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MATRICES WITH PRESCRIBED PRINCIPAL ELEMENTS AND SINGULAR VALUES

BY
GRACIANO N. DE OLIVEIRA

We shall be concerned with the following problem. Let $a_{11}, \ldots, a_{nn}$ be complex numbers and $\lambda_1, \ldots, \lambda_n$ nonnegative real numbers. Under what conditions does there exist an $n \times n$ complex matrix A with $a_{11}, \ldots, a_{nn}$ as principal elements and $\lambda_1, \ldots, \lambda_n$ as singular values? This problem has been suggested in [3] but, to our knowledge, has not yet been solved.

We can formulate another problem which apparently is more general. Given an $n \times n$ matrix $A = [a_{ij}]$, we shall say that $a_{1\sigma(1)}, \ldots, a_{n\sigma(n)}$, where σ is an element of the symmetric group S_n, is the σ-diagonal of A. We can ask: Under what conditions does there exist an $n \times n$ matrix A with $a_{11}, \ldots, a_{nn}$ as its σ-diagonal and with $\lambda_1, \ldots, \lambda_n$ as singular values? This problem can be reduced to the former. In fact, a permutation performed on the columns of a matrix does not alter its singular values because it is equivalent to multiplying on the right by an orthogonal matrix. Thus the elements $a_{11}, \ldots, a_{nn}$ can always be brought to the principal diagonal without changing the singular values.

There is no loss of generality if we suppose that $|a_{11}| \geq \cdots \geq |a_{nn}|$. In fact, if $a_{11}, \ldots, a_{nn}$ are the principal elements, we can always achieve this by performing a permutation on the rows and the same permutation on the columns of the matrix. These permutations leave the singular values unchanged.

Obviously we can also assume that $\lambda_1 \geq \cdots \geq \lambda_n$ without loss of generality.

A complete solution of the problem (even in the case $n=2$) seems to be difficult. We present a necessary condition but we do not know whether it is sufficient.

Let $A = [a_{ij}]$ be an $n \times n$ complex matrix with singular values $\lambda_1 \geq \cdots \geq \lambda_n$. Then, assuming that $|a_{11}| \geq \cdots \geq |a_{nn}|$ we have

$$|a_{11}|^2 + \cdots + |a_{kk}|^2 \leq \lambda_1^2 + \cdots + \lambda_k^2 \quad (k = 1, \ldots, n).$$

Moreover if for $k = k_0$ we have equality, then equality holds for every $k \leq k_0$. If for $k = n-1$ we have equality, then for $k = n$ we have also equality.

Proof. The characteristic roots of the Hermitian matrix $AA^* = [b_{ij}]$ will be $\lambda_1^2, \ldots, \lambda_n^2$. If $(x_1, \ldots, x_n)$ is a real n-vector by $\sum^{(k)} (x_1, \ldots, x_n)$ we denote the sum of its k greatest components. As it is well known [2] we have

$$\sum^{(k)} (b_{11}, \ldots, b_{nn}) \leq \lambda_1^2 + \cdots + \lambda_k^2 \quad (k = 1, \ldots, n) \tag{1}$$

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with equality for $k=n$. We have

(2) $$b_{ii} = \sum_{\alpha=1}^{n} |a_{i\alpha}|^2 \geq |a_{ii}|^2.$$

Therefore (1) gives

(3) $$|a_{11}|^2+\cdots+|a_{kk}|^2 \leq \lambda_1^2+\cdots+\lambda_k^2 \quad (k = 1,\ldots,n).$$

Suppose now that

(4) $$|a_{11}|^2+\cdots+|a_{k_0k_0}|^2 = \lambda_1^2+\cdots+\lambda_{k_0}^2.$$

Then (1) combined with (2) gives $b_{ii}=|a_{ii}|^2$, $i=1,\ldots,k_0$, or $a_{i\alpha}=0$, $i=1,\ldots,k_0$; $\alpha=1,\ldots,n$, $\alpha\neq i$.

The matrix $A^*A=[c_{ij}]$ has the same characteristic roots as AA^* and

(5) $$c_{ii} = \sum_{\alpha=1}^{n} |a_{\alpha i}|^2 \geq |a_{ii}|^2.$$

Also

(6) $$\sum^{(k)} (c_{11},\ldots,c_{nn}) \leq \lambda_1^2+\cdots+\lambda_k^2 \quad (k = 1,\ldots,n).$$

(4), (5), and (6) imply $a_{\alpha i}=0$, $i=1,\ldots,k_0$; $\alpha=1,\ldots,n$, $\alpha\neq i$. Thus the matrix A is of the form

$$A = \begin{bmatrix} D & 0 \\ 0 & B \end{bmatrix}$$

with

$$D = \operatorname{diag}(a_{11},\ldots,a_{k_0k_0}).$$

The singular values of A are $|a_{11}|,\ldots,|a_{k_0k_0}|$ and those of B. (4) and (3) for $k=k_0-1$ imply $|a_{k_0k_0}|\geq\lambda_{k_0}$. Therefore the k_0 greatest singular values of A are $|a_{11}|,\ldots,|a_{k_0k_0}|$. This is equivalent to saying that in (3) the sign "$=$" holds for every $k\leq k_0$. If $k_0=n-1$ then $B=[a_{nn}]$ and the last part of the proposition follows. The proof is complete.

We do not know if the above condition is sufficient. We think that a proof could be tried by induction. The case $n=1$ is trivial and different from the other cases because there are no nondiagonal elements. So, first of all a solution for the case $n=2$ should be found. Then a proof by induction could be tried using the interlacing inequalities for singular values [4]. We think that there might be some analogy between the solution of the present problem and the problem considered in [2].

Another problem can be posed if we wish to prescribe the principal elements, singular values and characteristic roots of an $n\times n$ matrix. A necessary and sufficient condition for the existence of a matrix with prescribed singular values and characteristic roots has been given in [1]. If the principal elements and the characteristic roots are prescribed the matrix exists if and only if the sum of the principal elements is equal to the sum of the characteristic roots.

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Instituto Gulbenkian de Ciencia, C.C.C.,
Oeiras, Portugal

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