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GENERALIZED EULER NUMBER SEQUENCES: ASYMPTOTIC ESTIMATES AND CONGRUENCES

D. J. LEEMING AND R. A. MACLEOD

1. Introduction. We define (as in [7]) integer sequences $\{E_n^{(k)}\}$, one for each positive integer $k \geqq 2$, by

$$(1.1) \quad \begin{cases} E_0^{(k)} = 1 \\ \sum_{j=1}^{k} \left(E^{(k)} + \omega_j^{(k)}\right)^n = \begin{cases} k, & n = 0 \\ 0, & n > 0 \end{cases} \end{cases}$$

where $\{\omega_j^{(k)}\}_{j=1}^{k}$ are the kth roots of unity and $(E^{(k)})^n$ is replaced by $E_n^{(k)}$ after multiplying out. We note that (1.1) implies $E_n^{(k)} = 0$, $n \neq 0 \pmod{k}$.

In [7], we considered some special properties of these number sequences, proved several congruences and conjectured several others. This paper is a continuation of the work presented in [7].

In Section 2 we demonstrate the asymptotic rate of growth of the numbers $\{E_{kn}^{(k)}\}$ by showing that

$$|E_{kn}^{(k)}|^{1/kn} \sim c_k n \quad (n \to \infty).$$

In Section 3 we present a large number of congruences (modulo 2048), some of which are proved or can be proved by the techniques presented herein, and other congruences which appear to be true on the basis of numerical evidence.

In Sections 4 and 5 we present two methods for proving such congruences. In Section 4, we give a method of successive approximations and, in Section 5, a method of undetermined coefficients. In Section 6 we use the techniques developed in the previous sections to prove the congruences given in Theorem 3.1.

2. An asymptotic estimate for the growth of $|E_{kn}^{(k)}|$. Leeming [6] has shown that in the case $k = 2$,

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$$\lim_{n\to\infty} \frac{|E_{2n}|^{1/2n}}{n} = \frac{4}{\pi e}.$$

We will show here that for each integer $k \geqq 2$ a similar asymptotic growth rate of the constants $\{|E_{kn}|^{1/nk}\}$ occurs as $n \to \infty$.

From ([7], Theorem 3.2) we have

$$(2.1) \quad \frac{1}{Q_k(z)} = \sum_{n=0}^{\infty} E_{kn}^{(k)} \frac{z^{kn}}{(kn)!} \quad \text{where} \quad Q_k(z) = \sum_{n=0}^{\infty} \frac{z^{kn}}{(kn)!}.$$

Since the zeros of an analytic function are isolated, let p_k be a zero of $Q_k(z)$ which is closest to the origin. Using the Cauchy-Hadamard theorem, we have

$$(2.2) \quad \overline{\lim_{n\to\infty}} \left[\frac{|E_{kn}^{(k)}|}{(kn)!}\right]^{1/kn} = \frac{1}{|p_k|}.$$

Therefore, we need to know something about the growth of the sequence $\{|p_k|\}_{k=2}^{\infty}$, which we prove in the next theorem.

THEOREM 2.1. *Let* $p_k^{(j)} = |p_k|e^{i(\theta_j + 2\pi j)/k}$, $(j = 0, 1, \dots, k - 1)$ *be the* k *zeros of* $Q_k(z)$ *nearest the origin. Then we have*

$$|p_k| = (1 + \delta_k)\left(\frac{k}{e}\right)$$

where $\delta_k > 0$ *and* $\lim_{k\to\infty} \delta_k = 0$, *hence*

$$\lim_{k\to\infty} \frac{|p_k|}{k} = \frac{1}{e}.$$

Proof. Mikusinski [9] has shown that for alternating series of the form

$$(2.3) \quad M_k(x) = \sum_{\nu=0}^{\infty} (-1)^{\nu} \frac{x^{k\nu}}{(k\nu)!} \quad k = 2, 3, \dots$$

the real zero q_k nearest the origin has asymptotic growth $q_k/k \sim 1/e$ $(k \to \infty)$. If k is even, let $\gamma_k = e^{\pi i/k}$ and we have $Q_k(z) = M_k(\gamma_k z)$; if k is odd, we have $Q_k(z) = M_k(-z)$, so that $q_k = |p_k|$.

Stirling's formula (see e.g. [5], p. 111) yields the inequalities

$$(2.4) \quad \sqrt{2\pi k}\left(1 + \frac{1}{12k}\right)\left(\frac{k}{e}\right) < k! < \sqrt{2\pi k}\left(1 + \frac{1}{6k}\right)\left(\frac{k}{e}\right), \quad k \geqq 2.$$

Therefore,

$$(2.5)\qquad (2\pi k)^{1/2k}\left(1+\frac{1}{12k}\right)^{1/k}\left(\frac{k}{e}\right) < (k!)^{1/k} < (2\pi k)^{1/2k}\left(1+\frac{1}{6k}\right)^{1/k}\left(\frac{k}{e}\right)$$

and it is evident that

$$(2.6)\qquad (1+\eta_k)\frac{k}{e} < (k!)^{1/k} < (1+\epsilon_k)\frac{k}{e}$$

where $\eta_k \to 0$, $\epsilon_k \to 0$ as $k \to \infty$.

Now Mikusinski [**9**] gives the estimate

$$(2.7)\qquad (k!)^{1/k} < |p_k| < \sqrt[k]{2}\,(k!)^{1/k}$$

and combining (2.6) and (2.7) establishes the theorem.

We now use Theorem 2.1 to obtain an asymptotic estimate for

$|E_{kn}^{(k)}|^{1/kn} \quad (n \to \infty)$.

THEOREM 2.2. *For $k \geqq 2$, we have*

$$(2.8)\qquad \overline{\lim_{n\to\infty}} \frac{|E_{kn}^{(k)}|^{1/kn}}{n} = c_k$$

where $\dfrac{4}{\pi e} \leqq c_k < 1$ *and* $\lim_{k\to\infty} c_k = 1$.

Proof. From (2.2) and (2.4) we have

$$(2.9)\qquad \lim_{n\to\infty} \frac{|E_{kn}^{(k)}|^{1/kn}}{n} = \frac{1}{|p_k|}\lim_{n\to\infty}\frac{[(kn)!]^{1/kn}}{n} = \frac{1}{|p_k|}\cdot\frac{k}{e}.$$

If we set $c_k = k/|p_k|e$, then (2.9) becomes (2.8). Now (2.7) is equivalent to

$$(2.10)\qquad \frac{k}{2^{1/k}(k!)^{1/k}e} < c_k < \frac{k}{(k!)^{1/k}e}$$

and since $Q_2(z) = \cosh z$, we have $c_2 = 4/\pi e \approx 0.468$. For $k = 3$, (2.10) gives $0.482 < c_3$, and hence $4/\pi e \leqq c_k < 1, \quad k \geqq 2$.

We note that numerical evidence indicates that $c_k \nearrow 1$ as $k \to \infty$. Selected values of $|p_k|$ and c_k are given in Table I below.

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TABLE I.

k	$\|p_k\|$	c_k	k	$\|p_k\|$	c_k
2	1.571	0.468	10	4.529	0.812
3	1.850	0.596	20	8.304	0.886
4	2.221	0.662	30	12.05	0.916
5	2.607	0.705	40	15.77	0.933
6	2.994	0.737	50	19.48	0.944
7	3.380	0.762	60	23.19	0.951
8	3.764	0.782	70	26.90	0.955

Added in press. M. E. Ismail has pointed out that using the method of Darboux (see Olver, Introduction to Asymptotics and Special Functions, Acad. Press, N.Y. 1974), (2.8) can be replaced by

$$\lim_{n\to\infty} \frac{|E_{kn}^{(k)}|^{1/kn}}{n} = c_k.$$

3. Congruences modulo powers of two. We present a collection of congruences modulo powers of two, especially $2^{11} = 2048$, for the number sequences $\{E_{kn}^{(k)}\}$, $0 \leqq n < \infty$, which were defined by (1.1). These congruences fall into three categories.

In the first category, we let $k = 2^t$, $t \geqq 2$, and we are able to prove the following theorem.

THEOREM 3.1. *For* $n = 1, 2, \ldots$, *we have the following congruences*:

$$E_{4n}^{(4)} \equiv 1 - 2n + 72\binom{n}{2} + 944\binom{n}{3} - 384\binom{n}{4} - 768\binom{n}{5} + 1024\binom{n}{6} \pmod{2048} \tag{3.1}$$

$$E_{8n}^{(8)} \equiv 1 - 2n + 584\binom{n}{2} + 944\binom{n}{3} - 384\binom{n}{4} - 768\binom{n}{5} + 1024\binom{n}{6} \pmod{2048} \tag{3.2}$$

$$E_{2^{t+1}n}^{(2^{t+1})} \equiv E_{2^t n}^{(2^t)} \pmod{2^{3t+2}}, \quad t \geqq 3. \tag{3.3}$$

In particular, since $3t + 2 \geqq 11$ for $t \geqq 3$, (3.2) is valid if 8 is replaced by 16, 32, $\cdots$. Certain other congruences are listed which may be proved by the same types of extensions of our work in [7] as we use to prove congruence (3.2) above. Two examples are given in the following theorem.

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THEOREM 3.2. *For $n = 1, 2, \cdots$ we have the following congruences* (mod 2048):

$$(3.4)\quad E_{3n}^{(3)} \equiv 3 + 156n - 552\binom{n}{2} - 328\binom{n}{3} + 32\binom{n}{4} + 128\binom{n}{5} + 832\binom{n}{6} - 768\binom{n}{7} + 512\binom{n}{8} - 512\binom{n}{9}$$

$$(3.5)\quad E_{6n}^{(6)} \equiv 3 - 164n - 800\binom{n}{2} - 32\binom{n}{3} - 256\binom{n}{4} - 512\binom{n}{5}.$$

In the second category, we consider the case $k = 2^t - 1$, $t = 2, 3, \cdots$, and obtain the result

$$(3.6)\quad E_{(2^t-1)n}^{(2^t-1)} \equiv (2^t - 1) - 2^t n + 2^{t+1}\binom{n+1}{3} \pmod{2^{t+2}}, \quad t \geqq 2.$$

In the third category are a number of congruences which have been verified for the first 50 or 25 values of n, but have not been proven. Representative examples are

$$(3.7)\quad E_{7n}^{(7)} \equiv 7 + 248n + 880\binom{n}{2} + 112\binom{n}{3} - 704\binom{n}{4} + 320\binom{n}{5} - 384\binom{n}{6} - 896\binom{n}{7} + 1024\binom{n}{8} \pmod{2048}, \quad 1 \leqq n \leqq 50$$

$$(3.8)\quad E_{14n}^{(14)} \equiv 7 - 8n + 384\binom{n}{2} + 512\binom{n}{3} + 1024\binom{n}{4} \pmod{2048}, \quad 1 \leqq n \leqq 50$$

$$(3.9)\quad E_{63n}^{(63)} \equiv 63 - 64n - 128\binom{n}{2} - 128\binom{n}{3} + 512\binom{n}{4} + 512\binom{n}{5} + 1024\binom{n}{6} + 1024\binom{n}{7} \pmod{2048}, \quad 1 \leqq n \leqq 25.$$

Before proving any of the congruences given in this section, we present two methods of obtaining such congruences. The first method might be termed a method of successive approximations, involving a study of odd and even patterns in

$$\left\{\binom{n}{m}, \ m \text{ fixed}, \ n = 0, 1, \cdots\right\}$$

and the second might be termed a method of undetermined coefficients.

4. Method of successive approximations and patterns for $\binom{n}{m}$, for

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fixed m. For this particular method, we proceed as follows. Assume we have a congruence modulo 2^t which is of the type $E_{kn}^{(k)} \equiv f_t(n)(\text{mod } 2^t)$ found to be valid over our numerical checking range ($n \leqq 25$ or $n \leqq 50$). We test the same congruence modulo 2^{t+1}. Normally there will be differences between $E_{kn}^{(k)}(\text{mod } 2^{t+1})$ and $f_t(n)(\text{mod } 2^{t+1})$. We adjust $f_t(n)$ to remove these differences by adding an appropriate linear combination of binomial coefficients to $f_t(n)$ to obtain a congruence $E_{kn}^{(k)} \equiv f_{t+1}(n)$ $(\text{mod } 2^{t+1})$.

We illustrate with the case $k = 6$. Suppose we know that the following is valid, $1 \leqq n \leqq 50$.

$$(4.1) \qquad E_{6n}^{(6)} \equiv 3 - 164n + 224\binom{n}{2} - 32\binom{n}{3} + 256\binom{n}{4} \ (\text{mod } 512).$$

Upon testing this same congruence modulo 1024, we find errors of 512 for $n = 4, 6, 12, 14, 20, 22, 28, 30, 36, 38, 44$ and 46. These are the integers ($\leqq 50$) which are congruent to 4 and 6 modulo 8. As we shall see below (Table II) the binomial coefficient $\binom{n+1}{5}$ is odd precisely when n is congruent to 4 and 6 modulo 8, and even otherwise. Therefore, if we add the term $512\binom{n+1}{5}$ to the congruence (4.1) we remove all of the errors modulo 1024. Since $\binom{n+1}{5} = \binom{n}{5} + \binom{n}{4}$, our congruence becomes

$$(4.2) \qquad E_{6n}^{(6)} \equiv 3 - 164n + 224\binom{n}{2} - 32\binom{n}{3} - 256\binom{n}{4} + 512\binom{n}{5} \quad (\text{mod } 1024).$$

Upon testing congruence (4.2) modulo 2048 ($1 \leqq n \leqq 50$) we find errors of 1024 for $n \equiv 2, 3, 5$, and $6 \ (\text{mod } 8)$. Since the binomial coefficient $\binom{n}{6}$ is odd if and only if $n \equiv 6 \ (\text{mod } 8)$ or $n \equiv 7 \ (\text{mod } 8)$, we add the term $1024\binom{n+1}{6} + 1024\binom{n+4}{6}$ and note that

$$\binom{n+1}{6} + \binom{n+4}{6} \equiv \left(\binom{n}{5} + \binom{n}{2}\right) \ (\text{mod } 2)$$

so (4.2) becomes

$$(4.3) \qquad E_{6n}^{(6)} \equiv 3 - 164n - 800\binom{n}{2} - 32\binom{n}{3} - 256\binom{n}{4} - 512\binom{n}{5} \quad (\text{mod } 2048)$$

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which is indeed valid for $1 \leqq n \leqq 50$, and can be proved for all values of n using the methods presented in [7].

We list below a collection of congruences which were obtained by the method described above. As will be seen, some we have been able to prove for $n \geqq 1$, (see Section 6), some could be proved by the methods of our paper [7] (marked with an asterisk), for some we can prove partial results, while for the rest we are unable to construct proofs. All congruences have been checked for "small" values of n, as listed, and all congruences are modulo $2048 = 2^{11}$.

$$(4.4)\quad E_{3n}^{(3)} \equiv 3 + 156n - 552\binom{n}{2} - 328\binom{n}{3} + 32\binom{n}{4} + 128\binom{n}{5} + 832\binom{n}{6} - 768\binom{n}{7} + 512\binom{n}{8} - 512\binom{n}{9}, \quad 1 \leqq n \leqq 50; *$$

$$(4.5)\quad E_{6n}^{(6)} \equiv 3 - 164n - 800\binom{n}{2} - 32\binom{n}{3} - 256\binom{n}{4} - 512\binom{n}{5}, \quad 1 \leqq n \leqq 50; *$$

$$(4.6)\quad E_{12n}^{(12)} \equiv 3 - 228n - 800\binom{n}{2} + 928\binom{n}{3} - 768\binom{n}{4} + 512\binom{n}{5}, \quad 1 \leqq n \leqq 50; *$$

$$(4.7)\quad E_{24n}^{(24)} \equiv 3 - 740n - 800\binom{n}{2} + 416\binom{n}{3} - 768\binom{n}{4} + 512\binom{n}{5}, \quad 1 \leqq n \leqq 40;$$

$$(4.8)\quad E_{4n}^{(4)} \equiv 1 - 2n + 72\binom{n}{2} + 944\binom{n}{3} - 384\binom{n}{4} - 768\binom{n}{5} + 1024\binom{n}{6}, \quad 1 \leqq n \leqq 50; *$$

$$(4.9)\quad E_{2^k n}^{(2^k)} \equiv 1 - 2n + 584\binom{n}{2} + 944\binom{n}{3} - 384\binom{n}{4} - 768\binom{n}{5} + 1024\binom{n}{6}, \quad k \geqq 3;$$

$$(4.10)\quad E_{7n}^{(7)} \equiv 7 + 248n + 880\binom{n}{2} + 112\binom{n}{3} - 704\binom{n}{4} + 320\binom{n}{5} - 384\binom{n}{6} - 896\binom{n}{7} + 1024\binom{n}{8}, \quad 1 \leqq n \leqq 50;$$

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(4.11) $$E_{14n}^{(14)} \equiv 7 - 8n + 384\binom{n}{2} + 512\binom{n}{3} + 1024\binom{n}{4}, \quad 1 \leqq n \leqq 50;$$

(4.12) $$E_{28n}^{(28)} \equiv 7 - 8n - 896\binom{n}{2} + 768\binom{n}{3} + 512\binom{n}{4} + 1024\binom{n}{5}, \quad 1 \leqq n \leqq 25;$$

(4.13) $$E_{15n}^{(15)} \equiv 15 - 16n - 32\binom{n}{2} + 992\binom{n}{3} - 384\binom{n}{4} - 384\binom{n}{5} - 256\binom{n}{6} + 768\binom{n}{7}, \quad 1 \leqq n \leqq 50;$$

(4.14) $$E_{31n}^{(31)} \equiv 31 - 32n + 960\binom{n}{2} + 960\binom{n}{3} - 768\binom{n}{4} - 768\binom{n}{5} - 512\binom{n}{6} - 512\binom{n}{7}, \quad 1 \leqq n \leqq 22;$$

(4.15) $$E_{63n}^{(63)} \equiv 63 - 64n - 128\binom{n}{2} - 128\binom{n}{3} + 512\binom{n}{4} + 512\binom{n}{5} + 1024\binom{n}{6} + 1024\binom{n}{7}, \quad 1 \leqq n \leqq 20.$$

We note that (4.10), (4.11) and (4.12) generalize parts (i), (iii) and (iv) of Conjecture 4.1 in [**7**].

In order to use the method of successive approximations we must determine precisely when $\binom{n}{m}$ is even and when it is odd, for fixed m and varying n, $n \geqq 0$. We make use of the following theorems.

THEOREM 4.1. ([**8**]). *If p is prime and $m = b_0 + b_1 p + \cdots + b_k p^k$, $n = a_0 + a_1 p + \cdots + a_k p^k$, then*

$$\binom{n}{m} = \binom{a_0}{b_0}\binom{a_1}{b_1} \cdots \binom{a_k}{b_k} \pmod{p}.$$

In particular, $\binom{n}{m}$ is odd if and only if $a_i \geqq b_i$, $i = 0, 1, \cdots, k$ in the base 2 representation of m and n.

THEOREM 4.2. ([**11**], [**10**]). *The length of the period modulo $t = p_1^{\alpha_1} \cdots p_l^{\alpha_l}$ of $\left\{\binom{n}{m}\right\}$ is given by*

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$$q_m = tp_1^{\beta_1} \cdots p_l^{\beta_l}$$

where each $\beta_i \quad (1 \leqq i \leqq l)$ *satisfies*

$$p_i^{\beta_i} \leqq m < p_i^{\beta_i + 1}.$$

In particular the length of the period modulo 2 *is* $2^{\beta+1}$ *where* $2^\beta \leqq m < 2^{\beta+1}$.

Suppose now that we have determined the pattern of odd and even blocks up to a power of two. The next theorem shows how to extend up to the following power of two.

THEOREM 4.3.

$$\binom{2^t + \alpha + \beta}{2^t + \alpha} \equiv \binom{\alpha + \beta}{\alpha} \pmod 2$$

for $0 \leqq \alpha \leqq 2^t - 1$, $0 \leqq \beta \leqq 2^t - \alpha$.

Proof. $\binom{2^t + \alpha + \beta}{2^t + \alpha} = \binom{2^t + \alpha + \beta}{\beta}$. Now, the period of $\binom{m}{\beta}$ is $2^{\gamma+1}$ where $2^\gamma \leqq \beta < 2^{\gamma+1}$. Suppose first that $2^{t-1} \leqq \beta < 2^t - \alpha$. Then the period (mod 2) of $\binom{m}{\beta}$ is 2^t. Hence

$$\binom{2^t + \alpha + \beta}{\beta} \equiv \binom{\alpha + \beta}{\beta} \pmod 2, \quad \text{which equals } \binom{\alpha + \beta}{\alpha}. \tag{4.16}$$

If, on the other hand, $0 \leqq \beta < 2^{t-1}$, then the period of $\binom{m}{\beta}$ is $< 2^t$, and a power of 2, so that (4.16) holds again.

We now use Theorem 4.3 as follows.

Case i. $\alpha = 2^{t-1} + \delta$, $0 \leqq \delta \leqq 2^{t-1} - 1$.

$$\binom{2^t + 2^{t-1} + \delta + \beta}{2^t + 2^{t-1} + \delta} \equiv \binom{2^{t-1} + \delta + \beta}{2^{t-1} + \delta},$$

$$0 \leqq \beta < 2^{t-1} - \delta \quad (\text{i.e., } \beta < \alpha).$$

That is, the remaining pattern for $\binom{n}{2^t + \alpha}$, after its $2^t + \alpha$ initial evens (zeros), is the same as the pattern for $\binom{n}{\alpha}$ remaining after its α initial evens (zeros).

Case ii. $\alpha = \delta$, $0 \leqq \delta \leqq 2^{t-1} - 1$.

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$$\binom{2^t + \alpha + \beta}{2^t + \alpha} = \binom{2^t + \delta + \beta}{2^t + \delta} \equiv \binom{\delta + \beta}{\delta}$$

(here, β can be large in terms of α). That is, the remaining pattern for $\binom{n}{2^t + \alpha}$, after its $2^t + \alpha$ initial evens (zeros), is a repeated version of the pattern for $\binom{n}{\alpha}$, including its α initial evens, where the repeating is backwards from 2^t, the symmetrizing discussed in the theorem.

The following table of patterns of even and odd blocks for $1 \leqq m \leqq 32$ shows how Theorem 4.3 is applied.

TABLE II

Patterns of even and odd blocks for $\binom{n}{m}$, fixed m, $(1 \leqq m \leqq 32)$, $n \geqq 0$.

m	even-odd block lengths	m	even-odd block lengths
1	1, 1	17	17, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1
2	2, 2	18	18, 2, 2, 2, 2, 2, 2, 2
3	3, 1	19	19, 1, 3, 1, 3, 1, 3, 1
4	4, 4	20	20, 4, 4, 4
5	5, 1, 1, 1	21	21, 1, 1, 1, 5, 1, 1, 1
6	6, 2	22	22, 2, 6, 2
7	7, 1	23	23, 1, 7, 1
8	8, 8	24	24, 8
9	9, 1, 1, 1, 1, 1, 1, 1	25	25, 1, 1, 1, 1, 1, 1, 1
10	10, 2, 2, 2	26	26, 2, 2, 2
11	11, 1, 3, 1	27	27, 1, 3, 1
12	12, 4	28	28, 4
13	13, 1, 1, 1	29	29, 1, 1, 1
14	14, 2	30	30, 2
15	15, 1	31	31, 1
16	16, 16	32	32, 32

For example, $\binom{n}{9}$ is even for n in the residue class 0, 1, 2, 3, 4, 5, 6, 7, 8, 10, 12, and 14 and odd for n in the residue class 9, 11, 13 and 15 (mod 16).

5. Undetermined coefficients. From Theorem 2.1 in [**7**], we have

$$(5.1) \qquad \sum_{s=0}^{n} \binom{tn}{ts} E_{ts}^{(t)} = \begin{cases} 1, & n = 0 \\ 0, & n > 0. \end{cases}$$

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TABLE III

Values of $E_{ts}^{(t)}$ and $a_{m,t}$, $0 \leqq s, m \leqq 5$

m, s	$E_{ts}^{(t)}$	$a_{m,t}$
0	1	1
1	-1	-2
2	$\binom{2t}{t}-1$	$\binom{2t}{t}+2$
3	$-\binom{3t}{t}\binom{2t}{t}+2\binom{3t}{t}-1$	$-\binom{3t}{t}\binom{2t}{t}+2\binom{3t}{t}-3\binom{2t}{t}-2$
4	$\binom{4t}{t}\binom{3t}{t}\binom{2t}{t}-2\binom{4t}{t}\binom{3t}{t}-\binom{4t}{2t}\binom{2t}{t}$ $+\binom{4t}{2t}+2\binom{4t}{t}-1$	$\binom{4t}{t}\binom{3t}{t}\binom{2t}{t}-2\binom{4t}{t}\binom{3t}{t}-\binom{4t}{2t}\binom{2t}{t}$ $+\binom{4t}{2t}+2\binom{4t}{t}+4\binom{3t}{t}\binom{2t}{t}-8\binom{3t}{t}+6\binom{2t}{t}+2$
5	$-\binom{5t}{t}\binom{4t}{t}\binom{3t}{t}\binom{2t}{t}+2\binom{5t}{t}\binom{4t}{t}\binom{3t}{t}$ $+\binom{5t}{t}\binom{4t}{2t}\binom{2t}{t}+\binom{5t}{2t}\binom{3t}{t}\binom{2t}{t}$ $-\binom{5t}{t}\binom{4t}{2t}-2\binom{5t}{t}\binom{4t}{t}-\binom{5t}{2t}\binom{2t}{t}$ $-2\binom{5t}{2t}\binom{3t}{t}+2\binom{5t}{t}+2\binom{5t}{2t}-1$	$-\binom{5t}{t}\binom{4t}{t}\binom{3t}{t}\binom{2t}{t}+2\binom{5t}{t}\binom{4t}{t}\binom{3t}{t}$ $+\binom{5t}{t}\binom{4t}{2t}\binom{2t}{t}+\binom{5t}{2t}\binom{3t}{t}\binom{2t}{t}-\binom{5t}{t}\binom{4t}{2t}$ $-2\binom{5t}{t}\binom{4t}{t}-\binom{5t}{2t}\binom{2t}{t}-2\binom{5t}{2t}\binom{3t}{t}+2\binom{5t}{t}$ $+2\binom{5t}{2t}-5\binom{4t}{t}\binom{3t}{t}\binom{2t}{t}+10\binom{4t}{t}\binom{3t}{t}$ $+5\binom{4t}{2t}\binom{2t}{t}-5\binom{4t}{2t}-10\binom{4t}{t}+20\binom{3t}{t}$ $-10\binom{3t}{t}\binom{2t}{t}-10\binom{2t}{t}-2$

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Suppose we write

$$(5.2)\qquad E_{tn}^{(t)} = a_{0,t}\binom{n}{0} + a_{1,t}\binom{n}{1} + \cdots + a_{n,t}\binom{n}{n} = \sum_{k=0}^{n} a_{k,t}\binom{n}{k}.$$

Then we have, by the usual inversion procedure,

$$\sum_{n=0}^{m} E_{tn}^{(t)}\binom{m}{n}(-1)^n = \sum_{n=0}^{m}\sum_{k=0}^{n}\binom{n}{k}a_{k,t}\binom{m}{n}(-1)^n$$

$$= \sum_{k=0}^{m} a_{k,t}\sum_{n=k}^{m}\binom{n}{k}\binom{m}{n}(-1)^n$$

$$= \sum_{k=0}^{m} a_{k,t}\binom{m}{k}(-1)^k\delta_{k,m} = (-1)^m a_{m,t}.$$

Therefore,

$$(5.3)\qquad a_{m,t} = (-1)^m \sum_{n=0}^{m} E_{tn}^{(t)}\binom{m}{n}(-1)^n.$$

We give in Table III the first six values of $E_{ts}^{(t)}$, that is, valid for all $t \geqq 2$ and $s = 0, 1, \cdots, 5$ and the first six values of $a_{m,t}$ determined respectively by (5.1) and (5.3).

We shall be using Table III for the case $t = 2^\alpha$. To this end, we require the residues modulo 2048 of each of the binomial coefficients occurring in Table III.

THEOREM 5.1. *For* $k = 3, 4, \cdots$

$$(5.4)\qquad \binom{2^{k+1}}{2^k} \equiv 582 \pmod{4096}.$$

Proof. The result is easily verified for $k = 3$. Assume (5.4) is true for k. Then we have

$$(5.5)\qquad \binom{2^{k+1}}{2^k} = 4096a_k + 582.$$

By a result of [**3**],

$$\binom{2^{k+2}}{2^{k+1}} - \binom{2^{k+1}}{2^k} = 2^{3k+3}H_k, \quad H_k \equiv 1 \pmod 2.$$

Hence we have

$$\binom{2^{k+2}}{2^{k+1}} = \binom{2^{k+1}}{2^k} + 2^{3k+3}H_k$$

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$$= 4096a_k + 582 + 2^{3k+3}H_k$$

$$= (4096)(a_k + 2^{3k-9}H_k) + 582.$$

The following generalization of Fjeldstad's result is a special case of a more general result of Jacobsthal [**4**].

LEMMA 5.1. *Let* $1 \leqq s \leqq q$, $k \geqq 1$ *be integers. Let* $m = 2^k q$, $n = 2^k s$ *and let* α, β, γ, δ *be respectively the largest power of* 2 *dividing* q, s, $q - s$, *and* $\binom{m}{n}$. *Then*

$$\binom{2m}{2n} - \binom{m}{n} = 2^{3k+1+\alpha+\beta+\gamma+\delta}H_k \quad (H_k \text{ an integer}).$$

In Table IV below, we give, for various values of positive integers s and q, $s \leqq q$, the power of 2 dividing $f(k + 1) - f(k)$ and the residue modulo 2048 of $f(k)$, where

$$f(k) = \binom{2^k q}{2^k s}.$$

TABLE IV

s	q	$f(k)$	$\alpha + \beta + \gamma$	δ	power of 2	residue mod 2048
1	2	$\binom{2^{k+1}}{2^k}$	1	1	$3k + 3$	582
1	3	$\binom{3 \cdot 2^k}{2^k}$	1	0	$3k + 2$	239
1	4	$\binom{4 \cdot 2^k}{2^k}$	2	2	$3k + 5$	1820
2	4	$\binom{4 \cdot 2^k}{2 \cdot 2^k}$	4	1	$3k + 6$	582
1	5	$\binom{5 \cdot 2^k}{2^k}$	2	0	$3k + 3$	237
2	5	$\binom{5 \cdot 2^k}{2 \cdot 2^k}$	1	1	$3k + 3$	1554
1	6	$\binom{6 \cdot 2^k}{2^k}$	1	1	$3k + 3$	898 ($k \geqq 3$)
2	6	$\binom{6 \cdot 2^k}{2 \cdot 2^k}$	4	0	$3k + 5$	239
3	6	$\binom{6 \cdot 2^k}{3 \cdot 2^k}$	1	2	$3k + 4$	1820 ($k \geqq 3$)

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The evaluations of δ depend upon the formula

$$\delta = \delta_{m,n} = s_2(n) + s_2(n - m) - s_2(m),$$

where $s_2(m)$ is the sum of the digits of m in base 2. The entries in the residue column are obtained as in the proof of Theorem 5.1. We note that the fourth and eighth entries could also be proven from the first and second by replacing k by $k + 1$.

From Tables III and IV we are now in a position to establish our congruences for $E_{sn}^{(n)}$ where $n = 2^k$, $k \geqq 3$. The terms $a_{m,t}$, $0 \leqq m \leqq 5$, are computed as follows;

$$a_{0,t} = 1;\ a_{1,t} = -2;$$

$$a_{2,t} \equiv 582 + 2 \equiv 584 \pmod{2048};$$

$$\begin{aligned} a_{3,t} &\equiv -239(582) + 2(239) - 3(582) - 2 \\ &\equiv 944 \pmod{2048}; \end{aligned}$$

$$\begin{aligned} a_{4,t} &\equiv 1820(239)(582) - 2(1820)(239) - (582)(582) + 582 \\ &\qquad + 2(1820) + 4(239)(582) - 8(239) + 6(582) + 2 \\ &\equiv 1664 \equiv -384 \pmod{2048}; \end{aligned}$$

$$\begin{aligned} a_{5,t} &\equiv -237(1820)(239)(582) + 2(237)(1820)(239) \\ &\qquad + 237(582)(582) + 1554(239)(582) - 237(582) \\ &\qquad - 2(237)(1820) - 1554(582) - 2(1554)(239) + 2(237) \\ &\qquad + 2(1554) - 5(1820)(239)(582) + 10(1820)(239) \\ &\qquad + 5(582)(582) - 5(582) - 10(1820) + 20(239) \\ &\qquad - 10(239)(582) - 10(582) - 2 \\ &\equiv -768 \pmod{2048}. \end{aligned}$$

The fact that $a_{6,t} \equiv 1024 \pmod{2048}$ would be established similarly. We also note that, for $k = 2$, we have

$$a_{2,t} = \binom{8}{4} + 2 = 72.$$

Since, for any choice of q and s ($s \leqq q$) in Lemma 5.1, we have $\alpha + \beta + \gamma \geqq 1$, we obtain the following lemma.

LEMMA 5.2. *Let* $1 \leqq s \leqq q$, $k \geqq 1$ *be integers. Then* 2^{3k+2} *is a divisor of*
$$\binom{2^{k+1}q}{2^{k+1}s} - \binom{2^kq}{2^ks}.$$

6. Proof of theorem 3.1. We prove parts (ii) and (iii) of Theorem 3.1.

Part (i) is proved in a similar manner to part (ii), although the proof is somewhat simpler. In order to prove the validity of (3.2), we require the following.

LEMMA 6.1. *For* $n = 2, 3, \cdots$ *we have the following identities*:

$$(6.1)\qquad \sum_{s=0}^{n} \binom{8n}{8s} = 2^{8n-3} + 2^{4n-2} + (-1)^n 2^{2n-1} \sum_{l=0}^{2n} 2^l \binom{4n}{2l},$$

$$(6.2)\qquad \sum_{s=0}^{n} \binom{8n-1}{8s} = 2^{8n-4} + 2^{4n-3} + (-1)^n 2^{2n-2} \sum_{l=0}^{2n} 2^l \binom{4n}{2l},$$

$$(6.3)\qquad \sum_{s=0}^{n} \binom{8n-2}{8s} = 2^{8n-5} + (-1)^n 2^{2n-2} \sum_{l=0}^{2n-1} 2^l \binom{4n-1}{2l},$$

$$(6.4)\qquad \sum_{s=0}^{n} \binom{8n-3}{8s} = 2^{8n-6} - 2^{4n-4} + (-1)^n 2^{2n-2} \sum_{l=0}^{2n-2} 2^l \binom{4n-2}{2l+1},$$

$$(6.5)\qquad \sum_{s=0}^{n} \binom{8n-4}{8s} = 2^{8n-7} - 2^{4n-4},$$

$$(6.6)\qquad \sum_{s=0}^{n} \binom{8n-5}{8s} = 2^{8n-8} - 2^{4n-5} - (-1)^n 2^{2n-3} \sum_{l=0}^{2n-2} 2^l \binom{4n-3}{2l},$$

$$(6.7)\qquad \sum_{s=0}^{n} \binom{8n-6}{8s} = 2^{8n-9} - (-1)^n 2^{2n-3} \sum_{l=0}^{2n-2} 2^l \binom{4n-3}{2l},$$

$$(6.8)\qquad \sum_{s=0}^{n} \binom{8n-7}{8s} = 2^{8n-10} + 2^{4n-6} - (-1)^n 2^{2n-3} \sum_{l=0}^{2n-2} 2^l \binom{4n-3}{2l+1}.$$

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Proof. Identity (6.1) is proved and (6.3) is stated in [7, Lemma 4.1.4]. Identity (6.2) can be proved using (6.1) and [7, Lemma 4.1.3(i)] after observing that

$$\sum_{s=0}^{n} \binom{8n-1}{8s} = \frac{1}{8n} \sum_{s=0}^{n} (8n - 8s) \binom{8n}{8s}.$$

Identity (6.5) follows by setting $l = 0$ and $j = 8$ in [7, Lemma 4.1.2(ii)] after replacing n by $8n - 4$. Identities (6.6) – (6.8) can be proved in a similar manner.

COROLLARY 6.1. *For* $n = 2, 3, \cdots$ *we have*

$$\sum_{s=0}^{n} \binom{8n}{8s} \equiv 0 \pmod{2^{2n-1}} \tag{6.9}$$

$$\sum_{s=0}^{n} \binom{8n-j}{8s} \equiv 0 \pmod{2^{2n-2}}, \quad j = 1, 2, 3 \tag{6.10}$$

$$\sum_{s=0}^{n} \binom{8n-4}{8s} \equiv 0 \pmod{2^{4n-4}} \tag{6.11}$$

$$\sum_{s=0}^{n} \binom{8n-j}{8s} \equiv 0 \pmod{2^{2n-3}}, \quad j = 5, 6, 7. \tag{6.12}$$

Proof. These are an immediate consequence of Lemma 5.1. It is also evident from (6.1) – (6.8) that the power of two on the right hand side of (6.9) – (6.12) is the largest possible.

LEMMA 6.2. *We have for positive integers n and k*

$$\sum_{s=0}^{n} s^4 \binom{kn}{ks} = nk^{-3}\Bigg\{(kn-1)(kn-2)(kn-3) \sum_{s=0}^{n} \binom{kn-4}{ks} + 6(kn-1)(kn-2) \sum_{s=0}^{n} \binom{kn-3}{ks} + 7(kn-1) \sum_{s=0}^{n} \binom{kn-2}{ks} + \frac{1}{2} \sum_{s=0}^{n} \binom{kn}{ks}\Bigg\} \tag{6.13}$$

$$\sum_{s=0}^{n} s^5 \binom{kn}{ks} = \frac{n^2}{4k^3} \Bigg\{(2k^3n^3 - 5k^2n^2 + 5) \sum_{s=0}^{n} \binom{kn}{ks} \tag{6.14}$$

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$$+ 10(7 - k^2n^2)(kn - 1) \sum_{s=0}^{n} \binom{kn-2}{ks}$$

$$+ 60(kn - 1)(kn - 2) \sum_{s=0}^{n} \binom{kn-3}{ks}$$

$$+ 10(kn - 1)(kn - 2)(kn - 3) \sum_{s=0}^{n} \binom{kn-4}{ks} \Bigg\}$$

$$(6.15) \quad \sum_{s=0}^{n} s^6 \binom{kn}{ks} = nk^{-5} \Bigg\{ (kn - 1) \cdots (kn - 5) \sum_{s=0}^{n} \binom{kn-6}{ks}$$

$$+ 15(kn - 1) \cdots (kn - 4) \sum_{s=0}^{n} \binom{kn-5}{ks}$$

$$+ 65(kn - 1)(kn - 2)(kn - 3) \sum_{s=0}^{n} \binom{kn-4}{ks}$$

$$+ 90(kn - 1)(kn - 2) \sum_{s=0}^{n} \binom{kn-3}{ks}$$

$$+ 31(kn - 1) \sum_{s=0}^{n} \binom{kn-2}{ks} + \frac{1}{2} \sum_{s=0}^{n} \binom{kn}{ks} \Bigg\}.$$

Proof. To obtain (6.13) we write

$$(6.16) \quad s^4 = sk^{-3}[(ks - 1)(ks - 2)(ks - 3) + 6(ks - 1)(ks - 2) + 7(ks - 1) + 1]$$

and use (6.16) in the left hand member of (6.13). Observing that for $j = 1, 2, \cdots, k - 1$

$$(6.17) \quad \sum_{s=0}^{n} (ks)(ks - 1) \cdots (ks - j + 1) \binom{kn}{ks} = (kn)(kn - 1) \cdots (kn - j + 1) \sum_{s=0}^{n} \binom{kn-j}{ks},$$

and using [7, Lemma 4.1.3], yields (6.13). A similar argument proves (6.14)

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and (6.15).

Proof of Theorem 3.1(ii). We now show that for $n = 1, 2, \cdots$,

$$E_{8n}^{(8)} \equiv 1 - 2n + 584\binom{n}{2} + 944\binom{n}{3} - 384\binom{n}{4} - 768\binom{n}{5} + 1024\binom{n}{6} \pmod{2048}. \tag{3.2}$$

From [7, Lemma 4.1.1] we need only show that

$$\sum_{s=0}^{n}\binom{8n}{8s} f(s) \equiv -1 + f(0) \pmod{2048}. \tag{6.18}$$

In this case

$$f(s) = \frac{1}{45}(45 - 9{,}342s + 15{,}916s^2 - 13{,}080s^3 + 7{,}600s^4 - 1{,}248s^5 + 64s^6) \tag{6.19}$$

so $f(0) = 1$, and (6.18) becomes

$$\sum_{s=0}^{n}\binom{8n}{8s} f(s) \equiv 0 \pmod{2048}. \tag{6.20}$$

Since the denominator in (6.19) is odd, the numerator of each term will contain all powers of two for the quotient. From (6.19) we have

$$\begin{aligned}\sum_{s=0}^{n}\binom{8n}{8s} f(s) &= \frac{1}{45}\sum_{s=0}^{n}\binom{8n}{8s}(45 - 9{,}342s + 15{,}916s^2 \\ &\qquad - 13{,}080s^3 + 7{,}600s^4 - 1{,}248s^5 + 64s^6) \\ &= \frac{1}{45}\sum_{l=0}^{6} T_l(n).\end{aligned} \tag{6.21}$$

We thus consider the expansion of

$$\sum_{s=0}^{n}\binom{8n}{8s} f(s)$$

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one term at a time. We show here in detail the case $l = 4$, with

$$T_4(n) = 7{,}600 \sum_{s=0}^{n} s^4 \binom{8n}{8s}.$$

Setting $k = 8$ in (6.13) we have

$$T_4(n) = \frac{7{,}600n}{8} \Bigg[(8n-1)(8n-2)(8n-3) \sum_{s=0}^{n} \binom{8n-4}{8s} + 6(8n-1)(8n-2) \sum_{s=0}^{n} \binom{8n-3}{8s} + 7(8n-1) \sum_{s=0}^{n} \binom{8n-2}{8s} + \frac{1}{2} \sum_{s=0}^{n} \binom{8n}{8s} \Bigg].$$

Thus,

(6.22) $\quad T_4(n) = 950n[a_1(n) + a_2(n) + a_3(n) + a_4(n)].$

Using (6.4) and (6.5) and observing that $2|(8n-1)(8n-2)$, we have $2^{4n-3}|a_1(n)$ and $2^{2n-1}|a_2(n)$. From (6.1) and (6.3) we get $2^{2n-2}|a_3(n)$ and $2^{2n-2}|a_2(n)$. Therefore, from (6.22) we have $2^{2n-1}|T_4(n)$. Similarly, we can show that $2^{2n-1}|T_0(n)$, $2^{2n-1}|T_1(n)$, $2^{2n-1}|T_2(n)$, $2^{2n-3}|T_3(n)$, $2^{2n-7}|T_5(n)$, $2^{2n-11}|T_6(n)$. Now, $2n - 11 \geqq 11$ when $n \geqq 11$, so using (6.21) we have shown (6.20) for $n \geqq 11$. A direct calculation confirms (6.20) for $1 \leqq n \leqq 10$. Therefore we have

$$\sum_{s=0}^{n} \binom{8n}{8s} f(s) \equiv 0 \pmod{2048}, \quad n \geqq 1$$

which proves (3.2) and hence part (ii) of Theorem 3.1.

Proof of Theorem 3.1(iii). We now show that, for $t \geqq 3$,

(3.3) $\quad E_{2^{t+1}n}^{(2^{t+1})} \equiv E_{2^t n}^{(2^t)} \pmod{2^{3t+2}}.$

Write

$$g_n(t) = E_{2^{t+1}n}^{(2^{t+1})} - E_{2^t n}^{(2^t)}.$$

Then we have

$$g_1(t) = E_{2^{t+1}}^{(2^{t+1})} - E_{2^t}^{(2^t)} = (-1) - (-1) = 0$$

$$g_2(t) = E_{2^{t+1}2}^{(2^{t+1})} - E_{2^t 2}^{(2^t)} = \left\{ \binom{2^{t+2}}{2^{t+1}} - 1 \right\} - \left\{ \binom{2^{t+1}}{2^t} - 1 \right\}$$

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$$= \binom{2^{t+2}}{2^{t+1}} - \binom{2^{t+1}}{2^t}$$
$$= 2^{3t+3} H_2(t)$$

by Fjeldstad's result ([**1**], p. 47 or [**3**]). Suppose now that $g_r(t) \equiv 0 \pmod{2^{3t+2}}$ for $r = 1, 2, \cdots, n-1$; $t \geqq 3$, or equivalently that

$$E_{2^{t+1}r}^{(2^{t+1})} = E_{2^t r}^{(2^t)} + 2^{3t+2} K_r, \quad r = 1, 2, \cdots, n-1; \ t \geqq 3.$$

Then by (5.1) we have (for $n \geqq 2$)

$$g_n(t) = -1 + \binom{2^{t+1}n}{2^{t+1}} - \sum_{r=2}^{n-1} \binom{2^{t+1}n}{2^{t+1}r} E_{2^{t+1}r}^{(2^{t+1})} - \left\{ -1 + \binom{2^t n}{2^t} - \sum_{r=2}^{n-1} \binom{2^t n}{2^t r} E_{2^t r}^{(2^t)} \right\}$$

$$= \binom{2^{t+1}n}{2^{t+1}} - \binom{2^t n}{2^t} - \sum_{r=2}^{n-1} \left\{ \binom{2^{t+1}n}{2^{t+1}r} E_{2^{t+1}r}^{(2^{t+1})} - \binom{2^t n}{2^t r} E_{2^t r}^{(2^t)} \right\}$$

$$= \binom{2^{t+1}n}{2^{t+1}} - \binom{2^t n}{2^t} - \sum_{r=2}^{n-1} \left\{ \binom{2^{t+1}n}{2^{t+1}r} \left(E_{2^t r}^{(2^t)} + 2^{3t+2} K_r \right) - \binom{2^t n}{2^t r} E_{2^t r}^{(2^t)} \right\}$$

$$= \binom{2^{t+1}n}{2^{t+1}} - \binom{2^t n}{2^t} - \sum_{r=2}^{n-1} E_{2^t r}^{(2^t)} \left\{ \binom{2^{t+1}n}{2^{t+1}r} - \binom{2^t n}{2^t r} \right\} - 2^{3t+2} \sum_{r=2}^{n-1} K_r \binom{2^{t+1}n}{2^{t+1}r}$$

$$= 2^{3t+2} K_n.$$

COROLLARY 6.2. *For all* $t \geqq 3$

$$E_{2^t n}^{(2^t)} \equiv 1 - 2n + 584\binom{n}{2} + 944\binom{n}{3} - 384\binom{n}{4} - 768\binom{n}{5} + 1024\binom{n}{6} \pmod{2048}. \tag{4.9}$$

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Proof. Since for $t \geqq 3$, $2^{3t+2} \geqq 2^{11} = 2048$, it suffices, from (3.3), to consider the case $t = 3$. But this is precisely (3.2) from Theorem 3.1.

Remark. For $t = 2$, the congruence is similar except that the coefficient of $\binom{n}{2}$ is 72 instead of 584 (see (4.8)). Also, Corollary 6.2 verifies Conjecture 4.1(vi) in [7] for $k = 3$ (the result for $k = 1$ and $k = 2$ is given in [**2**]).

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University of Victoria,
Victoria, British Columbia