the pull-back of $\frac{dx}{y}$ on $X(N, z)$ and $\pi_z^*\gamma$ a lift of γ to $X(N, z)$.

Remark 11 These relations are the motivation for the whole construction, taking into account that a similar relation can be worked out for the integral in the denominator, as we shall see in Section 9. Indeed, these relations allow us to interpret (17) as a quotient of periods defined over $\mathbb{Q}(z)$ (this is the point of the assumption $a, b, c \in \mathbb{Q}$). As Wolfart pointed out, this is a key tool for the study of the set of algebraic points at which the series takes algebraic values (the so-called *exceptional set*). Indeed, if $z \in \bar{\mathbb{Q}}$, then the abelian varieties and the periods are defined over $\bar{\mathbb{Q}}$ and one can apply a consequence ([23] Satz 2) of Wüstholz's Analytic Subgroup Theorem to get a necessary condition on the corresponding abelian varieties for this quotient to be algebraic. This is a central observation in Wolfart's work [21], [22]. An explicit condition for z to lie in the exceptional set is determined in [4] for two hypergeometric series with monodromy group isomorphic to $SL_2(\mathbb{Z})$ and in [3] for a wider family of these series.

In order to apply the construction of New Jacobian to construct an abelian variety on which $\mathcal{P}(z)$ with $z \neq 0, 1$ lives as a period, we first note the existence of an isomorphism between the curve $X(N, z)$ and a curve of the same shape as those defined in Section 2.

We define $C_N(z)$ to be the projective algebraic curve defined affinely by the equation

$$y^N = x^{N-A-B-C}(x-1)^B(x-z)^C,$$

$X_N(z)$ to be its desingularization and $\pi\colon X_N(z) \to C_N(z)$ the desingularization morphism. Then the map

$$\kappa\colon C(N, z) \to C_N(z)$$
$$(x_0 : x_1 : x_2) \mapsto (x_1 : x_0 : x_2)$$

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is well-defined as one can verify using the equations in projective coordinates

$$C(N,z): x_2^N = x_0^{N-A-B-C} x_1^A (x_0 - x_1)^B (x_0 - z x_1)^C \quad \text{and}$$
$$C_N(z): u_2^N = u_0^A u_1^{N-A-B-C} (u_1 - u_0)^B (u_1 - z u_0)^C.$$

Moreover, κ is clearly a morphism of algebraic varieties, which is equal to its inverse. Hence it is an isomorphism and the composition $\kappa \circ \pi_z \colon X(N,z) \to C_N(z)$ is a birational morphism. It follows that $X(N,z)$ is a nonsingular model of $C_N(z)$. By the uniqueness up to isomorphism of the desingularization, $X(N,z)$ and $X_N(z)$ are isomorphic (call this isomorphism $\tilde{\kappa}$). Following Definition 1, we will suppose that

$$N \nmid N-A-B-C,\ B,\ C,\ N-A \quad \text{and} \quad (N, N-A-B-C, B, C) = 1.$$

In particular, this implies a, b, $c-a$, $c-b \notin \mathbb{Z}$.

The two curves $X(N,z)$ and $X_N(z)$ being isomorphic, they have the same Euler characteristic. By Theorem 4.1, we find

$$\chi\big(X(N,z)(\mathbb{C})\big) = -2N + (N,A) + (N,B) + (N,C) + (N, N-A-B-C)$$
$$g[X(N,z)] = N + 1 - \frac{1}{2}[(N,A) + (N,B) + (N,C) + (N, N-A-B-C)].$$

Let $n \in \{0, \dots, N-1\}$ and ω_n denote the following (rational) differential form on $C_N(z)$

$$\frac{x^{a_0}(x-1)^{a_1}(x-z)^{a_2}\, dx}{y^n}.$$

Then, by the regularity conditions (13) Section 6.1, the pull-back $\pi^*\omega_n$ on $X_N(z)$ is regular exactly when the following four conditions hold

$$\begin{aligned} a_0 &\geq \frac{n(N-A-B-C) + (N, N-A-B-C)}{N} - 1, \\ a_1 &\geq \frac{nB + (N,B)}{N} - 1, \\ a_2 &\geq \frac{nC + (N,C)}{N} - 1, \\ a_0 + a_1 + a_2 &\leq \frac{n(N-A) + (N,A)}{N} - 1. \end{aligned} \tag{18}$$

For $n \in \{0, \dots, N-1\}$, let V_n be the isotypical component of $\Omega^1[X_N(z)]$ of character χ_n for the action of μ_N. Then, Theorem 6.2 implies

$$\dim V_n = \begin{cases} d_n & \text{if } d_n \geq 0 \\ 0 & \text{otherwise,} \end{cases} \tag{19}$$

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where d_n is equal to

$$d_n = \Big[\frac{n(A+B+C)-(N,N-A-B-C)}{N}\Big] + \Big[1-\frac{nA+(N,A)}{N}\Big] + \Big[1-\frac{nB+(N,B)}{N}\Big] + \Big[1-\frac{nC+(N,C)}{N}\Big].$$

In order to get d_n in this form, use that for $k \in \mathbb{Z}$, $[x+k] = [x]+k$.

Remark 12 Let η_n denote the differential form $y^{-n}x^{b_0}(1-x)^{b_1}(1-zx)^{b_2}\,dx$ on $C(N,z)$. We can use the conditions (18) to determine when $\pi_z^*\eta_n$ is holomorphic on $X(N,z)$, since $\pi_z^*\eta_n$ is holomorphic on $X(N,z)$ exactly when $\pi^*(\kappa^{-1})^*\eta_n$ is holomorphic on $X_N(z)$. This is the case exactly when the following four conditions hold

$$\begin{aligned} b_0 &\geq \frac{nA+(N,A)}{N}-1\\ b_1 &\geq \frac{nB+(N,B)}{N}-1\\ b_2 &\geq \frac{nC+(N,C)}{N}-1\\ b_0+b_1+b_2 &\leq \frac{n(A+B+C)-(N,N-A-B-C)}{N}-1. \end{aligned} \tag{20}$$

These conditions correct slightly the assertion of Wolfart ([22] Section 4) on the holomorphy conditions for differential 1-forms, while his assertion on the dimension of the isotypical components V_n for $n \in \{0,\dots,N-1\}$ is corrected by (19).

Consider now the case $(n,N)=1$. Theorem 6.7 implies

$$\dim V_n = -\Big\langle \frac{n(N-A)}{N}\Big\rangle + \Big\langle \frac{n(N-A-B-C)}{N}\Big\rangle + \Big\langle \frac{nB}{N}\Big\rangle + \Big\langle \frac{nC}{N}\Big\rangle.$$

Using that $\forall x \in \mathbb{R}\setminus\mathbb{Z}$, $\langle -x\rangle = 1-\langle x\rangle$, we obtain

$$\dim V_n = \Big\langle \frac{nA}{N}\Big\rangle + \Big\langle \frac{nB}{N}\Big\rangle + \Big\langle \frac{nC}{N}\Big\rangle - \Big\langle \frac{n(A+B+C)}{N}\Big\rangle.$$

Finally, in the case $(n,N)=1$, Theorem 6.8 implies

$$\dim V_n + \dim V_{N-n} = 2. \tag{21}$$

By definition, the isomorphism $\tilde{\kappa}$ makes the following diagram commute

$$\begin{array}{ccc} X(N,z) & \xrightarrow{\tilde{\kappa}} & X_N \\ {\scriptstyle \pi_z}\big\downarrow & & \big\downarrow{\scriptstyle \pi} \\ C(N,z) & \xrightarrow[\kappa]{} & C_N. \end{array}$$

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By (15) and (21), the vector space of newforms $\Omega^1[X_N(z)]_{\text{new}}$ has dimension $\varphi(N)$. It defines a vector subspace of $\Omega^1[X(N,z)]$ of the same dimension by pulling-back

$$\Omega^1[X(N,z)]_{\text{new}} := \tilde{\kappa}^*\big(\Omega^1[X_N(z)]_{\text{new}}\big).$$

This vector subspace defines a $\varphi(N)$-dimensional abelian subvariety $T_{abc}(z)$ of the Jacobian variety $X(N,z)$, which is the abelian variety considered by Wolfart in [21], [22].

If $\pi_z^*(\frac{dx}{y})$ is regular on $X(N,z)$ (*cf.* the regularity conditions (20)), we have

$$\pi_z^*\Big(\frac{dx}{y}\Big) = \tilde{\kappa}^*\Big(\pi^*(\kappa^{-1})^*\Big(\frac{dx}{y}\Big)\Big) \in \tilde{\kappa}^*\big(\Omega^1[X_N(z)]_{\text{new}}\big).$$

In this case, $\mathcal{P}(z)$ is, up to multiplication by an algebraic constant, a period on $T_{abc}(z)$.

9 Abelian Varieties Associated to the Beta Function

In order to interpret Euler's integral representation (2) of $F(a,b,c;z)$ with $a,b,c \in \bar{\mathbb{Q}}$ as a quotient of periods, we are left with the construction of an abelian variety on which the denominator of (2) lives as a period. We refer to Section 8 and specially to Remark 11 for a motivation. Also the procedure to derive this construction from the construction of the New Jacobian will be totally similar to that of Section 8. After all, it is only a (degenerated) specialization at $z = 0$.

The integral $\mathcal{P}(0)$ in the denominator of Euler's integral representation as it is formulated in (17) is the Beta function

$$B(b, c-b) = \int_0^1 x^{b-1}(1-x)^{c-b-1}\,dx.$$

Let $M := \operatorname{lcd}(b,c)$, $P := M(1-b)$, $Q := M(1+b-c)$ and $X(M,0)$ be the desingularization of the projective curve $C(M,0)$ defined affinely by

$$y^M = x^P(1-x)^Q.$$

If $b, c-b \notin \mathbb{Z}$, we have

$$\mathcal{P}(0) = \int_0^1 \frac{dx}{y} = k\int_\gamma \frac{dx}{y},$$

where $k \in \bar{\mathbb{Q}}$, $\frac{dx}{y}$ is a differential form on $C(M,0)$ and γ a lift on $C(M,0)$ under $(x,y) \mapsto x$ of a double contour loop around 0 and 1 in $\mathbb{C}$. By the same argument as in Section 8, it is sufficient to work on a curve isomorphic to $C(M,0)$. Hence, we define $C_M(0)$ to be the projective curve with affine equation

$$y^M = x^{M-P-Q}(x-1)^Q$$

and $X_M(0)$ to be its desingularization. The projective equations are

$$C(M,0): x_2^M = x_0^{M-P-Q}x_1^P(x_0-x_1)^Q \quad \text{and}$$

$$C_M(0): x_2^M = x_0^P x_1^{M-P-Q}(x_1-x_0)^Q$$

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and there is an isomorphism

$$\kappa\colon C(M,0) \to C_M(0)$$
$$(x_0 : x_1 : x_2) \mapsto (x_1 : x_0 : x_2).$$

The unicity up to isomorphism of the desingularization implies the existence of an isomorphism $\tilde{\kappa}\colon X(M,0) \to X_M(0)$ such that the following diagram commutes

$$\begin{array}{ccc} X(M,0) & \xrightarrow{\tilde{\kappa}} & X_M(0) \\ {\scriptstyle \pi_0}\downarrow & & \downarrow{\scriptstyle \pi} \\ C(M,0) & \xrightarrow[\kappa]{} & C_M(0). \end{array}$$

In order to apply our general construction, we will suppose that

$$M \nmid M-P-Q,\ Q,\ M-P \quad \text{and} \quad (M, M-P-Q, Q) = 1.$$

This implies $b, c, c-b \notin \mathbb{Z}$. Since the two curves are isomorphic, they have the same Euler characteristic. By Theorem 4.1, we have

$$\chi\big(X(M,0)(\mathbb{C})\big) = -M + (M,P) + (M,Q) + (M, M-P-Q)$$
$$g[X(M,0)] = 1 + \frac{1}{2}[M - (M,P) - (M,Q) - (M, M-P-Q)].$$

For $n \in \{0, \dots, M-1\}$, let ω_n be the (rational) differential form on C_M defined by

$$\frac{x^{a_0}(x-1)^{a_1}\, dx}{y^n}.$$

By the regularity conditions (13), $\pi^*\omega_n$ is regular on X_M if and only if

$$\begin{aligned} a_0 &\geq \frac{n(M-P-Q) + (M, M-P-Q)}{M} - 1, \\ a_1 &\geq \frac{nQ + (M,Q)}{M} - 1, \\ a_0 + a_1 &\leq \frac{n(M-P) - (M,P)}{M} - 1. \end{aligned} \tag{22}$$

Remark 13 For $n \in \{0, \dots, M-1\}$, let η_n be the (rational) differential 1-form $y^{-n}x^{b_0}(1-x)^{b_1}\, dx$ on $C(M,0)$. Then $\pi_0^*\eta_n$ is regular on $X(M,0)$ if and only if $\pi^*(\kappa^{-1})^*\eta_n$ is regular on $X_M(0)$. By the conditions (22), this is the case exactly when the following three conditions hold.

$$\begin{aligned} b_0 &\geq \frac{nP + (M,P)}{M} - 1, \\ b_1 &\geq \frac{nQ + (M,Q)}{M} - 1, \\ b_0 + b_1 &\leq \frac{n(P+Q) - (M, M-P-Q)}{M} - 1. \end{aligned}$$

For $n \in \{0, \dots, M-1\}$, let V_n be the isotypical component of $\Omega^1[X_M(0)]$ with character χ_n. Theorem 6.2 implies

$$\dim V_n = \begin{cases} d_n & \text{if } d_n \geq 0 \\ 0 & \text{otherwise,} \end{cases}$$

where

$$\begin{aligned} d_n &= \Big\lceil \frac{n(M-P)-(M,P)}{M} \Big\rceil \\ &\quad + \Big\lceil 1 - \frac{n(M-P-Q)+(M,M-P-Q)}{M} \Big\rceil + \Big\lceil 1 - \frac{nQ+(M,Q)}{M} \Big\rceil \\ &= \Big\lceil \frac{n(P+Q)-(M,M-P-Q)}{M} \Big\rceil \\ &\quad + \Big\lceil 1 - \frac{nP+(M,P)}{M} \Big\rceil + \Big\lceil 1 - \frac{nQ+(M,Q)}{M} \Big\rceil. \end{aligned}$$

In the case $(n, N) = 1$, it follows from Theorem 6.7 that

$$\begin{aligned} \dim V_n &= -\Big\langle \frac{n(M-P)}{M} \Big\rangle + \Big\langle \frac{n(M-P-Q)}{M} \Big\rangle \\ &\quad + \Big\langle \frac{nQ}{M} \Big\rangle \\ &= \Big\langle \frac{nP}{M} \Big\rangle + \Big\langle \frac{nQ}{M} \Big\rangle \\ &\quad - \Big\langle \frac{n(P+Q)}{M} \Big\rangle \end{aligned}$$

and from Theorem 6.8 that

$$\dim V_n + \dim V_{N-n} = 1.$$

Together with (15), this implies that the vector space $\Omega^1[X_M]_{\text{new}}$ of new differential forms on X_M has dimension $\frac{\varphi(M)}{2}$. Set $\Omega^1[X(M,0)]_{\text{new}} := \tilde{\kappa}^*(\Omega^1[X_M]_{\text{new}})$. Then $\Omega^1[X(M,0)]_{\text{new}}$ defines and abelian subvariety $T_{abc}(0)$ of $\operatorname{Jac}\big(X(M,0)\big)$ of dimension $\frac{\varphi(M)}{2}$. If $\pi_0^*(\frac{dx}{y})$ is regular on $X(M,0)$ (*cf.* Remark 13), then $\mathcal{P}(0) = B(b, c-b)$ is, up to multiplication by an algebraic factor, a period on $T_{abc}(0)$.

Remark 14 In its Appendix to [20], Rohrlich constructed an abelian variety on which the Beta function is a period as a quotient of the Jacobian of the Fermat curve. That construction and the one given here are isomorphic.

References

[1] P. Appell, *Sur les fonctions hypergéométriques de deux variables*. Jour. de Math., 3ème sér. **VIII** (1882), 173–216.

[2] N. Archinard, *Abelian Varieties and Identies for Hypergeometric Series*. PhD thesis, ETH Zurich, 2000. http://e-collection.ethbib.ethz.ch.

[3] ———, *Exceptional Sets of Hypergeometric Series*. J. Number Theory **101**(2003), 244–269.

[4] F. Beukers and J. Wolfart, *Algebraic values of hypergeometric functions*. In: New Advances in Transcendence Theory, (ed. A. Baker), Cambrige University Press, 1986.

[5] E. Brieskorn and H. Knörrer, *Ebene Algebraische Kurven*. Birkhäuser, 1981.

[6] C. Chevalley and A. Weil, *Über das Verhalten der Integrale 1. Gattung bei Automorphismen des Funktionenkörpers*. Abh. Hamburger Math. Sem. **10**(1934), 358–361.

[7] P. Cohen and J. Wolfart, *Modular embeddings for some non-arithmetic Fuchsian groups*. Acta Arith. **56**(1990), 93–110.

[8] P. B. Cohen and J. Wolfart, *Algebraic Appell-Lauricella functions*. In: Special Differential Equations (ed. M. Yoshida), Proceedings of the Taniguchi Workshop, 1991, 150–164,

[9] ———, *Fonctions hypergéométriques en plusieurs variables et espaces des modules de variétés abéliennes*. Ann. Sci. École Norm. Sup. (4) **26**(1993), 665–690.

[10] P. B. Cohen and G. Wüstholz, *Applications of the André-Oort Conjecture to transcendence*. In: A Panorama in Number Theory, ed. G. Wüstholz, Cambrige University Press, 2001.

[11] H. Darmon, *Modularity of fibres in rigid local systems*. Ann. of Math. **149**(1999), 1079–1086.

[12] ———, *Rigid local systems, Hilbert modular forms, and Fermat's last theorem*. Duke Math. J. **102**(2000), 413–449.

[13] P. Deligne and G. D. Mostow, *Monodromy of hypergeometric functions and non-lattice integral monodromy*. Inst. Hautes Études Sci. Publ. Math. **63**(1986), 5–89.

[14] S. Edixhoven and A. Yafaev, *Subvarieties of Shimura varieties*. Ann. of Math., to appear.

[15] G. Lauricella, *Sulle funzioni ipergeometriche a piu variabili*. Rendiconti di Palermo **VII**(1893), 111–158.

[16] I. R. Shafarevich, *Basic Algebraic Geometry*. Volumes 1 and 2, Springer-Verlag, 1994.

[17] H. Shiga and J. Wolfart, *Criteria for complex multiplication and transcendence properties of automorphic functions*. J. Reine Angew. Math. **463**(1995), 1–25.

[18] T. Terada, *Problème de Riemann et fonctions automorphes provenant de fonctions hypergéométriques de plusieurs variables*. J. Math. Kyoto Univ. **13-3**(1973), 557–578.

[19] M. van der Put and F. Ulmer, *Differential equations and finite groups*. J. Algebra **226**(2000), 920–966.

[20] H. Gross (with an appendix by D. Rohrlich), *On the periods of abelian integrals and a formula of Chowla and Selberg*. Invent. Math. **45**(1978), 193–211.

[21] J. Wolfart, *Fonctions hypergéometriques, arguments exceptionels et groupes de monodromie*. Publ. Math. Univ. P. et M. Curie, Problèmes Diophantiens **79**(1985–86), 1–24.

[22] ———, *Werte hypergeometrischer Funktionen*. Invent. Math. **92**(1988), 187–216.

[23] J. Wolfart and G. Wüstholz, *Der Ueberlagerungsradius gewisser algebraischer Kurven und die Werte der Betafunktion an rationalen Stellen*. Math. Ann. **273**(1985), 1–15.

[24] G. Wüstholz, *Algebraische Punkte auf analytischen Untergruppen algebraischer Gruppen*. Ann. Math. **129**(1989), 501–517.

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