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PERIODIC AND NIL POLYNOMIALS IN RINGS

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Let R be an associative ring and $f(x_1, \ldots, x_d)$ a polynomial in non-commuting variables. We say that f is periodic or nil in R if for all $r_1, \ldots, r_d \in R$ we have that $f(r_1, \ldots, r_d)$ is periodic, respectively nilpotent (recall that $a \in R$ is periodic if for some integer $n(a) > 1$, $a^{n(a)} = a$).

In [2, Theorem 3.12] it was proved that if R is a primitive ring and f a homogeneous polynomial periodic in R, then R is finite dimensional over its center F; moreover if f is not a polynomial identity of R, then F is algebraic over a finite field and $R \cong F_n$ with $n \leq \deg(f)$. In this note we shall prove that in case f is multilinear then f is a central polynomial for R and, so, $n \leq \frac{1}{2}[\deg(f)+2]$. It will follow that if R is any ring in which f is a multilinear periodic polynomial, then R satisfies a polynomial identity of degree $\leq 2\deg(f)$; moreover f is central in R/N, where N is the nil radical of R.

We shall also remark that if R is a ring with no non-zero nil right ideals and f is a multilinear polynomial which is nil in R then f vanishes in R. This result is known when R is a semisimple ring or R is a ring with no non-zero nil ideals which either satisfies a polynomial identity or is an algebra over an uncountable field (see [3]).

In what follows all rings will be algebras over C, a commutative ring with 1. We assume that $f(x_1, \ldots, x_d)$ is a multilinear polynomial in d noncommuting variables $x_1, \ldots, x_d$ with coefficients in C. Moreover if $c(f)$ denotes the ideal generated by the coefficients of f, we assume that $c(f)r \neq 0$ for all $0 \neq r \in R$.

1. Let R be a ring and R_n the ring of $n \times n$ matrices over R. By adjoining a unit element if necessary, and considering the elements of R as scalar matrices, we can write every matrix of R_n as $\sum a_{ij}e_{ij}$, where the $a_{ij} \in R$ and the e_{ij} $(i, j = 1, \ldots, n)$ are the usual matrix units.

Given a sequence $u = (A_1, \ldots, A_d)$ of matrices from R_n, the value of u is defined to be $|u| = A_1A_1 \cdots A_d$. If σ is a permutation of $\{1, \ldots, d\}$, we write $u^\sigma = (A_{\sigma(1)}, \ldots, A_{\sigma(d)})$. A sequence of the form $u = (a_1e_{i_1j_1}, a_2e_{i_2j_2}, \ldots, a_de_{i_dj_d})$, where the $a_i \in R$, is called simple. A simple sequence u is called even if for some σ, $|u^\sigma| = be_{ii} \neq 0$, and odd if for some σ, $|u^\sigma| = be_{ij} \neq 0$ where $i \neq j$. By [3, Lemma 1] these terms are well defined.

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We begin with the following

LEMMA 1. *Let R be a ring and $f(x_1, \ldots, x_d)$ a multilinear polynomial. If f vanishes for all odd substitutions from R_n, then $f(u) \in R$ for all substitutions from R_n.*

Proof. Let $u = (A_1, \ldots, A_d)$ be a sequence of matrices from R_n. Since f is multilinear and vanishes for all odd substitutions we can write $f(u) = \sum f(u^{(r)})$ where the $u^{(r)}$ are simple even sequences. By [3, Lemma 2] the $f(u^{(r)})$ are diagonal matrices; hence $f(u)$ is diagonal, say $f(u) = \sum b_i e_{ii}$.

Now, for $j \neq 1$, let φ be the automorphism of R_n given by $A \to (1+e_{1j})A(1-e_{1j})$. If u^φ is the image of the sequence u under φ, we have, as before, that $f(u^\varphi)$ is diagonal. But

$$f(u^\varphi) = f(u)^\varphi = (\sum b_i e_{ii})^\varphi = (1+e_{1j}) \sum b_i e_{ii}(1-e_{1j}) = \sum b_i e_{ii} + (b_j - b_1)e_{1j}.$$

Thus we must have $(b_j - b_1)e_{1j} = 0$; that is, $b_j = b_1$. As j varies between 2 and n, we get the desired conclusion.

An immediate consequence is the following

COROLLARY. *Let R be a ring and $f(x_1, \ldots, x_d)$ a multilinear polynomial. If f is periodic in R_n, then $f(u) \in R$ for all substitutions from R_n. Moreover if $n > 1$ then f vanishes in R_{n-1}.*

Proof. Suppose f is periodic in R_n and let u be an odd sequence in R_n. By [3, Lemma 2], $f(u) = be_{ij}$ for some $b \in R$, $i \neq j$. Thus, since $f(u)$ is both nilpotent and periodic, $f(u)$ must be zero. By Lemma 1, $f(u)$ is in R. Now, if $n > 1$, considering $R_{n-1} \subset R_n$, we get that f vanishes in R_{n-1}.

Thus we now have the

THEOREM 1. *Let R be a primitive ring and $f(x_1, \ldots, x_d)$ a multilinear polynomial which is periodic in R. Then f is central in R. If f is not a polynomial identity of R, then $R \cong F_n$ where F is a field algebraic over a finite field and $n \leq \frac{1}{2}[\deg(f) + 2]$.*

Proof. Suppose f is not a polynomial identity. By [2, Theorem 3.12] $R \cong F_n$ where F is a field algebraic over a finite field. Thus, by the above Corollary, f is central in F_n and, so, $n \leq \frac{1}{2}[\deg(f) + 2]$.

We finish the periodic case with

THEOREM 2. *Let R be a ring and $f(x_1, \ldots, x_d)$ a multilinear polynomial which is periodic in R. Then*

(1) *R satisfies a polynomial identity of degree $\leq 2 \deg(f)$*

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(2) *the ideal generated in R by the elements* $f(r_1, \ldots, r_d)r_{d+1} - r_{d+1}f(r_1, \ldots, r_d)$, $r_i \in R$, *is nil.*

Proof. Let J be the Jacobson radical of R. Since f is periodic, it vanishes in J. Since R/J is a subdirect product of primitive rings, applying Theorem 1 we also get that f is central in R/J. Thus for all $r_1, \ldots, r_{2d}$ in R,

$$f(f(r_1, \ldots, r_d)r_{d+1} - r_{d+1}f(r_1, \ldots, r_d), r_{d+2}, \ldots, r_{2d}) = 0,$$

and (1) follows.

If N is the nil radical of R, R/N is a subdirect product of prime rings R_α satisfying a polynomial identity. Since a prime ring satisfying a polynomial identity is an order in a finite dimensional central simple algebra, we can apply Theorem 1 to these algebras, getting that f is central in R/N.

2. We treat now multilinear nil polynomials

THEOREM 3. *Let R be a ring with no non-zero nil right ideals and let* $f(x_1, \ldots, x_d)$ *be a multilinear polynomial nil in R. Then f is a polynomial identity for R.*

Proof. Suppose f is not an identity for R. Let $r_1, \ldots, r_d$ in R be such that $f(r_1, \ldots, r_d) \neq 0$, and let k be minimal such that $f(r_1, \ldots, r_d)^k = 0$. Then $a = f(r_1, \ldots, r_d)^{k-1} \neq 0$ is such that $a^2 = 0$. By the proof of Lemma 6 in [1], aR satisfies a polynomial identity. Now, since $a \neq 0$ and R is semiprime there exists a prime ideal P with $a \notin P$. Then $\bar{R} = R/P$ is a prime ring with a non-zero right ideal, $\overline{aR}$, satisfying a polynomial identity. Hence $\bar{R}$ satisfies a generalized polynomial identity; by a theorem of Martindale [4], the central closure of $\bar{R}$ is a primitive ring with non-zero socle. By [5, Corollary 1 of Lemma 6], either $\bar{R}$ satisfies a polynomial identity (PI) or for every integer $n \geq 1$, $\bar{R}$ contains a subring $\bar{R}^{(n)}$ which is prime PI and does not satisfy any identity of degree $< 2n$. By a repeated application of [3, Theorem 7], the second possibility cannot occur and f vanishes in $\bar{R}$. Since $a = f(r_1, \ldots r_d)^{k-1} \notin P$ we get a contradiction.

Combining the above result with [3, Theorem 4], we have the

COROLLARY. *Let R be a ring with no non-zero nil right ideals. If* $f(x_1, \ldots, x_d)$ *is a multilinear polynomial nil in* R_n, *then f vanishes in* R_n.

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