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UNIFORMIZATION OF QUASI-UNIFORM SPACES

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This paper considers the question of when a quasi-uniform space has a compatible uniform structure. Typical of the sufficient conditions provided is the result that a quasi-uniform space whose conjugate topology is compact and R_0 is uniformizable.

1. Introduction

Since Pervin [7] gave a simple direct proof of the fact that every topological space is quasi-uniformizable, there has been considerable interest in the study of quasi-uniform spaces. However, it seems that the question of when a quasi-uniform space is uniformizable has received very little attention. The purpose of this note is to provide some results in that direction.

It will be recalled that a *quasi-uniformity* on a set X is a filter $\mathcal{U}$ on $X \times X$ such that the diagonal Δ is contained in each member of $\mathcal{U}$, and for each $U \in \mathcal{U}$ there is a $V \in \mathcal{U}$ such that $V \circ V \subset U$. If $\mathcal{U}$ is a quasi-uniformity on X, then the topology $T(\mathcal{U})$ defined by the neighbourhood system $\{\mathcal{U}[x] : x \in X\}$ is the topology induced by $\mathcal{U}$, where $\mathcal{U}[x] = \{U[x] : U \in \mathcal{U}\}$. Each quasi-uniformity $\mathcal{U}$ on X induces a *conjugate quasi-uniformity* $\mathcal{U}^{-1}$ on X defined by $\mathcal{U}^{-1} = \{U^{-1} : U \in \mathcal{U}\}$. Thus a quasi-uniformity $\mathcal{U}$ on X generates two topologies $T(\mathcal{U})$ and $T(\mathcal{U}^{-1})$ on X, and we are in the setting of bitopological spaces (X, T_1, T_2) in the sense of Kelly [3], whose terms and notation we adopt

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here. A bitopological space (X, T_1, T_2) is *quasi-uniformizable* if there is a pair U and U^{-1} of conjugate quasi-uniformities on X such that $T_1 = T(U)$ and $T_2 = T(U^{-1})$, see Lane [4]. The bitopological space (X, T_1, T_2) is *pairwise completely regular* if for each T_i closed set C and each point $x \notin C$ there is a real valued function $f : X \to [0, 1]$ such that $f(x) = 0$, $f(C) = \{1\}$, f is T_i upper semi-continuous and T_j lower semi-continuous, for $i, j = 1, 2$, $i \neq j$. (X, T_1, T_2) is *pairwise regular* if for each T_i closed set C and each point $x \notin C$ there is a T_i open set U and a T_j open set V disjoint from U such that $x \in U$ and $C \subset V$, for $i, j = 1, 2$, $i \neq j$.

In the bitopological space (X, T_1, T_2) we say that

(i) T_1 is *(countably) paracompact with respect to* T_2 if each (countable) T_1 open cover of X has a T_1 open refinement which is T_2 locally finite,

(ii) T_1 is *locally (countably) compact with respect to* T_2 if each point of X has a T_2 neighbourhood which is T_1 (countably) compact.

A topological space (X, T) is R_0 if it satisfies one of the following equivalent conditions:

(i) $x \in U \in T$ implies $\mathrm{cl}\{x\} \subset U$,

(ii) $\mathrm{cl}\{x\} = \bigcap\{U \mid U \text{ is an open neighbourhood of } x\}$ for each $x \in X$,

(iii) $\mathrm{cl}\{x\} = \mathrm{cl}\{y\}$ or $\mathrm{cl}\{x\} \cap \mathrm{cl}\{y\} = \emptyset$ for $x, y \in X$.

The sufficiency conditions that we give for the uniformizability of $(X; U)$ are in terms of the relationship between $T(U)$ and $T(U^{-1})$, or solely conditions on the conjugate space. Our (quasi) uniform notation is standard and follows Murdeshwar and Naimpally [5].

We are not able to apply most of the results of Section 2 of [8] to

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the quasi-uniform situation because a quasi-uniformizable bitopological space is not necessarily pairwise Hausdorff. In Section 2 of this paper we prove similar results, but for considerably different hypotheses. In particular, the condition that one of the spaces be R_0 is crucial in the extension of our previous results for quasi-metric spaces to quasi-pseudometric spaces in Section 4.

2. Bitopological results

LEMMA 1. *If* (X, T_1, T_2) *is pairwise regular then* $T_i\text{cl}\{p\} = \bigcap\{U : U \in T_j \text{ and } p \in U\}$ *for each point* $p \in X$ *, where* $i, j = 1, 2$ *and* $i \neq j$ *.*

Proof. Let $p \in U \in T_j$. Then pairwise regularity gives a $V \in T_j$ such that $p \in V \in T_i$ cl $V \subset U$. If $B = \bigcap\{U : U \in T_j \text{ and } p \in U\}$, then $B = \bigcap\{T_i \text{ cl } U : U \in T_j \text{ and } p \in U\}$. Thus B is T_i closed, so that $T_i \text{ cl}\{p\} \subset B$.

Now suppose $q \in B - T_i \text{ cl}\{p\}$. Then $L_1 = X - T_i \text{ cl}\{p\}$ is T_i open and $q \in L_1$. So there is, by pairwise regularity, an $L_2 \in T_i$ such that $q \in L_2 \subset T_j \text{ cl } L_2 \subset L_1$. Thus $T_j \text{ cl}\{q\} \subset L_1$, so that $T_j \text{ cl}\{q\} \cap T_i \text{ cl}\{p\} = \emptyset$. Then if $U' = X - T_j \text{ cl}\{q\}$, we have $p \in U' \in T_j$ and $q \notin U'$, contradiction. Thus $T_i \text{ cl}\{p\} = B$. □

LEMMA 2. *Let* (X, T_1, T_2) *be pairwise regular and* (X, T_1) *be* R_0 *. Then every* T_1 *closed* T_1 *compact subset of* X *is* T_2 *closed.*

Proof. Let $A \subset X$ be T_1 closed and T_1 compact. Then $x \in X - A$ implies $T_1 \text{ cl}\{x\} \subset X - A$. By Lemma 1,

$$T_1 \text{ cl}\{x\} = \bigcap\{U : U \in T_2 \text{ and } x \in U\} = \bigcap\{T_1 \text{ cl } U : U \in T_2 \text{ and } x \in U\} .$$

Since $A \subset X - T_1 \text{ cl}\{x\}$, we have that $\{X-T_1 \text{ cl } U : U \in T_2 \text{ and } x \in U\}$ is a T_1 open cover of A . Hence there is an integer n and T_2 open sets $U_1, U_2, \ldots, U_n$ such that $A \subset \bigcup\{X-T_1 \text{ cl } U_k : k = 1, 2, \ldots, n\}$ and

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$x \in V = \bigcap\{U_k : k = 1, 2, \ldots, n\}$. Thus V is T_2 open and $V \cap A = \emptyset$, so that A is T_2 closed. □

PROPOSITION 1. *If* (X, T_1, T_2) *is pairwise regular,* (X, T_1) *is* R_0 *and* T_1 *is locally compact with respect to* T_2 *, then* $T_1 \subset T_2$.

Proof. Let $x \in P \in T_1$, and A be a T_2 neighbourhood of x which is T_1 compact. By pairwise regularity, there is a $Q \in T_2$ such that $x \in Q \subset T_1 \text{ cl } Q \subset A$. Thus $T_1 \text{ cl } Q$ is a T_2 neighbourhood of x which is T_1 compact. Now let $S = Q - P$. Then $T_1 \text{ cl } S$ is T_1 compact and $x \notin T_1 \text{ cl } S$. By Lemma 2, $T_1 \text{ cl } S$ is T_2 closed, so that $V = X - T_1 \text{ cl } S$ is T_2 open, contains x and $V \cap S = \emptyset$. Then $U = Q \cap V$ is T_2 open and $x \in U \subset P$, so that $P \in T_2$. □

PROPOSITION 2. *If* (X, T_1, T_2) *is pairwise regular,* (X, T_1) *is* R_0 *and* T_1 *is paracompact with respect to* T_2 *, then* $T_1 \subset T_2$.

Proof. Let $x \in V \in T_1$. Then $T_1 \text{ cl}\{x\} \subset V$, and

$$T_1 \text{ cl}\{x\} = \bigcap\{U : U \in T_2 \text{ and } x \in U\} = \bigcap\{T_1 \text{ cl } U : U \in T_2 \text{ and } x \in U\} .$$

Thus $X - T_1 \text{ cl}\{x\} = \bigcup\{X - T_1 \text{ cl } U : U \in T_2 \text{ and } x \in U\}$, so that $C = \{V, \{X - T_1 \text{ cl } U : U \in T_2 \text{ and } x \in U\}\}$ is a T_1 open cover of X . Now the proof follows that of Proposition 4 of [8]. □

COROLLARY 1. *If* (X, T_1, T_2) *is pairwise regular and* (X, T_1) *is* R_0 *and compact, then* $T_1 \subset T_2$.

Proof. If (X, T_1) is compact, then T_1 is paracompact with respect to T_2 for any topology T_2 on X . □

The next four results show that we can weaken the compactness conditions in the previous three results to the corresponding countable compactness conditions provided that we add an additional hypothesis. We omit the proofs which are obvious modifications of those above.

LEMMA 3. *Let* (X, T_1, T_2) *be pairwise regular,* (X, T_1) *be* R_0 *and* (X, T_2) *be first countable. Then every* T_1 *closed* T_1 *countably compact subset of* X *is* T_2 *closed.*

PROPOSITION 3. *If* (X, T_1, T_2) *is pairwise regular,* (X, T_1) *is* R_0, (X, T_2) *is first countable and* T_1 *is locally countably compact with respect to* T_2, *then* $T_1 \subset T_2$.

PROPOSITION 4. *If* (X, T_1, T_2) *is pairwise regular,* (X, T_1) *is* R_0, (X, T_2) *is first countable and* T_1 *is countably paracompact with respect to* T_2, *then* $T_1 \subset T_2$.

COROLLARY 2. *If* (X, T_1, T_2) *is pairwise regular,* (X, T_1) *is* R_0 *and countably compact, and* (X, T_2) *is first countable, then* $T_1 \subset T_2$.

Again, we can weaken the compactness conditions to corresponding Lindelof properties if we add an additional hypothesis. A topological space is called a P-space if each G_δ set is open.

LEMMA 4. *Let* (X, T_1, T_2) *be pairwise regular,* (X, T_1) *be* R_0 *and* (X, T_2) *be a* P-*space. Then every* T_1 *closed* T_1 *Lindelof subset of* X *is* T_2 *closed.*

PROPOSITION 5. *If* (X, T_1, T_2) *is pairwise regular,* (X, T_1) *is* R_0, (X, T_2) *is a* P-*space and* T_1 *is locally Lindelof with respect to* T_2, *then* $T_1 \subset T_2$.

PROPOSITION 6. *If* (X, T_1, T_2) *is pairwise regular,* (X, T_1) *is* R_0, (X, T_2) *is a* P-*space and* T_1 *is paraLindelof with respect to* T_2, *then* $T_1 \subset T_2$.

COROLLARY 3. *If* (X, T_1, T_2) *is pairwise regular,* (X, T_1) *is* R_0 *and Lindelöf, and* (X, T_2) *is a* P-*space, then* $T_1 \subset T_2$.

3. Quasi-uniform applications

The characterization of quasi-uniformizable bitopological spaces in Lemma 5 was given independently by Fletcher [1] and Lane [4].

LEMMA 5. *(X, T_1, T_2) is quasi-uniformizable if and only if it is pairwise completely regular.*

It follows immediately from Lemma 5 that if (X, T_1, T_2) is quasi-uniformizable then it is pairwise regular.

LEMMA 6. *If (X, T_1, T_2) is quasi-uniformizable and $T_1 \subset T_2$ then (X, T_2) is uniformizable.*

Murdeshwar and Naimpally [5, Theorem 1.47] proved Lemma 6 by showing that if $T_1 = T(U)$ and $T_2 = T(U^{-1})$ and $T_1 \subset T_2$ then the supremum $U \vee U^{-1}$ of U and U^{-1} is a uniformity which generates the topology T_2. A subbasis for $U \vee U^{-1}$ is $\{U, U^{-1} : U \in U\}$, and $U \vee U^{-1}$ is the smallest quasi-uniformity which is finer than both U and U^{-1}.

Since the proofs of the results in this section use Lemma 6 to provide a uniformity compatible with the given quasi-uniform space in each case, the results are more than existence theorems. They are constructive results, in that the given quasi-uniform space is uniformizable by the supremum of the given quasi-uniformity and its conjugate.

THEOREM 1. *Every quasi-uniform space whose conjugate topology is compact and R_0 is uniformizable.*

Proof. This follows from Lemma 5, Corollary 1 and Lemma 6. □

THEOREM 2. *If (X, T_1, T_2) is quasi-uniformizable and (X, T_1) is R_0, then (X, T_2) is uniformizable if either*

(i) T_1 is locally compact with respect to T_2, or

(ii) T_1 is paracompact with respect to T_2.

Proof. *(i)* follows from Proposition 1 and Lemmas 5 and 6, and *(ii)*

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from Proposition 2 and Lemmas 5 and 6. □

Using Corollary 2 we can obtain

THEOREM 3. *Every first countable quasi-uniform space whose conjugate topology is* R_0 *and countably compact is uniformizable.*

Corollary 3 yields the following result.

THEOREM 4. *Every quasi-uniform* P*-space whose conjugate topology is* R_0 *and Lindelof is uniformizable.*

In each case there is an analogue of Theorem 2 which we do not state.

In view of the fact that R_0 is a conjugate invariant property [5, Theorem 5.1], the R_0 hypothesis in each of the four theorems above can be on the given quasi-uniform space rather than on its conjugate.

Now we provide an example to show that the R_0 condition in our results cannot be replaced by T_0. First we make some comments about Pervin's quasi-uniformity and its conjugate. If (X, T) is a topological space, the family $\{S_G : G \in T\}$ of subsets of $X \times X$ generates a quasi-uniformity U on X compatible with T, where $S_G = (G \times G) \cup (X-G) \times X$, since $S_G[x] = G$ if $x \in G$, and X if $x \notin G$; see Pervin [7]. The conjugate U^{-1} is generated by the family $\left\{S_G^{-1} : G \in T\right\}$, and $S_G^{-1} = (G \times G) \cup X \times (X-G)$. Furthermore, $S_G^{-1}[x] = X$ if $x \in G$, and $X - G$ if $x \notin G$.

EXAMPLE 1. Let $X = \{a, b\}$ and $T_1 = \{\emptyset, X, \{a\}\}$. Then (X, T_1) is not uniformizable because it is not regular, since $\{b\}$ is T_1 closed and $a \notin \{b\}$ but they cannot be separated by disjoint T_1 open sets. Let U be the Pervin quasi-uniformity for T_1 and U^{-1} its conjugate. Then $T_2 = T(U^{-1})$ is defined by the neighbourhood system $\{X-G : G \in T_1\}$. Thus $T_2 = \{X, \emptyset, \{b\}\}$. Hence (X, T_1) is a non-uniformizable quasi-uniform space whose conjugate topology is T_0 and compact.

Our second example shows that the compactness conditions in our results cannot be relaxed too far.

EXAMPLE 2 (Patty [6, Example 2.6]). Let X be the set consisting of $\{1/n : n \text{ a positive integer}\}$ together with two points 0 and $0'$. Define the quasi-metric p on $X \times X$ by $p(x, x) = 0$ for each $x \in X$, $p(x, 0) = 1$ if $x \neq 0$, $p(x, 0') = 1$ if $x \neq 0'$, $p(0, 1/n) = p(0', 1/n) = 1/n$ and $p(1/n, 1/m) = |(1/n)-(1/m)|$. Let U be the quasi-uniformity and T_1 the topology induced on X by p. Define q by $q(x, y) = p(y, x)$ for $x, y \in X$.

Then q induces the conjugate quasi-uniformity U^{-1} of U, and the topology T_2 on X. Then (X, U) is a quasi-uniform space which is not uniformizable, because it is not Hausdorff and hence, being T_1, it is not completely regular. However, its conjugate topology T_2 is discrete and countable, and so has all the following properties: Lindelof, second countable, separable, paracompact, locally compact, σ-compact, meta-compact, countably paracompact, and it is a k-space. Thus no combination of these properties can replace the compactness conditions in Theorems 1, 2 and 3. Furthermore, (X, T_1) is not a P-space but its conjugate topology is R_0 and Lindelof. Thus the P-space condition cannot be dropped from Theorem 4.

4. Quasi-pseudometric applications

The following result was proved by Kelly [3, Proposition 4.2 and Lemma 4.3].

LEMMA 7. *If* p *and* q *are a pair of conjugate quasi-pseudometrics on* X *then*

(i) (X, T_p, T_q) *is pairwise regular,*

(ii) if $T_p \subset T_q$ *then* T_q *is pseudometrizable.*

THEOREM 5. *Every quasi-pseudometric space whose conjugate topology is* R_0 *and countably compact is pseudometrizable.*

Proof. Follows from Lemma 7 and Corollary 2. □

We observe that Theorems 1 and 4 of [8] are immediate corollaries of Theorem 5, since any quasi-metric space is T_1 and therefore R_0. Similarly Theorems 2 and 3 of [8] follow immediately from the following result.

THEOREM 6. *If* (X, T_1, T_2) *is quasi-pseudometrizable, and* (X, T_1) *is* R_0, *then* (X, T_2) *is pseudometrizable if either*

(i) T_1 *is locally countably compact with respect to* T_2, *or*

(ii) T_1 *is countably paracompact with respect to* T_2.

Proof. *(i)* follows from Lemma 7 and Proposition 3, and *(ii)* from Lemma 7 and Proposition 4. □

The next example shows that the R_0 condition is crucial in these results.

EXAMPLE 3. Let X be the interval $[0, 1]$, and define a quasi-pseudometric p on $X \times X$ by

$$p(x, y) = \begin{cases} x - y & \text{if } x \geq y, \\ 0 & \text{if } x < y. \end{cases}$$

Then p induces the topology on X consisting of rays open to the right, and its conjugate q induces the topology on X consisting of rays open to the left. Each of these topologies is compact and T_0 but not T_1, and hence not regular, and therefore not pseudometrizable. Neither of these topologies is R_0.

References

[1] Peter Fletcher, "Pairwise uniform spaces", Abstract 65T-328, *Notices Amer. Math. Soc.* 12 (1965), 612.

[2] Peter Fletcher, Hughes B. Hoyle, III and C.W. Patty, "The comparison of topologies", *Duke Math. J.* 36 (1969), 325-331.

[3] J.C. Kelly, "Bitopological spaces", *Proc. London Math. Soc.* (3) **13** (1963), 71-89.

[4] E.P.Lane, "Bitopological spaces and quasi-uniform spaces", *Proc. London Math. Soc.* (3) **17** (1967), 241-256.

[5] M.G. Murdeshwar and S.A. Naimpally, *Quasi-uniform topological spaces* (Noordhoff, Groningen, 1966).

[6] C.W. Patty, "Bitopological spaces", *Duke Math. J.* **34** (1967), 387-391.

[7] William J. Pervin, "Quasi-uniformization of topological spaces", *Math. Ann.* **147** (1962), 316-317.

[8] T.G. Raghavan and I.L. Reilly, "Metrizability of quasi-metric spaces", *J. London Math. Soc.* (2) **15** (1977), 169-172.

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