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FIXED POINT PRINCIPLES FOR CONES OF A LINEAR NORMED SPACE

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0. Introduction. In [**8**] and [**9**], Krasnosel'skiĭ proved several fundamental fixed point principles for operators leaving invariant a cone in a Banach space. In [**11**], Nussbaum extended one of the results, the theorem about compression and expansion of a cone, to condensing maps and he applied this theorem to prove the existence of periodic solutions of nonlinear autonomous functional differential equations.

Nussbaum's proof makes an essential use of the difficult Zabreiko and Krasnosel'skiĭ, and Steinlein (mod p)-theorem for the fixed point index [**13–16**]. In [**6**], Fournier and Peitgen proved two different versions of this theorem for completely continuous maps each one being sufficient for Nussbaum's applications. The proofs of these two theorems are much less involved and, although they are different, they make use of the same easier generalized Lefschetz number calculations (see [**12**] for (mod p) and [**5**] for compact attractor). The proofs are divided into two complementary parts, the first one gives the following results.

Let P be a cone in a Banach space and let $T:P \to P$ be a completely continuous map. Denote $S_r = \{x \in P: \|x\| = r\}$ and $B_r = \{x \in P: \|x\| < r\}$.

THEOREM 0.1. *Assume that*

(0.1.1) *there exists m such that $T^i(S_r) \subset B_r$ whenever $i \geqq m$;*

(0.1.2) *there exists n such that $T^n(B_r) \subset B_r$.*

Then ind $(P, T, B_r) = 1$.

THEOREM 0.2. *Assume that*

(0.2.1) *there exists m such that $T^m(S_r) \subset B_r$;*

(0.2.2) *for any $x \in B_r$, there exists $n = n(x) \in \mathbf{N}$ such that $T^i(x) \in B_r$ for $i \geqq n$.*

Then ind $(P, T, B_r) = 1$.

Using Nussbaum's result, we would have a similar theorem.

THEOREM 0.3. *Assume that*

(0.3.1) *there exists $m = p^t$ (where p is a prime and t is an integer) such that $T^m(S_r) \subset B_r$;*

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(0.3.2) $T^m(x) = x \in B_r$ *implies* $T(x) \in B_r$.
Then ind $(P, T, B_r) = 1 \pmod{p}$.

In all three cases, T has a fixed point in B_r.

Let us notice that Theorem 0.3 implies Theorem 0.1; conditions (0.2.1) and (0.2.2) imply condition (0.3.2); thus if, in (0.2.1), m is a power of a prime, the existence of a fixed point is implied by Theorem 0.3.

Furthermore, if we delete one condition from any theorem above, that theorem is no longer true (see examples (2.7.1), (2.7.2), (2.7.3) of [**6**]).

The objects of this paper are to give a common proof of Theorem 0.1 and 0.2 and their complementary theorems, and to weaken, as much as possible, the conditions on $T\big|_{B_r}$ and $T\big|_{P\setminus B_r}$. Our answer to this problem is given in Propositions 4.1, 4.2 and 4.3. Let us notice that, in Proposition 4.1, condition (0.3.2) is no longer satisfied.

1. Lefschetz number. In this paper, we shall make an essential use of the notion of the generalized Lefschetz number in the sense given by Leray [**10**] and of the fixed point index for metric ANR's developed in [**7**].

Let E be a graded vector space over the field of rational numbers, ϕ an endomorphism of degree zero of E and

$$N(\phi) = \cup \{\ker (\phi^n) : n > 0\}.$$

Then ϕ is said to be a *Leray endomorphism* if and only if $\tilde{E} = E/N(\phi)$ is of finite type, that is (i) dim $E_q < \infty$ for all q, and (ii) $E_q \neq 0$ only for a finite number of q. In that case, one defines $\mathrm{Tr}(\phi) = \mathrm{trace}\ (\tilde{\phi})$, where $\tilde{\phi} : \tilde{E} \to \tilde{E}$ is the induced endomorphism.

Let H denote the singular homology functor with rational coefficients and f_* denote $H(f)$, where $f : X \to X$ is a continuous map; f is said to be a *Lefschetz map* if and only if f_* is a Leray endomorphism and, in that case, the *generalized Lefschetz number* of f is defined to be

$$\Lambda(f) = \sum_q (-1)^q \,\mathrm{Tr}\ (f_{*q}).$$

A topological space X is *acyclic* if dim $H_0(X) = 1$ and for all $q > 0$ $H_q(X) = 0$. Notice that if X is acyclic, it follows that $\Lambda(f : X \to X) = 1$. Any contractible space is acyclic (X is *contractible* if 1_X is homotopic to a constant map).

The reason for using singular homology is that it has compact support. We use this essential fact in the following lemma.

LEMMA 1.1. *Let* $f : X \to X$ *be a map and* $Y \subset X$ *be a subset of* X *such that* (i) $H_q(Y) = 0$ *for all* $q > m$ *and* (ii) dim $(H_q(Y)) < \infty$ *for all* q. *Assume that, for any compact subset* K *of* X, *there exists a map* $g : X \to X$ *and an* *integer n such that g is homotopic to f^n and $g(K) \subset Y$; then f is a Lefschetz map and*

$$\Lambda(f) = \sum_{q=0}^{m} (-1)^q \operatorname{Tr}(f_{*q}).$$

Furthermore, if $f_{*q}(i_{*q}(H_q(Y))) \subset i_{*q}(H_q(Y))$ for all $q \leqq m$, where $i: Y \to X$ is the inclusion, it follows that

$$\Lambda(f) = \sum_{q=0}^{m} (-1)^q \operatorname{Tr}(f_{*q}|_{i_{*q}(H_q(Y))}).$$

Proof. It suffices to prove that

$$\operatorname{tr}(\tilde{f}_{*q}) = \operatorname{tr}(\tilde{f}_{*q}: \tilde{A}_q \to \tilde{A}_q)$$

where A is the smallest f-invariant subspace of $H(X)$ containing $i_*H(Y)$ whenever this last map is defined. (Notice that if $q > m$, the latter is 0.)

Consider the following commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \tilde{A}_q & \longrightarrow & \widetilde{H_q(X)} & \longrightarrow & \widetilde{H_q(X)}\big/A_q & \longrightarrow & 0 \\
 & & \uparrow \tilde{f}'_{*q} & & \uparrow \tilde{f}_{*q} & & \uparrow \tilde{f}''_{*q} & & \\
0 & \longrightarrow & \tilde{A}_q & \longrightarrow & \widetilde{H_q(X)} & \longrightarrow & \widetilde{H_q(X)}\big/A_q & \longrightarrow & 0
\end{array}$$

where $\tilde{f}_{*q}{}'$ and $\tilde{f}_{*q}{}''$ are contractions of $\tilde{f}_{*q}$. (This diagram is obtained from the similar diagram without the $\sim$'s). By a property of the ordinary trace, we obtain that

$$\operatorname{tr}(\tilde{f}_{*q}) = \operatorname{tr}(\tilde{f}_{*q}{}') + \operatorname{tr}(f_{*q}{}'')$$

if the last two traces are defined. If $\operatorname{tr}(\tilde{f}_{*q}{}'') = 0$ and $\dim A_q < \infty$, the lemma is proved. Thus it remains to show that

$$N(f_{*q}{}'') = H_q(X)\big/A_q$$

and $\dim A_q < \infty$.

Let $a \in H_q(X)$. There exist a compact subset K of X and $b \in H_q(X)$ such that $j_{*q}(b) = a$, where $j: K \to X$ is the inclusion. Since the following diagram is commutative

$$\begin{array}{ccc}
Y & \xrightarrow{i} & X \\
g' \uparrow & & \uparrow g \\
K & \xrightarrow{j} & X
\end{array}$$

where g' is a contraction of g, it follows that

$$f_{*q}{}^n(a) = g_{*q}(a) = g_{*q} \circ j_{*q}(b) = i_{*q} \circ g_{*q}{}'(b) \in i_{*q}H_q(Y) \subset A_q.$$

Thus $a + A_q \in N(f_{*q}'')$ and if $\mathscr{B}$ is a basis of $i_{*q}H_q(Y)$ there exists an integer n such that $\cup \{f_{*q}'(\mathscr{B}):1 \leqq i \leqq n\}$ generates A_q.

2. Fixed point index. A map $f:X \to Y$ is *locally compact* provided that for any $x \in X$, there exists a neighbourhood $\mathscr{U}$ of x such that $f(\mathscr{U})$ is contained in a compact subset of Y. Consider the map $f:\mathscr{U} \to X$ where $\mathscr{U}$ is an open subset of X. Denote by Fix (f) the set of fixed points of f (that is Fix $(f) = \{x \in X:f(x) = x\}$).

A map $f:\mathscr{U} \to X$ is called *admissible* provided (i) $\mathscr{U}$ is an open subset of X and (ii) Fix (f) is compact. A homotopy $h:\mathscr{U} \times I \to X$ is called *admissible* provided (i) $\mathscr{U}$ is an open subset of X and (ii)

$$\text{Fix }(h) = \cup \{\text{Fix }(h_t):t \in I\}$$

is compact.

Let X be an ANR (cf. [**1**]) and let $f:\mathscr{U} \to X$ be a locally compact, admissible map. Then the number ind $(X, f, \mathscr{U})$ is defined ([**2**], [**7**]) and is called the *fixed point index of the map* $f:\mathscr{U} \to X$.

This index satisfies a number of properties among which are the following:

Additivity: Assume that $\mathscr{U} = \mathscr{U}_1 \cup \mathscr{U}_2$ and that

$$\text{Fix }(f) \cap \mathscr{U}_1 \cap \mathscr{U}_2 = \emptyset.$$

Then

$$\text{ind }(X, f, \mathscr{U}) = \text{ind }(X, f, \mathscr{U}_1) + \text{ind }(X, f, \mathscr{U}_2).$$

Excision: Let $\mathscr{U}' \subset \mathscr{U}$ and Fix $(f) \subset \mathscr{U}'$. Then

$$f' = f\Big|_{\mathscr{U}'} :\mathscr{U}' \to X$$

is admissible and ind $(X, f, \mathscr{U}) =$ ind $(X', f', \mathscr{U}')$.

Fixed points: If ind $(X, f, \mathscr{U}) \neq 0$, it follows that Fix $(f) \neq \emptyset$.

Homotopy: Let $h:\mathscr{U} \times I \to X$ be an admissible homotopy; then

$$\text{ind }(X, h_0, \mathscr{U}) = \text{ind }(X, h_1, \mathscr{U}).$$

Normality: Let $\mathscr{U} = X$ and f be a compact map; then ind $(X, f, X) = \Lambda(f)$.

Let M be a compact subset of X; M is said to be a *compact attractor* for the map $f:X \to X$ provided the closure of the set $\{f^n(x)\}_{m \in \mathbf{N}}$ intersects M, for any $x \in X$.

Assume $f:X \to X$ is a map with a compact attractor M; then ([**4**], [**5**]) M has an f-invariant open neighbourhood V such that $f(V)$ is relatively compact in V and $\Lambda(f:V \to V) = \Lambda(f)$. Since f has no fixed points in $X \backslash M$, we obtain the following property of the fixed point index:

Normality: Assume that $\mathscr{U} = X$ and that f has a compact attractor. Then ind $(X, f, X) = \Lambda(f)$.

3. Main results.

Notations 3.1. Let A be a set. Denote by $\mathring{A}$, the interior of A, ∂A the boundary of A, and $\bar{A}$ the closure of A.

Definition 3.2. $r:X \to Y$ is a *retraction* if $Y \subset X$ and $r \circ i = 1_Y$, where $i: Y \to X$ is the inclusion.

In this section, we shall prove the following theorem.

THEOREM 3.3. *Let $f:X \to X$ be a locally compact map where X is an* ANR. *Let A be a closed acyclic subset of X such that $f(\partial A)$ is relatively compact and there exists a retracting r onto ∂A which is homotopic to 1_A. Furthermore assume that the following conditions are satisfied:*

(3.3.1) *for any $a \in A$, there exists an integer $n(a) \geqq 1$ such that $f^{n(a)}(a) \in \mathring{A}$*

(3.3.2) *there exists an integer n such that $f^n(\partial A) \subset A$*

(3.3.3) *for any $a \in \partial A$ and for any integer $1 \leqq i \leqq n$ (where n is the same as in condition* (ii)) *there exists an integer m and V a neighbourhood of a such that*

$$f^m(V \cap \partial A) \subset A \text{ and } m \equiv i \pmod{n}.$$

Then ind $(X, f \circ \rho, \mathring{A}) = 1$, *where $\rho:X \to X$ is defined by $\rho\big|_A = r$ and $\rho\big|_{X \setminus A} = 1_{X \setminus A}$. If furthermore $f(A)$ is relatively compact, then*

ind $(X, f, \mathring{A}) = 1$.

Notice that condition (3.3.3) can be replaced by the following simpler but less general condition.

(3.3.3a) for any $a \in \partial A$ and for any integer $1 \leqq i \leqq n$, there exists an integer m such that

$$f^m(a) \in \mathring{A} \text{ and } m \equiv i \pmod{n}.$$

In order to prove this theorem, we shall need the following lemmas. But first let us introduce some definitions used in these lemmas:

$$W = \bigcup \{f^{-i}(\mathring{A}) : i \geqq 1\}$$

$$\rho:X \to X \text{ such that } \rho\big|_A = r \quad \text{and} \quad \rho\big|_{X \setminus \mathring{A}} = 1_{X \setminus \mathring{A}}$$

$$g:X \to X \text{ such that } g = f \circ \rho.$$

LEMMA 3.4. *Under the assumptions of condition* (i) *of Theorem* 3.3, *$f(W) \subset W$, $\rho(W) \subset W$, $g(W) \subset W$, $\rho \sim 1_W$ and $g \sim f: W \to W$.*

Proof. This is evident from the fact that A is in W.

LEMMA 3.5. *Under the assumptions of condition* (3.3.1) *of Theorem* 3.3, $W = \cup \{g^{-i}(\mathring{A}): i \geqq 1\}$.

Proof. If $a \in W$, it follows that $\rho(a) \in W$. Thus there exists an integer $n \geqq 1$, minimal with respect to the property $f^n(\rho(a)) \in \mathring{A}$; hence $g^n(a) \in \mathring{A}$ since $g\Big|_{X\setminus\mathring{A}} = f\Big|_{X\setminus\mathring{A}}$.

LEMMA 3.6. *If $f(\partial A)$ is relatively compact and $\partial A \subset \cup_1^n f^{-i}(A)$, then $g: W \to W$ has a compact attractor.*

Proof. Consider the compact set $M = \overline{f(\partial A)}$. First, $M \subset W$: In fact, by (ii), $f(\partial A) \subset \cup_{i=1}^n f^{-i+1}(A)$ hence

$$f(\partial A) \subset \bigcup_{i=1}^{n} f^{-i+1}(A) \subset \bigcup_{i=1}^{n} f^{-i+1}(W) \subset W.$$

Finally M is a compact attractor; if $a \in W$ there exists n such that $g^n(a) \in A$ thus $g^{n+1}(a) \in \overline{g(A)} = M$.

LEMMA 3.7. *Under the assumptions of Theorem* 3.3, $\Lambda(g: W \to W) = 1$.

Proof. We shall apply Lemma 1.1 with $Y = A$.

(A) Let us show that for any compact subset K of W, there exists an integer k and a map $h: W \to W$ such that h is homotopic to g^k and $h(K) \subset A$.

By Lemma 3.5, there exists j such that $K \subset \cup \{g^{-i}(A): i = 1, \ldots, j-1\}$. Hence if $x \in K$, there exists $d \leqq j$ such that $g^d(x) \in A$ and $g^i(x) \notin A$ for all $d < i \leqq j$. Let $b = \rho \circ g^d(x) \in \partial A$; we have that $g^i(x) = f^{i-d}(b)$ for all $d < i \leqq j$ hence $n > j - d$, since $f^n(b) \in f^n(\partial A) \subset A$. Consequently

$$g^j(K) = \cup \{f^i(K_i), \ i = 0 \ldots n-1\}$$

where $K_i = \rho \circ g^{j-i}(K) \cap \partial A$ is compact. Since $f^n \circ \rho(A) \subset A$ and $f^n \circ \rho \sim g^n: W \to W$ it is sufficient to prove that for every K_i, there exists s such that

$$(f^n \circ \rho)^s \, (f^i(K_i)) \subset A.$$

Let $x \in K_i$. By condition (iii), let $m \equiv i \pmod n$ and let V be an open neighbourhood of x such that $f^m(V \cap \partial A) \subset A$. Denote by t the integer such that $m = tn + i$. Then

$$f^{tn} \circ f^i(V \cap \partial A) = f^m(V \cap \partial A) \subset A$$

and since $(f^n \circ \rho)\ (A) \subset A$, it follows that

$$(f^n \circ \rho)^t \circ f^i(V \cap \partial A) \subset A.$$

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Finally since V and t depend on x, which is arbitrary, we can cover K_i by a finite number of such V's; and if s is the maximum of the corresponding t's, we have that

$$(f^n \circ \rho)^s \circ f^i(K_i) \subset A$$

since $(f^n \circ \rho)(A) \subset A$.

(B) Let us show that $\Lambda(g: W \to W) = 1$.

Since $\dim H_0(A) = 1$, the homology class $[x]$ of the constant map x generates $H_0(A)$ for any $x \in A$. It is sufficient to prove that $f_{*0}([x]) = [x]$: because, since $g \sim f$ and so $g_* = f_*$, we have that

$$g_{*0} \circ j_{*0}(H_0(A)) \subset j_{*0}(H_0(A));$$

hence by Lemma 1.1 and by (A), we obtain that

$$\Lambda(g) = \operatorname{Tr}\left(g_{*0}\Big|_{j_{*0}(H_0(A))}\right) = 1.$$

Let us now prove our assertion. Notice that $[x] = [y]$ for any $x, y \in A$. Take $x \in \partial A$; we have that $f_{*0}{}^n([x]) = [f^n(x)] = [x]$ since $f^n(x) \in A$. Furthermore, take $m \equiv 1 \pmod{n}$ such that $f^m(x) \in A$; thus $f_{*0}{}^m([x]) = [f^m(x)] = [x]$. Take k such that $m = kn + 1$; then

$$f_{*0}[x] = f((f_{*0}{}^n)^k[x]) = f_{*0}^{kn+1}[x] = f_{*0}{}^m[x] = [x].$$

Proof of Theorem 3.3. By Lemmas 3.6 and 3.7 and the normality property of the index,

$$\operatorname{ind}(W, g, W) = \Lambda(g: W \to W) = 1$$

(g is locally compact since $g = f \circ \rho$ and f is locally compact). By Lemma 3.5, g has no fixed points in $W \backslash \mathring{A}$, thus by excision,

$$\operatorname{ind}(X, g, \mathring{A}) = \operatorname{ind}(W, g, \mathring{A}) = \operatorname{ind}(W, g, W) = 1.$$

This proves our first assertion.

Finally let us show the homotopy property of the index,

$$\operatorname{ind}(X, g, \mathring{A}) = \operatorname{ind}(X, f, \mathring{A}),$$

provided that $f(A)$ is relatively compact.

In fact, the homotopy $h(x, t) = tf(x) + (1 - t)g(x)$ is a homotopy between f and g, which is compact since $\overline{g(A)} = \overline{f(\partial A)} \subset \overline{f(A)}$ are compact. Furthermore $h(x, t) = f(x)$ for any $x \in \partial A$ and $t \in [0, 1]$; thus h_t has no fixed points on ∂A, and

$$\operatorname{Fix}(h: \mathring{A} \times I \to X) = \operatorname{Fix}(h: A \times I \to X)$$

which is a closed subset of the compact $A \cap h(A \times I)$.

This concludes the proof of Theorem 3.3.

4. Applications. We shall try to apply the preceding theorem in order to obtain certain fixed point principles of the Krasnosel'skiĭ type ([**8**], [**9**]) for mappings of cones in a linear normed space. Our results although similar to those of ([**11**]), are independent of these last results. However, they will generalize Theorems 2.5 and 2.6 of [**6**].

We shall call a *wedge* a subset P of a linear normed space E which is a convex subset of E and satisfies the following conditions:

(i) $a \in P$ implies that $ta \in P$ for all $t \geqq 0$

(ii) there exists $y \in P$ such that $-y \notin P$.

Notice that P is an ANR (cf. [**3**]).

PROPOSITION 4.1. *Let P be a wedge of E and let $f: P \to P$ be a completely continuous map (i.e., a map which is compact on bounded subsets). Assume that there exists $r > 0$ such that*

(4.1.1) *for all $a \in \bar{B}_r$ there exists $n_a > 0$ such that $f^{n_a}(a) \in B_r$*

(4.1.2) *there exists $n > 0$ such that* (i) $f^n(S_r) \subset \bar{B}_r$, *and* (ii) *for any $a \in S_r$ and $i \in \mathbf{N}$, there exists $m \in \mathbf{N}$ and $\epsilon > 0$ such that $m \equiv i \pmod{n}$ and*

$$f^m(N_\epsilon(a) \cap S_r) \subset \bar{B}_r.$$

Then $\operatorname{ind}(P, f, B_r) = 1$.

Proof. It is sufficient to verify the conditions of Theorem 3.3 for $A = \bar{B}_r$. Since $\bar{B}_r$ is convex, it is contractible, hence acyclic. Since $\bar{B}_r$ is bounded, $f(\bar{B}_r)$ is relatively compact. Notice that the retraction from $\bar{B}_r$ onto S_r is homotopic to the identity. Finally the conditions (i)–(iii) of Theorem 3.3 are immediate from conditions (i) and (ii) of the Proposition.

PROPOSITION 4.2. *Let P be a wedge of E and let $f: P \to P$ be a completely continuous map. Assume that there exists $r > 0$ such that*

(4.2.1) *for all $a \in P\backslash\bar{B}_r$, there exists n_a such that*

$$f^{n_a}(a) \in P\backslash\bar{B}_r$$

(4.2.2) *there exists $n > 0$ such that* (i) $f^n(S_r) \subset P\backslash B_r$ *and* (ii) *for any $a \in S_r$ and $i \in \mathbf{N}$, there exists $m \in \mathbf{N}$ and $\epsilon > 0$ such that $m \equiv i \pmod{n}$ and $f^m(N_\epsilon(a) \cap S_r) \subset P\backslash B_r$.*

Then $\operatorname{ind}(P, f, B_r) = 0$.

Proof. We apply Theorem 3.3 with $A = P\backslash B_r$.

First there exists the radial retraction of $P\backslash\bar{B}_r$ onto S_r. Since $P\backslash B_r$ is contractible (cf. [**6**]), $P\backslash B_r$ is acyclic. Furthermore $g(P\backslash B_r) \subset f(S_r)$ which is relatively compact. Hence by Theorem 3.3,

$$\operatorname{ind}(P, g, P\backslash\bar{B}_r) = 1.$$

Moreover $\operatorname{ind}(P, g, P) = 1$: In fact, $\overline{g(P)} = \overline{f(\bar{B}_r)}$ is a compact subset

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of P, hence g is compact. Therefore

$$\text{ind}\,(P, g, P) = \Lambda(g:P \to P).$$

Furthermore $\Lambda(g:P \to P) = 1$ since P is acyclic.

It follows that

$$\begin{aligned}\text{ind}\,(P,f,B_r) = \text{ind}\,(P,g,B_r) &= \text{ind}\,(P,g,P) - \text{ind}\,(P,g,P\backslash\bar{B}_r)\\ &= 1 - 1 = 0\end{aligned}$$

since $f\big|_{\bar{B}_r} = g\big|_{\bar{B}_r}$ has no fixed points on S_r.

PROPOSITION 4.3. *Let P be a wedge of E and let $f:P \to P$ be a completely continuous map. Assume that conditions* (4.1.1) and (4.1.2) *are satisfied for r and that conditions* (4.2.1) *and* (4.2.2) *are satisfied for R. Let*

$$\mathscr{U} = \{x \in P: \min\{r, R\} < \|x\| < \max\{r, R\}\}.$$

Then

$$\text{ind}\,(P,f,\mathscr{U}) = \begin{cases} -1, \text{ if } r < R \text{ (expansion)} \\ +1, \text{ if } r > R \text{ (compression)};\end{cases}$$

thus, f has a fixed point in $\mathscr{U}$.

Notice that we can replace, in Propositions 4.1, 4.2 and 4.3, the sphere S by a closed convex subset.

Remark 4.4. Let H be a hyperplane such that $H \cap P = \{0\}$ and let $y \in P$ such that $-y \notin P$ and $S_r = (ry + H) \cap P$ is bounded for all $r > 0$. Call

$$B_r = (\text{co}\,(S_r \cup \{0\}))\backslash S_r;$$

then S_r, $\bar{B}_r$ and $P\backslash B_r$ are convex subsets and the boundary of $\bar{B}_r$ and $P\backslash B_r$, in P, is S_r: in fact,

$$\bar{B}_r = \cup_{s \leqq r} S_s \quad \text{and} \quad P = \cup_{s \geqq 0} S_s.$$

Since S_r is an ANR, the two retractions from B_r and $P\backslash B_r$, respectively, onto S_r exist and Propositions 4.1, 4.2 and 4.3 are still valid for these new definitions of S_r, B_r and $P\backslash B_r$.

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