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# CO-RANK OF A COMPOSITION OPERATOR

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ABSTRACT. A composition operator $C_T$ on $L^2(X, \Sigma, m)$ is a bounded linear transformation induced by a mapping $T: X \to X$ via $C_T f = f \circ T$.

If $m$ has no atoms then the co-rank of $C_T$ (i.e., $\dim \overline{R(C_T)}^{\perp}$) is either zero or infinite. As a corollary, when $m$ has no atoms, $C_T$ is a Fredholm operator iff it is invertible.

Let $(X, \Sigma, m)$ be a sigma-finite measure space and $T: X \to X$ a $\Sigma$-measurable mapping. Then $T$ induces a bounded linear *composition operator* $C_T$ on $L^2(X, \Sigma, m)$ via $C_T f = f \circ T$ iff (i) the measure $mT^{-1} = m \circ T^{-1}$ is absolutely continuous with respect to $m$ and (ii) the Radon-Nikodym derivative $h = [dmT^{-1}/dm]$ is in $L^\infty(m)$. In this case $\|C_T\| = \|h\|_\infty^{1/2}$ ([1], pp. 663–665, and [4]). We shall assume in what follows that these conditions are satisfied.

It proves useful ([6]) to consider the sigma field $T^{-1}(\Sigma) = \{T^{-1}E : E \in \Sigma\}$. (Since $L^2(m)$ consists of equivalence classes of functions equal a.e. $[m]$, we will, strictly speaking, consider the relative completion of $T^{-1}(\Sigma)$ in $\Sigma$, i.e., the sigma field generated by $T^{-1}(\Sigma)$ and $\{F \in \Sigma: mF = 0\}$, once again calling it $T^{-1}(\Sigma)$.) We will use the fact that the closure $\overline{R(C_T)}$ of the range of $C_T$ equals the subspace of $L^2(m)$ consisting of $T^{-1}(\Sigma)$-measurable functions [2].

Recall that $G \in \Sigma$ is called an *atom* of $m$ in case (i) $mG > 0$ and (ii) $F \in \Sigma, F \subset G$ imply $mF = 0$ or $mF = mG$.

1. LEMMA. *If $m$ has no atoms then $mT^{-1}$ has no atoms in $\Sigma$ of finite $mT^{-1}$-measure.*

PROOF. Suppose the contrary, that is, that there exists $G \in \Sigma$ with $0 < mT^{-1}G < \infty$ such that $G$ is an atom of $mT^{-1}$. Since

$$0 < mT^{-1}G = \int_G hdm,$$

there exists $F \in \Sigma$ with $F \subset G$ and $mF > 0$ such that $h$ is (essentially) bounded below by, say, $\delta > 0$ on $F$. Since $m$ has no atoms we can choose $E \in \Sigma$, $E \subset F$ such that $0 < mE < mF$. We have

$$mT^{-1}E = \int_E hdm \geq \delta mE > 0$$

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and so $mT^{-1}E = mT^{-1}G$. This implies, in particular, that

$$0 = mT^{-1}(F\backslash E) = \int_{F\backslash E} h\,dm \geq \delta m(F\backslash E) > 0,$$

a contradiction. □

2. EXAMPLE. *Under the hypotheses of the Lemma, $mT^{-1}$ may have atoms of infinite $mT^{-1}$-measure. Let $X = [0, \infty)$, $\Sigma$ = Borel sets and let m be Lebesgue measure. Define T on $[n, n + 1)$ for each $n \geq 0$ by*

$$T(x) = \begin{cases} x - n, & n \text{ even} \\ n + 1 - x, & n \text{ odd}. \end{cases}$$

*Then* $(0, 1)$ *is an atom of* $mT^{-1}$ *(as is any subset of* $(0, 1)$ *with positive m-measure) and* $mT^{-1}(0, 1) = \infty$.

In general, if $m$ is a measure with no atoms and $G_0 \in \Sigma$ with $0 < mG_0 < \infty$, then one can choose $G_1 \in \Sigma$ such that $G_1 \subset G_0$ and $0 < mG_1 < mG_0$. Similarly, there exists $G_2 \in \Sigma$ with $G_2 \subset G_1$ and $0 < mG_2 < mG_1$. Proceeding inductively we obtain a strictly decreasing sequence of subsets $G_n$ of $G_0$, each with positive measure. Setting $F_n = G_n \backslash G_{n+1}$ for $n \geq 0$ yields the partition

$$G_0 = \bigcup_{n\geq 0} F_n \cup \bigcap_{n\geq 0} G_n$$

consisting of disjoint $\Sigma$-measurable subsets of $G_0$, each having positive measure (except possibly the intersection). We use this simple construction below.

3. THEOREM. *If m has no atoms, the dimension of* $\overline{R(C_T)}^{\perp}$ *is either zero or infinite.*

PROOF. Assume $f \in \overline{R(C_T)}^{\perp}, f \neq 0$. Let $P$ denote the projection on $\overline{R(C_T)}$. Note that $P(|f|) \geq 0$ a.e. $[m]$. For if $N = \{x \in X : P(|f|)(x) < 0\}$ then $N \in T^{-1}(\Sigma)$, as $P(|f|)$ is $T^{-1}(\Sigma)$-measurable. By sigma-finiteness we may write

$$N = \bigcup_{k\geq 1} N_k$$

where $N_k \in T^{-1}(\Sigma)$ and $mN_k < \infty$ for each $k$. Since $\chi_{N_k} \in \overline{R(C_T)}$,

$$0 \geq \int_{N_k} P(|f|)\,dm = \int_{N_k} |f|\,dm \geq 0.$$

Equality holds and so $mN_k = 0$ for all $k$. In general, if $A \in T^{-1}(\Sigma)$ then, again by sigma-finiteness, we can choose $A_n \in T^{-1}(\Sigma)$ with $mA_n < \infty$ and $A_n \subset A_{n+1}$ for each $n$ such that

$$A = \bigcup_{n\geq 1} A_n.$$

As above,

$$\int_{A_n} P(|f|)\,dm = \int_{A_n} |f|\,dm$$

for each $n$ so that

$$\int_A P(|f|)dm = \int_A |f|dm$$

by the Monotone Convergence Theorem. In particular,

$$\int P(|f|)dm = \int |f|dm > 0.$$

(Both integrals may be infinite.) It follows that we can choose $A \in T^{-1}(\Sigma)$, $0 < mA < \infty$ and $\delta > 0$ such that $P(|F|) \geq \delta$ a.e. on $A$. Furthermore, from the definition of $T^{-1}(\Sigma)$ as a relative completion, there exists $E \in \Sigma$ with $0 < mT^{-1}E < \infty$ such that $m(A \triangle T^{-1}E) = 0$. By the Lemma and comments above, we may write

$$T^{-1}E = \bigcup_{n \geq 1} T^{-1}E_n$$

where $E_n \in \Sigma$, $mT^{-1}E_n > 0$ and $E_n \cap E_m = \phi$ for $n \neq m$. Set $f_n = f\chi_{T^{-1}E_n}$, $n \geq 1$. We have

$$\begin{aligned} \int |f_n|dm &= \int_{T^{-1}E_n} |f|dm \\ &= \int_{T^{-1}E_n} P(|f|)dm \\ &\geq \delta m T^{-1}E_n > 0 \end{aligned}$$

for each $n$. Therefore the $f_n$, having disjoint supports, constitute a sequence of non-zero orthogonal elements of $\overline{R(C_T)}^{\perp}$. □

It has been observed elsewhere ([3], [5]) that if $m$ has no atoms, the nullity of $C_T$ (i.e., dim Ker$(C_T)$) is either zero or infinite. This follows from the basic relation

$$(*) \qquad \|C_T f\|^2 = \int |f \circ T|^2 dm = \int |f|^2 h dm.$$

For if $C_T f = 0, f \neq 0$, then $h$ must vanish on a set $E \in \Sigma$ of positive $m$-measure. It then follows that Ker$(C_T)$ contains the infinite dimensional subspace of elements of $L^2(m)$ supported on $E$.

Recall that an operator with closed range and finite co-rank and nullity is called a *Fredholm* operator. In these terms, the Theorem has the following consequence.

4. Corollary. *If $m$ has no atoms, $C_T$ is Fredholm iff it is invertible.*

The left shift operator on the space $\ell^2(\mathbb{N})$ of square summable complex sequences indexed by $\mathbb{N} = \{1, 2, 3, \ldots\}$ ($C_T$ induced by $T(n) = n + 1$, $m$ being the counting measure) shows the situation to be quite different for atomic measures. In that case, dim Ker$(C_T) = 1$ and $R(C_T) = \ell^2(\mathbb{N})$, so that $C_T$ is Fredholm but not invertible.

The Corollary was proved in [3] for the special case $X = [0, 1]$, $\Sigma =$ Borel sets and $m =$ Lebesgue measure using techniques less elementary than those used here.

5. EXAMPLE. *Take* $(X, \Sigma, m)$ *as in Example 2. Set* $T(x) = e^x - 1$, $x \geq 0$. *Then* $T^{-1}(x) = \ln(x + 1)$ *and* $h(x) = 1/(x + 1)$ on $[0, \infty)$. *By Eqn.* (*), $h > 0$ *a.e. implies* $\text{Ker}(C_T) = \{0\}$. *Since* $T^{-1}(\Sigma)$ *contains all open subintervals and, therefore, all open subsets of* $[0, \infty)$, *it follows that* $T^{-1}(\Sigma) = \Sigma$ *and so* $C_T$ *has dense range. However, since* $h$ *is not essentially bounded below,* $C_T$ *is not bounded below, this again following from Eqn.* (*). *Hence,* $C_T$ *is neither invertible nor, by the Corollary, a Fredholm operator. We conclude that* $R(C_T)$ *is not closed.*

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