Proceedings of the Edinburgh Mathematical Society (1984) **27**, 303–311 ©

ON THE UNIQUE SOLVABILITY OF A CLASS OF MODIFIED BOUNDARY INTEGRAL EQUATIONS

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(Received 2nd March 1984)

1. Introduction

In a recent paper the authors considered the transmission problem for the Helmholtz equation by using a reformulation of the problem in terms of a pair of coupled boundary integral equations with modified Green's functions as kernels. In this note we settle the question of the unique solvability of these modified boundary integral equations.

Modified Green's functions were introduced by Jones [3] to eliminate the problems of non uniqueness in boundary integral equation formulations of the exterior Dirichlet and Neumann problems for the Helmholtz equation. Specifically it is shown in [3] that if the free space Green's function is modified by the addition of a linear combination of radiating wave functions then the coefficients in the modification can be so chosen that the boundary integral equations, obtained by either a layer ansatz or the Green's Theorem approach, are uniquely solvable for all real values of the wave number. The arguments in [3] were refined by Ursell [14] and also by the authors [6], [7] where it was shown that the coefficients could be chosen to ensure that a number of different optimality conditions are satisfied. These ideas were extended to the Robin problem in [1].

The Transmission problem may also be cast in terms of boundary integral equations using either a layer ansatz [2], [12] or the Green's Theorem method [4]. Although in neither of these particular approaches does the problem of non uniqueness present itself [2], [12], nevertheless modifying the Green's function could still be of advantage for numerical purposes.

The idea of choosing a modification of the Green's function which minimized the norm of an integral operator was considered in [7] and in [5] it was shown, for a particular example, that the procedure produced well conditioned operators. The extension of this idea to the Transmission problem was reported in [8] and extended and more general versions appear in [9] and [10]. We shall show here that, subject to only mild restrictions on the coefficients appearing in the modification of the Green's function, the modified boundary integral equations are uniquely solvable.

2. Notation and statement of the problem

Let $B_i \subset \mathbb{R}^n$, $n=2$ or 3, be a bounded domain with a smooth closed boundary ∂B and let B_e be the simply connected exterior of ∂B. Arbitrary points in $\mathbb{R}^n$ are denoted by p and q whilst the unit normal to ∂B at p, written $\hat{n}_p$, is assumed to be directed into B_e.

With respect to a Euclidean coordinate system with origin in B_i the distance of any point p from the origin will be written $r_p := |p|$. We shall use ∂_{n_p} and ∂_{n_q} to denote derivatives at p and q in the direction of the unit normals $\hat{n}_p$ and $\hat{n}_q$ respectively.

The Transmission problem for the Helmholtz equation can be stated in the following manner. Determine functions ϕ_e and ϕ_i defined in B_e and B_i respectively which satisfy

$$\begin{aligned}(\Delta+k_e^2)\phi_e=0 \quad \text{in} \quad B_e \\ (\Delta+k_i^2)\phi_i=0 \quad \text{in} \quad B_i\end{aligned} \tag{2.1}$$

$$\left\{\frac{\partial}{\partial r_p}-ik_e\right\}\phi_= = \begin{cases} o(r_p^{-1/2}) \text{ in } \mathbb{R}^2 \\ o(r_p^{-1}) \text{ in } \mathbb{R}^3 \end{cases} \tag{2.2}$$

$$\partial_n\phi_e=\partial_n\phi_i+g \quad \text{on} \quad \partial B \tag{2.3}$$

$$\mu_e\phi_e=\mu_i\phi_i+f \quad \text{on} \quad \partial B \tag{2.4}$$

where μ_e, μ_i are known constants and f, g are arbitrary continuous functions.

A uniqueness theorem for this problem is known for the case when μ_e and μ_i are constants; see [**2**] for the case when k_e, k_i, μ_e, μ_i are real and [**12**] when they are complex.

In order to reformulate this problem in terms of boundary integral equations we define the free space Green's functions

$$\gamma_o^e(p,q) := \begin{cases} -\dfrac{i}{2}H_o^{(1)}(k_e|p-q|) & \text{in} \quad \mathbb{R}^2 \\ -\dfrac{e^{ik_e|p-q|}}{2\pi|p-q|} & \text{in} \quad \mathbb{R}^3 \end{cases} \tag{2.5}$$

$$\gamma_o^i(p,q) := \begin{cases} -\dfrac{i}{2}H_o^{(1)}(k_i|p-q|) & \text{in} \quad \mathbb{R}^2 \\ -\dfrac{e^{ik_i|p-q|}}{2\pi|p-q|} & \text{in} \quad \mathbb{R}^3 \end{cases} \tag{2.6}$$

We shall denote by $\{V^e(p)\}$ and $\{V^i(p)\}$ families of radiating solutions of $(\Delta+k_e^2)V=0$ and $(\Delta+k_i^2)V=0$ respectively in B_e and V_l^e satisfies the radiation condition (2.2) whilst V_l^i satisfies (2.2) with k_e replaced by k_i. We also define families of regular solutions $\{U_l^e(p)\}$ and $\{U_l^i(p)\}$ of $(\Delta+k_e^2)U=0$ and $(\Delta+k_i^2)U=0$ respectively in B_i. Much of the succeeding analysis may be carried out without specifying further these various families. However considerable simplification can be made by taking V_l^e and V_l^i to be radiating spherical or cylindrical wave functions with wave number k_e and k_i respectively and U_l^e and U_l^i to be regular spherical or cylindrical wave functions associated with k_e and k_i respectively. Furthermore we assume a normalisation of the functions so that the unimodified Green's functions have the expansions

$$\gamma_o^{e,i}(p,q)=\sum_{|l|=0}^{\infty} V_l^{e,i}(p>)U_l^{e,i}(p<) \tag{2.7}$$

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where

$$p> = \begin{cases} p & \text{if} \quad r_p > r_q \\ q & \text{if} \quad r_p < r_q \end{cases} \qquad p< = \begin{cases} p & \text{if} \quad r_p < r_q \\ q & \text{if} \quad r_p > r_q \end{cases}.$$

Explicitly, if we take l to be the multi-index

$$l = \begin{cases} (n,j), \ n \geqq 0, \ 0 \leqq j \leqq 1 & \text{in} \quad \mathbb{R}^2 \\ (n,m,j), \ n \geqq 0, \ 0 \leqq m \leqq n, \ 0 \leqq j \leqq 1 & \text{in} \quad \mathbb{R}^3 \end{cases}$$

with

$$|l| = \begin{cases} n+j & \text{in} \quad \mathbb{R}^2 \\ n+m+j & \text{in} \quad \mathbb{R}^3 \end{cases}$$

then

$$V_l^{e,i} = \left(-\frac{i\varepsilon_n}{2} \right)^{1/2} H_n^{(1)}(k_{e,i} r)(j \sin n\phi + (1-j)\cos n\phi) \quad \text{in} \quad \mathbb{R}^2$$

$$= \left(-\frac{ik_{e,i}}{2\pi} \varepsilon_m (2n+1) \frac{(n-m)!}{(n+m)!} \right)^{1/2} h_n^{(1)}(k_{e,i} r) P_n^m(\cos\theta)$$

$$\times (j \sin m\phi + (1-j)\cos m\phi) \quad \text{in} \quad \mathbb{R}^3$$

$$U_l^{e,i} = \left(-\frac{i\varepsilon_n}{2} \right)^{1/2} J_n(k_{e,i} r)(j \sin n\phi + (1-j)\cos n\phi) \quad \text{in} \quad \mathbb{R}^2$$

$$= \left(-\frac{ik_{e,i}}{2\pi} \varepsilon_m (2n+1) \frac{(n-m)!}{(n+m)!} \right)^{1/2} j_n(k_{e,i} r) P_n^m(\cos\theta)$$

$$\times (j \sin m\phi + (1-j)\cos m) \quad \text{in} \quad \mathbb{R}^3$$

where $\varepsilon_o = 1$ and $\varepsilon_m = 2$, $m > 0$ and P_n^m are associated Legendre functions, J_n and $H_n^{(1)}$ are Bessel and Hankel functions respectively and j_n and $h_n^{(1)}$ are spherical Bessel and Hankel functions respectively.

We now define modified Green's functions

$$\gamma_N^e(p,q) := \gamma_o^e(p,q) + \sum_{|l|=0}^{N-1} \alpha_l^e V_l^e(p) V_l^e(q) \tag{2.8}$$

and

$$\gamma_N^i(p,q) := \gamma_o^i(p,q) + \sum_{|l|=0}^{N-1} \alpha_l^i U_l^i(p) U_l^i(q) \tag{2.9}$$

where the summation is absent when $N = 0$.

In terms of these modified Green's functions we define the single and double layers

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$S_N^{e,i}$ and $D_N^{e,i}$ by

$$(S_N^{e,i}\phi)(p)=\int_{\partial B}\gamma_N^{e,i}(p,q)\phi(q)\,ds_q \tag{2.10}$$

$$(D_N^{e,i}\phi)(p):=\int_{\partial B}\partial_{n_q}\gamma_N^{e,i}(p,q)\phi(q)\,ds_q. \tag{2.11}$$

We remark that S_N^i and D_N^i are defined everywhere in $\mathbb{R}^n$, $n=2,3$, whereas S_N^e and D_N^e are not. This is because there are no non-trivial radiating solutions of the Helmholtz equation defined throughout $\mathbb{R}^n$, $n=2,3$, thus V_i^e, hence S_N^e and D_N^e must be singular in B_i.

We also define the boundary integral operators $K_N^{e,i}$ by

$$(K_N^{e,i}\phi)(p):=\int_{\partial B}\partial_{n_p}\gamma_N^{e,i}(p,q)\phi(q)\,ds_q,\quad p\in\partial B. \tag{2.12}$$

In terms of these integral operators the layers satisfy, on ∂B, the following jump conditions.

As p approaches ∂B from B_i

$$\partial_{n_p}S_N^{e,i}\phi=(-I+K_N^{e,i})\phi \tag{2.13}$$

$$D_N^{e,i}\phi=(I+\overline{K_N^{e,i*}})\phi. \tag{2.14}$$

As p approaches ∂B from B_e

$$\partial_{n_p}S_N^{e,i}\phi=(I+K_N^{e,i})\phi \tag{2.15}$$

$$D_N^{e,i}\phi=(-I+\overline{K_N^{e,i*}})\phi. \tag{2.16}$$

To reduce the Transmission problem (2.1)–(2.4) to a boundary integral equation problem we use the layer approach and assume

$$\begin{aligned}\phi_e&=D_N^e\phi_1+S_N^e\mu_e\phi_2 \quad \text{in} \quad B_e\\ \phi_i&=D_N^i\phi_1+S_N^i\mu_i\phi_2 \quad \text{in} \quad B_i\end{aligned} \tag{2.17}$$

where ϕ_1,ϕ_2 are unknown continuous functions defined on ∂B.

With this ansatz using the jump conditions (2.13) to (2.16) and the transmission conditions (2.3) and (2.4) we obtain [9], [10] the following matrix boundary integral equation

$$(I-B_N)\Phi=F \tag{2.18}$$

where

$$\Phi := \begin{bmatrix} \phi_1 \\ \phi_2 \end{bmatrix} \quad \text{and} \quad F := \frac{1}{\mu_e + \mu_i} \begin{bmatrix} -f \\ g \end{bmatrix} \tag{2.19}$$

with

$$B_N := \frac{1}{\mu_e + \mu_i} \begin{bmatrix} \mu_e \overline{K_N^{e*}} - \mu_i \overline{K_N^{i*}} & \mu_e S_N^e \mu_e - \mu_i S_N^i \mu_i \\ \partial_n D_N^i - \partial_n D_N^e & K_N^i \mu_i - K_N^e \mu_e \end{bmatrix}. \tag{2.20}$$

Because the difference $(\partial_n D_N^i - \partial_n D_N^e)$ is at most weakly singular, as indeed are all the other operators in B_N, it follows that B_N is a compact operator on either $L_2(\partial B) \oplus L_2(\partial B)$ or $C(\partial B) \oplus C(\partial B)$. Therefore (2.18) is uniquely solvable, for any F, provided there are no nontrivial solutions of

$$(I - B_N)\Phi = 0. \tag{2.21}$$

3. A uniqueness theorem

In this section we establish a uniqueness theorem for the boundary integral equations (2.18). The proof combines features of uniqueness theorem proofs for the integral equation formulation of boundary value problems using modified Green's functions **[6]** on the one hand and those for transmission problems using unmodified Green's functions on the other **[2]**. To be specific we prove

Theorem 3.1. *If*

(i) μ_e, μ_i, k_e, k_i *are real positive constants*

(ii) $1 - |2\alpha_l^e + 1|^2 > 0$ *and* $\operatorname{Re} \alpha_l^i > -1 \; \forall l$
or
$1 - |2\alpha_l^e + 1|^2 < 0$ *and* $\operatorname{Re} \alpha_l^i < -1 \; \forall l$

(iii) $-(\mu_e + \mu_i)\phi_1 + (\mu_e \overline{K_N^{e*}} - \mu_i \overline{K_N^{i*}})\phi_1 + (\mu_e S_N^e \mu_e - \mu_i S_N^i \mu_i)\phi_2 = 0$
$(\mu_e + \mu_i)\phi_2 + (\partial_n D_N^e - \partial_n D_N^i)\phi_1 + (K_N^e \mu_e - K_N^i \mu_i)\phi_2 = 0$
holding on ∂B

then

$$\phi_1 = \phi_2 = 0 \quad on \quad \partial B.$$

Proof. Assume that ϕ_1, ϕ_2 are solutions of (iii) and define

$$\psi_+^e := D_N^e \phi_1 + S_N^e \mu_e \phi_2 \quad \text{in} \quad B_e. \tag{3.1}$$

$$\psi_-^e := D_N^e \phi_1 + S_N^e \mu_e \phi_2 \quad \text{in} \quad B_i \backslash \{0\} \tag{3.2}$$

$$\psi_+^i := D_N^i \phi_1 + S_N^i \mu_i \phi_2 \quad \text{in} \quad B_e \tag{3.3}$$

$$\psi^i_- := D^i_N\phi_1 + S^i_N\mu_i\phi_2 \quad \text{in} \quad B_i. \tag{3.4}$$

The jump conditions (2.13)–(2.16) together with (iii) indicate that ψ^e_+ and ψ^i_- solve the homogeneous transmission problem

$$\begin{aligned}(\Delta + k_e^2)\psi^e_+ &= 0 \quad \text{in} \quad B_e \\ (\Delta + k_i^2)\psi^i_- &= 0 \quad \text{in} \quad B_i\end{aligned} \tag{3.5}$$

$$\mu_e\psi^e_+ = \mu_i\psi^i_- \quad \text{and} \quad \partial_n\psi^e_+ = \partial_n\psi^i_- \quad \text{on} \quad \partial B$$

ψ^e_+ satisfies the radiation condition (2.2).

Since there are no nontrivial solutions of the homogeneous Transmission problem provided (i) is satisfied, [**12**], it follows that $\psi^e_+ \equiv 0$ in B_e and $\psi^i_- = 0$ in B_i.

Again, using the jump conditions we find that on ∂B

$$\psi^e_+ = (-I + \overline{K^{e*}_N})\phi_1 + S^e_N\mu_e\phi_2 = 0 \tag{3.6}$$

$$\frac{\partial\psi^e_+}{\partial n} = \partial_n D^e_N\phi_1 + \mu_e(I + K^e_N)\phi_2 = 0 \tag{3.7}$$

$$\psi^i_- = (I + \overline{K^{i*}_N})\phi_1 + S^i_N\mu_i\phi_2 = 0 \tag{3.8}$$

$$\frac{\partial\psi^i_-}{\partial n} = \partial_n D^i_N\phi_1 + \mu_i(-I + K^i_N)\phi_2 = 0. \tag{3.9}$$

Applying the jump conditions to ψ^e_- and ψ^i_+ and using (3.6) to (3.8) we obtain

$$\psi^e_- = \psi^e_+ + 2\phi_1 = 2\phi_1 \tag{3.10}$$

$$\psi^i_+ = \psi^i_- - 2\phi_1 = -2\phi_1 \tag{3.11}$$

$$\frac{\partial\psi^e_-}{\partial n} = \frac{\partial\psi^e_+}{\partial n} - 2\mu_e\phi_2 = -2\mu_e\phi_2 \tag{3.12}$$

$$\frac{\partial\psi^i_+}{\partial n} = \frac{\partial\psi^i_-}{\partial n} + 2\mu_i\phi_2 = 2\mu_i\phi_2. \tag{3.13}$$

From these equations we infer

$$\psi^e_- + \psi^i_+ = 0 \quad \text{on} \quad \partial B \tag{3.14}$$

and

$$\mu_i\frac{\partial\psi^e_-}{\partial n} + \mu_e\frac{\partial\psi^i_+}{\partial n} = 0 \quad \text{on} \quad \partial B. \tag{3.15}$$

Using (3.14) it follows that

$$\int_{\partial B} \{(\psi^e_- + \psi^i_+)\overline{\partial_n \psi^e_-} - \partial_n \psi^e_-(\overline{\psi^e_- + \psi^i_+})\} \, ds = 0 \tag{3.16}$$

which on using (3.15) reduces to

$$\int_{\partial B} \{(\psi^e_- \overline{\partial_n \psi^e_-} - \overline{\psi^e_-} \partial_n \psi^e_-) - \frac{\mu_e}{\mu_i}(\psi^i_+ \overline{\partial_n \psi^i_+} - \overline{\psi^i_+} \partial_n \psi^i_+)\} \, ds = 0. \tag{3.17}$$

Now choose B_a and B_A to be balls with centres in B_i and radius a and A respectively such that $\bar{B}_a \subset B_i \subset \bar{B}_i \subset B_A$. Then Green's Theorem applies to the region contained between B_a and B_A yields, on using (3.17)

$$\int_{\partial B_a} (\psi^e_- \overline{\partial_n \psi^e_-} - \overline{\psi^e_-} \partial_n \psi^e_-) \, ds - \frac{\mu_e}{\mu_i} \int_{\partial B_A} (\psi^i_+ \overline{\partial_n \psi^i_+} - \overline{\psi^i_+} \, \partial_n \psi^i_+) \, ds = 0 \tag{3.18}$$

where

$$\partial_n = \left.\frac{\partial}{\partial r}\right|_{r=a} \quad \text{on} \quad \partial B_a \quad \text{and} \quad \partial_n = \left.\frac{\partial}{\partial r}\right|_{r=A} \quad \text{on} \quad \partial B_A.$$

Recalling the definition of the modified Green's functions (2.8),(2.9) we may write ψ^e_- and ψ^i_+ in the form

$$\psi^e_- = D^e_o \phi_1 + S^e_o \mu_e \phi_2 + \sum_{|l|=0}^{N-1} \alpha^e_l V^e_l C^e_l \tag{3.19}$$

$$\psi^i_+ = D^i_o \phi_1 + S^i_o \mu_i \phi_2 + \sum_{|l|=0}^{N-1} \alpha^i_l U^i_l C^i_l \tag{3.20}$$

where

$$C^e_l := \int_{\partial B} (\partial_n V^e_l \phi_1 + V^e_l \mu_e \phi_2) ds$$

$$C^i_l := \int_{\partial B} (\partial_n U^i \phi_1 + U^i_l \mu_i \phi_2) ds.$$

Introducing the expansion (2.7) of the unmodified Green function (3.19) and (3.20) become

$$\psi^e_- = \sum_{|l|=0}^{\infty} C^e_l [U^e_l + \alpha^e_l V^e_l] \quad 0 < r_p \leqq a \tag{3.21}$$

$$\psi^i_+ = \sum_{|l|=0}^{\infty} C^i_l [V^i_l + \alpha^i_l U^i_l] \quad r_p \geqq A \tag{3.22}$$

where we set $\alpha^e_l = \alpha^i_l = 0$ for $|l| \geqq N$.

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Using these expressions for ψ^e_- and ψ^i_+ together with the orthogonality relations for spherical wave functions the two parts of (3.18) can be written after a straightforward but lengthy calculation, in the form

$$\int_{\partial B} (\psi^e_- \partial_n \overline{\psi^e_-} - \overline{\psi^e_-} \partial_n \psi^e_-) ds = -i \sum_{|l|=0}^{N-1} |C^e_l|^2 \{1 - |2\alpha^e_l + 1|^2\} \tag{3.23}$$

$$\int_{\partial B_A} (\psi^i_+ \overline{\partial_n \psi^i_+} - \overline{\psi^i_+} \partial_n \psi^i_+) ds = -4i \sum_{|l|=0}^{\infty} |C^i_l|^2 (1 + \operatorname{Re} \alpha^i_l). \tag{3.24}$$

Substituting (3.23) and (3.24) into (3.18) yields

$$i \sum_{|l|=0}^{N-1} |C^e_l|^2 (1 - |2\alpha^e_l + 1|^2) + \frac{\mu_e}{\mu_i} 4i \sum_{|l|=0}^{\infty} |C^i_l|^2 (1 + \operatorname{Re} \alpha^i_l) = 0. \tag{3.25}$$

Thus the restrictions (i) and (ii) imply that

$$\begin{aligned} C^i_l &= 0 \quad \text{for all } l \\ C^e_l &= 0 \quad \text{for} \quad 0 \leq |l| \leq N-1. \end{aligned} \tag{3.26}$$

However (3.26) together with (3.19) and (3.22) indicate that

$$\psi^e_- = D^e_o \phi_1 + S^e_o \mu_e \phi_2 \quad \text{in} \quad B_e \tag{3.27}$$

$$\psi^i_+ = 0 \quad \text{for} \quad r > A. \tag{3.28}$$

By analytic continuation we then see that ψ^i_+ vanishes identically in B_e and, in particular,

$$\psi^i_+ = \partial_n \psi^i_+ = 0 \quad \text{on} \quad \partial B.$$

This result with conditions (3.10)–(3.13) then guarantees that

$$\phi_1 = \phi_2 = 0$$

which completes the proof of the theorem.

Acknowledgement. This work was supported by NATO Research Grant No. 0351/82 and NSF Grant MCS-82-02-033.

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