ON NORMED ALGEBRAS WHICH SATISFY A REALITY CONDITION

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1. Introduction. Various results exist which permit real Banach algebras satisfying some sort of "reality condition" to be identified with the algebra of all continuous real-valued functions on a suitable compact space (or with the algebra of all continuous real-valued functions that "vanish at infinity" on a suitable non-compact, locally compact space in case the algebra has no unit). (Terminology follows **(4)**, so that compact and locally compact spaces must be Hausdorff spaces, in addition to satisfying the usual requirements.) Kadison established such a result in **(6**, Theorem 6.6**)** for any Banach algebra with unit, if the algebra satisfies an appropriate reality condition and has a norm N such that $N(x)^2 = N(x^2)$ for all x. Another such theorem was obtained by Segal in **(8**, Theorem 1**)**, for commutative Banach algebras with unit, if the norm N satisfies the conditions $N(u^2) = N(u)^2$ and $N(u^2 - v^2) \leqslant \max(N(u^2), N(v^2))$ for all u and v; here the second condition imposed upon the norm serves as a reality condition.

In this note, we shall consider commutative, connected, complete normed algebras, over a non-discrete field with absolute value, such that the norm of the algebra *satisfies a polynomial identity* on the algebra in the sense of **(3)**. The algebras are also subjected to a fairly strong reality condition: *it is assumed that whenever I is a closed two-sided ideal such that xy in I implies x in I or y in I, then $x^2 + y^2$ in I implies that x and y are in I.* With these restrictions, it is shown in Theorem 2 that such an algebra is algebraically and topologically isomorphic to the ring of all continuous real-valued functions on a suitable compact space if the algebra has a unit, and to the ring of all continuous real-valued functions which "vanish at infinity" on a suitable non-compact, locally compact space if the algebra is without unit.

The method of proof used in this note is similar to that in **(3)**.

2. Preliminaries. Familiarity with the contents of **(2)** and of **(3)** is presupposed in this note.

If ρ is a real number such that $0 < \rho \leqslant 1$, then the field of all real numbers, normed with the ρth power of the ordinary absolute value, will be denoted by the symbol $\mathfrak{R}^{(\rho)}$. Clearly, $\mathfrak{R}^{(\rho)}$ is a connected field with absolute value and is complete.

Our definitions of compact and locally compact spaces will agree with those in **(4**, chapter I**)**; that is, a *compact* space is a Hausdorff space for which

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every open covering has a finite subcovering, and a *locally compact* space is a Hausdorff space such that each point is interior to at least one compact subspace.

If Φ is a compact space and ρ is a real number with $0 < \rho \leqslant 1$, the symbol $C(\Phi; \mathfrak{R}; \rho)$ will denote the set of all continuous real-valued functions $x(\phi)$ on Φ, with algebraic operations defined in the obvious way, and with the norm N such that $N(x) = \sup\{|x(\phi)|^\rho | \phi \in \Phi\}$ for all x in $C(\Phi; \mathfrak{R}; \rho)$. In case Φ is a locally compact space which is not compact, then $C(\Phi; \mathfrak{R}; \rho)$ will be the set of all continuous real-valued functions on Φ which "vanish at infinity," if ρ is a real number with $0 < \rho \leqslant 1$; the definitions for the algebraic operations and for the norm remain the same as before. Then, whenever Φ is a locally compact space and ρ is a real number with $0 < \rho \leqslant 1$, the system $C(\Phi; \mathfrak{R}; \rho)$ is defined and is a connected, commutative, complete normed algebra over $\mathfrak{R}^{(\rho)}$. The norm defined on $C(\Phi; \mathfrak{R}; \rho)$ is simply the ρth power of the usual supremum norm, and it follows that the topology in $C(\Phi; \mathfrak{R}; \rho)$ is the topology of uniform convergence on Φ. It may also be noted that the norm of $C(\Phi; \mathfrak{R}; \rho)$ is *power multiplicative.* (See **(2)** for a definition of that concept.)

If Φ is a locally compact space, it is possible to construct a compact space Ψ which contains every point of Φ and precisely one point not in Φ. In case Φ is locally compact but not compact, this is achieved by the standard "one point compactification" (see **(4**, chapter I, § 10, Theorem 3**)**, for instance), in which a "point at infinity" ω is added to Φ to produce the compact space Ψ. The case of a compact space Φ is handled by adjoining an isolated point ω to Φ in order to form Ψ. In either case, a compact space Ψ is obtained by the addition of a single point ω to Φ.

Conversely, if a point ω is selected in a compact space Ψ, then the complement, Φ, of ω in Ψ is open and hence locally compact. (See **(4**, chapter I, § 10, Proposition 10**)**.) Thus, there exists a one-to-one correspondence between locally compact spaces Φ, and pairs consisting of a compact space Ψ and a selected point ω in Ψ. (The correspondence is actually between equivalence classes, where the equivalence relation is homeomorphism.) This correspondence is related in an interesting way to the algebras of continuous functions which were introduced above. Let ρ be a real number with $0 < \rho \leqslant 1$, let Ψ be a compact space with a selected point ω in Ψ, and let $\hat{A}$ be the subalgebra of $C(\Psi; \mathfrak{R}; \rho)$ consisting of those functions in $C(\Psi; \mathfrak{R}; \rho)$ which vanish at ω. If Φ is the complement of ω in Ψ, and if τ is the mapping which carries every function in $\hat{A}$ into its restriction to Φ, then it is easily verified that τ is a norm-preserving isomorphism of $\hat{A}$ onto $C(\Phi; \mathfrak{R}; \rho)$, as algebras over $\mathfrak{R}^{(\rho)}$.

The principal result of this section will now be proved; this result will show that if a complete normed algebra over an $\mathfrak{R}^{(\rho)}$ has sufficiently many homomorphisms into $\mathfrak{R}^{(\rho)}$, as algebras over $\mathfrak{R}^{(\rho)}$, then the algebra is a $C(\Phi; \mathfrak{R}; \rho)$ for a suitable locally compact space Φ. This theorem will be used in the next section in proving the principal theorem.

THEOREM 1. *Let A be a non-zero, complete normed algebra over $\mathfrak{R}^{(\rho)}$, for a fixed real number ρ such that $0 < \rho \leqslant 1$. Suppose that for every non-zero c in A there is a homomorphism ϕ of A into $\mathfrak{R}^{(\rho)}$, as algebras over $\mathfrak{R}^{(\rho)}$, such that $||\phi(x)|| \leqslant N(x)$ for all x in A, and $||\phi(c)|| = N(c)$, where N is the norm of A. Then there exists a locally compact space Φ and a norm-preserving isomorphism σ of A onto $C(\Phi; \mathfrak{R}; \rho)$, as algebras over $\mathfrak{R}^{(\rho)}$, such that Φ is compact if and only if A has a unit element.*

Proof. Let Ψ be the set of all homomorphisms ϕ of A into $\mathfrak{R}^{(\rho)}$, as algebras over $\mathfrak{R}^{(\rho)}$, such that $||\phi(x)|| \leqslant N(x)$ for all x in A. The zero homomorphism of A into $\mathfrak{R}^{(\rho)}$ will be denoted by ω, and clearly ω belongs to Ψ; the complement of ω in Ψ will be written Φ.

If ϕ_0 is in Ψ, if $x_1, \ldots, x_n$ are elements of A, and if ϵ is a positive number, then the symbol $\{\phi_0; x_1, \ldots, x_n; \epsilon\}$ will represent the set of all ϕ in Ψ such that $|\phi(x_i) - \phi_0(x_i)| < \epsilon$ for $i = 1, \ldots, n$. Let $\mathscr{U}$ be the family of all sets $\{\phi_0; x_1, \ldots, x_n; \epsilon\}$ as ϕ_0 ranges over the elements of Ψ, as $\{x_1, \ldots, x_n\}$ ranges over all finite subsets of A, and as ϵ ranges over all positive numbers. Then if $\mathscr{U}$ is used as a base for open sets in Ψ we obtain a topological space Ψ.

Now, let $\mathscr{E}_x$ be the closed interval $\{t| \, |t|^\rho \leqslant N(x)\}$, for each non-zero x in A. If $\mathscr{E}$ is the cartesian product of the $\mathscr{E}_x$ as x ranges over all non-zero elements of A, then $\mathscr{E}$ is compact since each $\mathscr{E}_x$ is compact. But Ψ may be identified with a subset of $\mathscr{E}$ in an obvious way, by letting the $\mathscr{E}_x$-co-ordinate of ϕ be $\phi(x)$, if ϕ is in Ψ. Then the topology which was previously defined in Ψ is clearly the relative topology of Ψ as a subset of $\mathscr{E}$. It is easily shown that Ψ is a closed set in $\mathscr{E}$, so that Ψ is compact.

For x in A, define $\hat{x}(\phi) = \phi(x)$ for all ϕ in Ψ, so that $\hat{x}$ is a real-valued function defined on Ψ. Because of the nature of the topology which was introduced in Ψ, each $\hat{x}$ must be continuous on Ψ. The mapping $x \rightarrow \hat{x}$ then becomes a homomorphism of A into $C(\Psi; \mathfrak{R}; \rho)$, as algebras over $\mathfrak{R}^{(\rho)}$.

If c is a non-zero element of A, then $N(c) \geqslant ||\phi(c)|| = |\phi(c)|^\rho = |\hat{c}(\phi)|^\rho$ for all ϕ in Ψ, and it follows that $N(c) \geqslant \sup\{|\hat{c}(\phi)|^\rho \, |\phi \in \Psi\}$. On the other hand, the hypothesis of this theorem asserts the existence of a ϕ in Ψ such that $N(c) = ||\phi(c)|| = |\phi(c)|^\rho = |\hat{c}(\phi)|^\rho$, so that $N(c) = \sup\{|\hat{c}(\phi)|^\rho \, |\phi \in \Psi\}$. Thus, the mapping $x \rightarrow \hat{x}$ is norm-preserving, and is consequently an isomorphism since its kernel would have to be zero.

Let $\hat{A}$ be the image in $C(\Psi; \mathfrak{R}; \rho)$ of the mapping $x \rightarrow \hat{x}$, so that $\hat{A}$ is a subalgebra of $C(\Psi; \mathfrak{R}; \rho)$. Since A is complete and is isometric to $\hat{A}$, it follows that $\hat{A}$ is also complete, and $\hat{A}$ must therefore be uniformly closed in $C(\Psi; \mathfrak{R}; \rho)$. Also, each element of $\hat{A}$ clearly vanishes at ω. Finally, if ϕ_1 and ϕ_2 are distinct elements of Ψ, then $\phi_1(x) \neq \phi_2(x)$ for some x in A, whence $\hat{x}(\phi_1) \neq \hat{x}(\phi_2)$ for such an x. That is, every pair of distinct points of Ψ may be distinguished by an element of $\hat{A}$; in the language of **(9)**, $\hat{A}$ is a *separating family* for Ψ. Corollary 2 of Theorem 5 in **(9)** then shows that $\hat{A}$ is precisely the subalgebra of all elements of $C(\Psi; \mathfrak{R}; \rho)$ which vanish at ω.

But the discussion preceding this theorem then shows that there is a norm-preserving isomorphism τ of $\hat{A}$ onto $C(\Phi; \mathfrak{R}; \rho)$, as algebras over $\mathfrak{R}^{(\rho)}$. Let σ be the mapping obtained by following the mapping $x \rightarrow \hat{x}$ by the mapping τ. Then σ is a norm-preserving isomorphism of A onto $C(\Phi; \mathfrak{R}; \rho)$, as algebras over $\mathfrak{R}^{(\rho)}$.

In case A has a unit element e, then $\{\omega; e; 1/2\}$ is an open set in Ψ and contains only ω, so that Φ is closed in Ψ, and Φ is therefore compact. Conversely, if Φ is compact, then the constant function 1 belongs to $C(\Phi; \mathfrak{R}; \rho)$ and acts as a unit element there; thus, A must have a unit element also, for A is isomorphic to $C(\Phi; \mathfrak{R}; \rho)$. This completes the proof.

Note. In Theorem 1, the space Φ consists of all non-zero homomorphisms ϕ of A into $\mathfrak{R}^{(\rho)}$, as algebras over $\mathfrak{R}^{(\rho)}$, such that $||\phi(x)|| \leqslant N(x)$ for all x in A. However, if ϕ is a non-zero homomorphism of A into $\mathfrak{R}^{(\rho)}$ such that $||\phi(x)|| \leqslant N(x)$ for all x in A, then the inequality $||\phi(y) - \phi(z)|| = ||\phi(y - z)|| \leqslant N(y - z)$ shows that ϕ is continuous, and it follows easily that ϕ is a homomorphism of A into $\mathfrak{R}^{(\rho)}$,as algebras over $\mathfrak{R}^{(\rho)}$. Thus, the space Φ can be described as the set of all non-zero homomorphisms ϕ of A into $\mathfrak{R}^{(\rho)}$, such that $||\phi(x)|| \leqslant N(x)$ for all x in A. (That is, the elements of Φ may be regarded either as algebra-homomorphisms or as ring-homomorphisms.)

3. Normed algebras which satisfy a reality condition. A two-sided ideal I in a ring R will be called a *prime* ideal if xy in I implies x in I or y in I; a two-sided ideal I in a ring R will be called a *real* ideal if $x^2 + y^2$ in I implies that x and y are in I. We shall say that a topological ring R *satisfies the reality condition* if every closed prime ideal in R is a real ideal. Kadison has considered a weaker type of reality condition in **(6)**, where "strictly real" algebras are considered; the present methods seem to require a stronger reality condition, however, such as the one we have just given.

In this section, Theorem 1 will be used in order to obtain a representation theorem for connected, commutative, complete normed algebras which satisfy the reality condition and which have a norm that satisfies a polynomial identity in the sense of **(3)**. It will be shown that such an algebra can be identified with a $C(\Phi; \mathfrak{R}; \rho)$, for some locally compact space Φ and for some real number ρ with $0 < \rho \leqslant 1$ .First, some useful lemmas are obtained.

LEMMA 1. *Let E be an archimedean field with absolute value, such that E is a normed algebra over a complete field F with absolute value. Then E is complete.*

The proof is routine, and is left to the reader.

LEMMA 2. *Let A be a connected normed algebra over a complete field F with absolute value, and let E be a field with absolute value such that E is a normed algebra over F. Suppose that there is a continuous non-zero homomorphism ϕ of A into E, as algebras over F, and suppose that the zero ideal is a real ideal in E. Then there is a real number ρ with $0 < \rho \leqslant 1$, such that there exist unique isometries of E and of F onto $\mathfrak{R}^{(\rho)}$.*

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Proof. $\phi(A)$ is the continuous image of the connected set A, and is therefore connected in E. Since ϕ is not the zero homomorphism, $\phi(A)$ is a connected set which contains a non-zero element in addition to zero. The field E is not totally disconnected, then, and Lemma 15 of **(1)** implies that E is archimedean. Lemma 1 shows that E is also complete.

Let ρ be the exponent for E, as defined in § 3 of **(3)**. Then there is an isometry σ of E into $\mathfrak{C}^{(\rho)}$. Because E is complete, σ is actually an isometry of E onto $\mathfrak{R}^{(\rho)}$ or $\mathfrak{C}^{(\rho)}$. But the zero ideal is not a real ideal in $\mathfrak{C}^{(\rho)}$, so that σ is an isometry of E onto $\mathfrak{R}^{(\rho)}$. There is an obvious isometry of F into E, and F is therefore isometric to a complete subfield of $\mathfrak{R}^{(\rho)}$, whence F is isometric to $\mathfrak{R}^{(\rho)}$.

Finally, if L is any field with absolute value such that L is isometric to some $\mathfrak{R}^{(\rho)}$, then there is precisely one isometry of L onto $\mathfrak{R}^{(\rho)}$. For, all isometries of L onto $\mathfrak{R}^{(\rho)}$ must agree on the prime field of L and therefore on the closure of the prime field of L. But L is the closure of its prime field, since L is isometric to $\mathfrak{R}^{(\rho)}$, which is the closure of its prime field. Thus, all isometries of L onto $\mathfrak{R}^{(\rho)}$ must coincide. In particular, the isometries of E and of F onto $\mathfrak{R}^{(\rho)}$ are uniquely determined.

THEOREM 2. *Let A be a non-zero, connected, commutative, complete normed algebra, over a non-discrete field K with absolute value. Suppose that A satisfies the reality condition, and suppose that the norm of A satisfies a polynomial identity on A (or a polynomial identity of order greater than* 1 *on a neighbourhood of zero in A). Then there exist a real number ρ, with $0 < \rho \leqslant 1$, and a locally compact space Φ, such that there is an isometry σ of A onto $C(\Phi; \mathfrak{R}; \rho)$, and such that Φ is compact and if only if A has a unit element.*

Proof. If F is the completion of K, then **(4**, chapter IX, § 3, no. 7**)** indicates that the completion of A is a normed algebra over F, so that A may be regarded as a normed algebra over F since A is complete.

The norm, N, of A is power multiplicative, by Theorem 1 of **(3)**. If c is a non-zero element of A, let $\mathcal{N}$ be the set of all power multiplicative pseudonorms N' subordinate to N and such that: (i) $N'(c) = N(c)$, (ii) $N'(cx) = N'(c) \cdot N'(x)$ for all x in A, and (iii) $N'(kx) = ||k|| \cdot N'(x)$ whenever k is in F and x is in A. It is easily verified that $\mathcal{N}$ contains N_c and is a *hereditary system*, in the terminology of **(2)**. As in Lemma 4 of **(2)**, it can be shown that $\mathcal{N}$ contains a minimal element N', and **(2**, Lemma 2**)** shows that N' is a pseudo absolute value.

The null ideal $I(N')$ is an ideal and a subalgebra of A, so that $A/I(N')$ is an algebra over F. Let $\bar{N}$ be the function defined on $A/I(N')$ such that the value $\bar{N}(X)$ assumed by $\bar{N}$ on the residue class X (modulo $I(N')$) is the same as the constant value assumed by N' on the elements of X. Then the algebra $\bar{A} = A/I(N')$, with the function $\bar{N}$ as its norm, becomes a commutative normed algebra over F, and $\bar{N}$ is an absolute value on $\bar{A}$. If η is the natural mapping of A onto $\bar{A}$, then η is a homomorphism of A onto $\bar{A}$, as algebras over F, and $\bar{N}(\eta(x)) = N'(x) \leqslant N(x)$ for all x in A, while $\bar{N}(\eta(c)) = N'(c) = N(c) \neq 0$. The fact that $\bar{N}(\eta(c)) \neq 0$ shows that $\bar{A}$ contains a non-zero element, $\eta(c)$. Thus, $\bar{A}$ is a non-zero commutative algebra and has no proper zero-divisors since $\bar{N}$ is an absolute value.

A standard procedure permits the formation of a field E of quotients for $\bar{A}$, and E also is an algebra over F. If $a, s \in \bar{A}$, with $s \neq 0$, let $||[a/s]|| = \bar{N}(a)/\bar{N}(s)$ define a norm in E. Then E becomes a field with absolute value such that E is a normed algebra over F. The natural mapping ξ of $\bar{A}$ into E is clearly a norm-preserving isomorphism of $\bar{A}$ into E, as algebras over F. Let ϕ be the mapping obtained by applying first η, and then ξ. It is evident that ϕ is a homomorphism of A into E, as algebras over F. Also, $||\phi(c)|| = ||\xi(\eta(c))|| = \bar{N}(\eta(c)) = N'(c) = N(c) \neq 0$, so that $\phi(c) \neq 0$, whence ϕ is a non-zero homomorphism. For x and y in A, we have $||\phi(x) - \phi(y)|| = ||\phi(x - y)|| = ||\xi(\eta(x - y))|| = \bar{N}(\eta(x - y)) = N'(x - y) \leqslant N(x - y))$, and this inequality shows that ϕ is continuous.

Lemma 5 of **(2)** shows that the subordinate pseudonorm N' of A must be continuous on A. Thus, $I(N')$ must be a closed set since it is the pre-image under N' of the closed set 0. Also, $I(N')$ is a prime ideal since N' is a pseudo absolute value. But A satisfies the reality condition, and the closed prime ideal $I(N')$ is therefore a real ideal in A. It follows easily that the zero ideal of $\bar{A}$ is a real ideal; that is, if X and Y in $\bar{A}$ are such that $X^2 + Y^2 = 0$, then $X = Y = 0$. Then E also has the property that if X and Y in E are such that $X^2 + Y^2 = 0$, then $X = Y = 0$. Thus, the zero ideal of E is a real ideal.

Lemma 2 may now be applied, so that there are unique isometries of E and of F onto $\mathfrak{R}^{(\rho)}$, for some real number ρ with $0 < \rho \leqslant 1$. We note that ρ must be the exponent for F, and ρ is consequently also the exponent for K; thus, ρ is determined by K. We shall now identify E and F with $\mathfrak{R}^{(\rho)}$, so that A is a non-zero, complete normed algebra over $\mathfrak{R}^{(\rho)}$; and for each non-zero c in A we have a homomorphism ϕ of A into $\mathfrak{R}^{(\rho)}$, as algebras over $\mathfrak{R}^{(\rho)}$, such that $||\phi(c)|| = N(c)$, as we have shown above. Finally, each ϕ also is such that $||\phi(x)|| = ||\xi(\eta(x))|| = \bar{N}(\eta(x)) = N'(x) \leqslant N(x)$ for all x in A, and A therefore satisfies the hypotheses of Theorem 1. It follows that there exist a locally compact space Φ and a norm-preserving isomorphism σ of A onto $C(\Phi; \mathfrak{R}; \rho)$, as algebras over $\mathfrak{R}^{(\rho)}$, such that Φ is compact if and only if A has a unit element. Since A was not given originally as an algebra over $\mathfrak{R}^{(\rho)}$, it is preferable to describe σ as a ring-isomorphism rather than an algebra-isomorphism in the statement of the present theorem.

It seems desirable to describe the space Φ in terms of the normed algebra as it is originally given in Theorem 2. The note at the end of § 2 permits us to describe Φ as the set of all non-zero homomorphisms ϕ of A into $\mathfrak{R}^{(\rho)}$ (as rings), such that $||\phi(x)|| \leqslant N(x)$ for all x in A, where ρ is the exponent for K.

The question might be raised whether any of the hypotheses in Theorem 2 may be dropped if the existence of an isometry of A *into* a $C(\Phi; \mathfrak{R}; \rho)$ is regarded as a satisfactory conclusion.

First, the reality condition cannot be dropped if we desire such a conclusion. For, the field $\mathfrak{C}$ of complex numbers satisfies every hypothesis of Theorem 2, except for the reality condition; but $\mathfrak{C}$ contains an element i, the negative of whose square is a non-zero idempotent, while no $C(\Phi; \mathfrak{R}; \rho)$ can contain such an element. Thus, $\mathfrak{C}$ can never be isometric to a subring of a $C(\Phi; \mathfrak{R}; \rho)$.

Connectedness of A is also essential in Theorem 2. For the field P_3 of 3-adic numbers (see example 9 in **(1)**, for instance) is a normed algebra over itself and satisfies all hypotheses of the theorem other than connectedness; but P_3 is non-archimedean and therefore cannot be isometric to a subring of a $C(\Phi; \mathfrak{R}; \rho)$, since every such subring is archimedean if it is not zero.

Completeness of A cannot be dropped as a hypothesis of Theorem 2, either, for Dieudonné constructed in **(5)** a dense, connected subfield F of $\mathfrak{C}$, such that F is algebraically isomorphic to the field of real numbers; such a field, as a normed algebra over itself, with the ordinary absolute value as the norm, satisfies every hypothesis of the theorem excepting completeness. But if there were an isometry of F into a $C(\Phi; \mathfrak{R}; \rho)$, completion would yield an isometry of $\mathfrak{C}$, the completion of F, into $C(\Phi; \mathfrak{R}; \rho)$, and this is impossible. Thus, F satisfies all conditions of Theorem 2 except for completeness, and F cannot be isometric to a subring of a $C(\Phi; \mathfrak{R}; \rho)$.

Note 1. The assumption that K is non-discrete is used in Theorem 2 only to show that the norm of the algebra is power multiplicative, so that this theorem remains true if the assumption that K is non-discrete is dropped, provided that the hypothesis that the norm satisfies a polynomial identity is replaced by the stronger assumption that the norm is power multiplicative.

Note 2. The hypothesis of connectedness can be replaced in Theorem 2 by the assumption that the field K is archimedean, for the completion of K would then be connected, and it would follow that A is connected since it is a normed algebra over the completion of K.

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