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INTERSECTIONS OF ARC-CLUSTER SETS FOR MEROMORPHIC FUNCTIONS

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1. Introduction

Let D and C denote the open unit disk and the unit circle in the complex plane, respectively; and let f be a function from D into the Riemann sphere Ω. An arc $\gamma \subset D$ is said to be an *arc at* $p \in C$ if $\gamma \cup \{p\}$ is a Jordan arc; and, for each t $(0 < t < 1)$, the component of $\gamma \cap \{z\colon t \leq |z| < 1\}$ which has p as a limit point is said to be a *terminal subarc of* γ. If γ is an arc at p, the *arc-cluster set* $C(f, p, \gamma)$ is the set of all points $a \in \Omega$ for which there exists a sequence $\{z_k\} \subset \gamma$ with $z_k \to p$ and $f(z_k) \to a$.

We say that the function f has the *n-arc property at* $p \in C$, for some integer n $(n \geq 2)$, if there exist n arcs $\gamma_1, \cdots, \gamma_n$ at p for which the intersection of all n of the sets $C(f, p, \gamma_j)$ $(j = 1, \cdots, n)$ is empty; if, in addition, the arcs $\gamma_1, \cdots, \gamma_n$ can be chosen to be mutually disjoint, we say that f has the *n-separated-arc property at* p.

A point $p \in C$ at which f has the 2-arc property is called an *ambiguous point of* f. Bagemihl's ambiguous point theorem ([1], p. 380, Theorem 2) states that the set of ambiguous points of an arbitrary function from D into Ω is countable.

Gresser ([3], p. 145, Theorem 2) has proved the existence of a meromorphic function in D having the 3-separated-arc property at each point of a perfect subset of C. Hence, in view of Bagemihl's ambiguous point theorem, a meromorphic function in D having the 3-arc property (or even the 3-separated-arc property) at a point $p \in C$ need not have the 2-arc property at p. However, we show that if a meromorphic function f in D has the 3-arc property at a point p, then in a certain sense f is very close

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to having the 2-arc property at p. The other main result of this paper states that a meromorphic function f in D has the n-arc property (resp., the n-separated-arc property) at p for some integer n $(n>3)$ if and only if f has the 3-arc property (resp., the 3-separated-arc property) at p.

2. Preliminary Results

A *region* is any non-empty open connected subset of Ω; and a region G is a *Jordan region* if the boundary of G, denoted ∂G, is the union of a finite number (>0) of mutually disjoint Jordan curves. A *continuum* is any non-empty closed connected subset of Ω; and a *proper continuum* is any continuum properly contained in Ω.

THEOREM 1. *Let T be a countable subset of Ω and let $K_1, \cdots, K_n$ be n $(n \geq 3)$ proper continua with $\cap_{j=1}^{n} K_j = \phi$. Then there exist n Jordan regions $G_1, \cdots, G_n$ for which the following conditions hold:*

(1) $K_j \subset G_j$ $(j=1, \cdots, n)$,

(2) $\cap_{j=1}^{n} \bar{G}_j = \phi$ (*the bar denotes closure*),

(3) $\partial G_j \cap T = \phi$ $(j=1, \cdots, n)$,

(4) *card* $[\partial G_1 \cap \partial G_2] < \aleph_0$

and

(5) $\partial G_1 \cap \partial G_2 \cap \partial G_3 = \phi$.

Proof. Let χ denote the chordal metric on Ω. There clearly exists a number $\varepsilon > 0$ for which $\cap_{j=1}^{n} [K_j]_\varepsilon = \phi$ and $[K_j]_\varepsilon \neq \Omega$ $(j=1, \cdots, n)$, where $[K_j]_\varepsilon$ denotes the set of all points $a \in \Omega$ satisfying $\chi(a, K_j) < \varepsilon$. Using the Heine-Borel Theorem, for each $j=1, \cdots, n$ we obtain finitely many open spherical caps $S(j,1), \cdots, S(j, n_j)$ having centers in K_j with

$$K_j \subset \cup_{k=1}^{n_j} S(j,k) \subset \cup_{k=1}^{n_j} \bar{S}(j,k) \subset [K_j]_\varepsilon.$$

Since K_j is connected, each set

$$0_j = \cup_{k=1}^{n_j} S(j,k) \quad (j=1, \cdots, n)$$

is a region. Since for each $j=1, \cdots, n$ the set $\{S(j,k)\}_{k=1}^{n_j}$ is finite, we can choose open spherical caps $S_*(j,k)$ $(k=1, \cdots, n_j)$ with

$$S(j,k) \subset S_*(j,k) \subset \bar{S}_*(j,k) \subset [K_j]_\varepsilon$$

such that the region

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$$0_j^* = \cup_{k=1}^{n_j} S_*(j,k)$$

is a Jordan region and

$$\{\text{radius } S_*(1,k)\}_{k=1}^{n_1} \cap \{\text{radius } S_*(2,k)\}_{k=1}^{n_2} = \phi.$$

From the latter condition it follows that

$$\text{card } [\partial 0_1^* \cap \partial 0_2^*] < \aleph_0.$$

Consequently, we can rechoose (if necessary) some of the caps $S_*(3,k)$ $(k=1,\cdots,n_3)$ so that

$$\partial S_*(3,k) \cap [\partial 0_1^* \cap \partial 0_2^*] = \phi \quad (k=1,\cdots,n_3).$$

Furthermore, since T is countable, we can rechoose (if necessary) some of the caps $S_*(j,k)$ so that

$$\partial S_*(j,k) \cap T = \phi \quad (j=1,\cdots,n;\ k=1,\cdots,n_j).$$

If we now set $G_j = 0_j^*$ $(j=1,\cdots,n)$ all five conditions of the theorem can readily be verified and the proof is complete.

We say that the arcs $\gamma_1,\cdots,\gamma_n$ at $p\in C$ are *ordered arcs* if for each $j=1,\cdots,n-1$ there exist an arc $\tau_j\subset D$ and a point $q\in C$ $(q\neq p)$ such that (1) $\tau_j \cup \{p,q\}$ is a Jordan arc and (2) γ_j and γ_{j+1} are, relative to an observer at p, contained in the left and right components of $D-\tau_j$, respectively. Then we say that the arc γ at p is *between the ordered arcs* γ_1 and γ_2 at p provided: if α is an arc in D for which $\alpha\cup\gamma_1\cup\gamma_2\cup\{p\}$ is a Jordan curve with interior domain Δ, then there exists a terminal subarc γ' of γ with $\gamma'\subset\Delta\cup\gamma_2$.

THEOREM 2. *Let f be meromorphic in D, let G_1 and G_2 be Jordan regions with*

$$\partial G_j \cap [\{f(z): f'(z)=0\} \cup \{\infty\}] = \phi \quad (j=1,2),$$

and let γ_1, γ_2 be a pair of ordered arcs at $p\in C$ with $C(f,p,\gamma_j)\subset G_j$ $(j=1,2)$. Then either p is an ambiguous point of f or there exists an arc γ at p between γ_1 and γ_2 with

$$C(f,p,\gamma) \subset (G_1\cap G_2) \cup \partial G_1.$$

Proof. There is no loss of generality in assuming that $\overline{f(\gamma_j)}\subset G_j$ $(j=1,2)$. Choose a Jordan arc $\alpha\subset D$ for which $\alpha\cup\gamma_1\cup\gamma_2\cup\{p\}$ is a Jordan curve Γ,

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and let Δ be the interior domain of Γ. Denote by Λ the set of components λ of $\Delta \cap f^{-1}(\partial G_1)$ which satisfy $\bar{\lambda} \cap \gamma_2 \neq \phi$. Then each component $\lambda \in \Lambda$ is a homeomorphic image of the open interval $(0, 1)$.

Consider the following cases: (I) There exists a terminal subarc γ_2' of γ_2 such that $\gamma_2' \cap \bar{\lambda} = \phi$ for each $\lambda \in \Lambda$.

(Ia) $f(\gamma_2') \subset \bar{G}_1$. Then for $\gamma = \gamma_2$ we have

$$C(f, p, \gamma) \subset \bar{G}_1 \cap G_2 \subset (G_1 \cap G_2) \cup \partial G_1.$$

(Ib) $f(\gamma_2') \subset \Omega - G_1$. Then

$$C(f, p, \gamma_1) \cap C(f, p, \gamma_2) = \phi$$

and p is an ambiguous point of f. (II) For each terminal subarc γ_2' of γ_2 there exists a component $\lambda \in \Lambda$ with $\gamma_2' \cap \bar{\lambda} \neq \phi$. Then, since f is a local homeomorphism on $f^{-1}(\partial G_1)$, $\bar{\lambda} \cap \alpha \neq \phi$ for at most finitely many $\lambda \in \Lambda$. Consequently, there exists an arc γ at p with

$$\gamma \subset [\gamma_2 \cap f^{-1}(G_1)] \cup (\bigcup_{\lambda \in \Lambda} \bar{\lambda}),$$

and it follows that

$$C(f, p, \gamma) \subset (G_2 \cap \bar{G}_1) \cup \partial G_1 = (G_1 \cap G_2) \cup \partial G_1.$$

Thus we have established the theorem in both cases (I) and (II), and the theorem is proved.

We say that the arcs γ_1, γ_2 at $p \in C$ are *intersecting arcs* if every neighborhood of p contains a point of the intersection $\gamma_1 \cap \gamma_2$. We now give an analogue of Theorem 2 for intersecting arcs.

THEOREM 2*. *Let f be meromorphic in D, let G_1 and G_2 be Jordan regions with*

$$\partial G_j \cap [\{f(z) : f'(z) = 0\} \cup \{\infty\}] = \phi \quad (j = 1, 2),$$

and let γ_1, γ_2 be a pair of intersecting arcs at $p \in C$ with $C(f, p, \gamma_j) \subset G_j$ $(j = 1, 2)$. Then there exists an arc γ at p with

$$C(f, p, \gamma) \subset (G_1 \cap G_2) \cup \partial G_1.$$

Proof. As in the proof of Theorem 2, we assume that $\overline{f(\gamma_j)} \subset G_j$ $(j=1, 2)$. Set $Q = \gamma_1 \cap \gamma_2$ and note that $\overline{f(Q)} \subset G_1 \cap G_2$. Let z, z' be a pair of points in Q for which the open subarc τ of γ_2 between z and z' satisfies $\tau \cap Q = \phi$.

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Let τ_* be the closed subarc of γ_1 joining z to z'. Then $\tau \cup \tau_*$ is a Jordan curve, and we let Δ denote its interior domain.

Let Λ denote the set of components λ of $\Delta \cap f^{-1}(\partial G_1)$ satisfying $\bar{\lambda} \cap \tau \neq \phi$. Then, since $z, z' \in f^{-1}(G_1 \cap G_2)$, it is easy to see that there exists a Jordan arc $\rho_{z,z'}$ joining z to z' such that

$$\rho_{z,z'} \subset [\tau \cap f^{-1}(G_1)] \cup (\bigcup_{\lambda \in \Lambda} \bar{\lambda}).$$

It follows that

$$\overline{f(\rho_{z,z'})} \subset (G_2 \cap \bar{G}_1) \cup \partial G_1 = (G_1 \cap G_2) \cup \partial G_1.$$

Set $M = \cup \rho_{z,z'}$ where the union is taken over all pairs $z, z' \in Q$ for which the open subarc τ of γ_2 between z and z' satisfies $\tau \cap Q = \phi$. Since $Q \cup M \cup \{p\}$ is locally connected, it follows ([4], p. 27, Theorem 4.1) that there exists an arc γ at p with $\gamma \subset Q \cup M$. Then, since

$$\overline{f(\gamma)} \subset (G_1 \cap G_2) \cup \partial G_1,$$

the proof is complete.

3. The n-Separated-Arc Property

THEOREM 3. *If f is meromorphic in D, then f has the n-separated-arc property $(n > 3)$ at $p \in C$ if and only if f has the 3-separated-arc property at p.*

Proof. If f has the 3-separated-arc property at p, then it is obvious that f has the n-separated-arc property at p for all n $(n > 3)$. Thus, we need only prove that if f has the n-separated-arc property $(n > 3)$ at p, then f has the $(n-1)$-separated-arc property at p.

Suppose $\gamma_1, \cdots, \gamma_n$ are n ordered arcs at p for which the intersection of all n of the sets $C(f, p, \gamma_j)$ $(j = 1, \cdots, n)$ is empty; and, to avoid the trivial case, assume that the intersection of any $n-1$ of them is non-empty. By Theorem 1 there exist Jordan regions G_j $(j = 1, \cdots, n)$ for which

(1) $C(f, p, \gamma_j) \subset G_j$ $(j = 1, \cdots, n)$,

(2) $\cap_{j=1}^{n} \bar{G}_j = \phi$,

(3) $\partial G_j \cap [\{f(z) : f'(z) = 0\} \cup \{\infty\}] = \phi$ $(j = 1, \cdots, n)$

and

(4) $\partial G_1 \cap \partial G_2 \cap \partial G_3 = \phi$.

We assume that p is not an ambiguous point of f, in which case there would be nothing to prove. Due to conditions (1) and (3) we can apply Theorem 2 to obtain arcs σ_j $(j = 1, \cdots, n-1)$ at p between the corresponding arcs γ_j and γ_{j+1} such that

$$C(f, p, \sigma_j) \subset (G_j \cap G_{j+1}) \cup \partial G_j.$$

Since the arcs $\gamma_1, \cdots, \gamma_n$ are ordered, for each $j = 1, \cdots, n-1$ we can choose a terminal subarc σ_j^* of σ_j in such a way that the arcs $\sigma_1^*, \cdots, \sigma_{n-1}^*$ are mutually disjoint. Then with the aid of conditions (2) and (4) we obtain the relations

$$\begin{aligned} \cap_{j=1}^{n-1} C(f, p, \sigma_j^*) &\subset \cap_{j=1}^{n-1} [(G_j \cap G_{j+1}) \cup \partial G_j] \\ &= \cap_{j=1}^{n-1} \partial G_j = \phi. \end{aligned}$$

That is, f has the $(n-1)$-separated-arc property at p as was to be shown.

THEOREM 4. *Let f be meromorphic in D. If f has the 3-separated-arc property at $p \in C$, then there exist disjoint arcs σ_1 and σ_2 at p for which*

$$\mathit{card}\ [C(f, p, \sigma_1) \cap C(f, p, \sigma_2)] < \aleph_0.$$

Proof. Suppose $\gamma_1, \gamma_2, \gamma_3$ are ordered arcs at p with

$$C(f, p, \gamma_1) \cap C(f, p, \gamma_2) \cap C(f, p, \gamma_3) = \phi.$$

If p is an ambiguous point of f, we are finished; hence we assume that p is not an ambiguous point of f. By Theorem 1 there exist Jordan regions G_1, G_2, G_3 for which

(1) $C(f, p, \gamma_j) \subset G_j$ $(j = 1, 2, 3)$,

(2) $\bar{G}_1 \cap \bar{G}_2 \cap \bar{G}_3 = \phi$,

(3) $\partial G_j \cap [\{f(z) : f'(z) = 0\} \cup \{\infty\}] = \phi$ $(j = 1, 2, 3)$

and

(4) card $[\partial G_1 \cap \partial G_2] < \aleph_0$.

By Theorem 2 there exist arcs σ_j $(j = 1, 2)$ at p between the corresponding arcs γ_j and γ_{j+1} such that

$$C(f, p, \sigma_j) \subset (G_j \cap G_{j+1}) \cup \partial G_j.$$

Since the arcs γ_1, γ_2, γ_3 are ordered, we may assume that $\sigma_1 \cap \sigma_2 = \phi$. Then, using condition (2) we obtain the relations

$$C(f, p, \sigma_1) \cap C(f, p, \sigma_2) \subset [(G_1 \cap G_2) \cup \partial G_1] \cap [(G_2 \cap G_3) \cup \partial G_2]$$
$$= \partial G_1 \cap \partial G_2;$$

and, in view of condition (4), the proof is complete.

Remark. In effect, Gresser ([3], p. 145, proof of Theorem 2) has proved the existence of a meromorphic function μ in D with the following property: there exists a triangle in Ω with sides s_1, s_2, s_3 and a perfect subset C' of C such that for each point $p \in C'$ there exist three mutually disjoint chords ρ_1, ρ_2, ρ_3 at p with

$$C(\mu, p, \rho_j) = s_j \quad (j = 1, 2, 3).$$

The function μ serves as an illustrative example of Theorem 4 in that

$$C(\mu, p, \rho_1) \cap C(\mu, p, \rho_2) \cap C(\mu, p, \rho_3) = \phi$$

and, for $i \neq j$,

$$\text{card } [C(\mu, p, \rho_i) \cap C(\mu, p, \rho_j)] = 1.$$

4. The *n*-Arc Property

By following the same line of proof as in the proofs of Theorems 3 and 4 with Theorem 2* playing the role of Theorem 2, we establish the following results.

THEOREM 5. *If f is meromorphic in D, then f has the n-arc property ($n > 3$) at $p \in C$ if and only if f has the 3-arc property at p.*

THEOREM 6. *Let f be meromorphic in D. If f has the 3-arc property at $p \in C$, then there exist arcs σ_1 and σ_2 at p for which*

$$\textit{card } [C(f, p, \sigma_1) \cap C(f, p, \sigma_2)] < \aleph_0.$$

Remark. Theorem 6 is exemplified by the modular function m mapping D onto the universal covering surface of $\Omega - \{0, 1, \infty\}$. Bagemihl, Piranian and Young ([2], p. 30, proof of Theorem 3) have shown that for each $p \in C$ there exist three arcs (any two of which are intersecting arcs) γ_1, γ_2, γ_3 at p such that

$$C(m, p, \gamma_1) \cap C(m, p, \gamma_2) \cap C(m, p, \gamma_3) = \phi$$

and, for $i \neq j$,

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$$\text{card}\ [C(m, p, \gamma_i) \cap C(m, p, \gamma_j)] \leq 4.$$

If we set $\Pi(f, p) = \cap C(f, p, \gamma)$ where the intersection is taken over all arcs γ at p, the next result follows from Theorems 5 and 6 and the fact that $\Pi(f, p) = \phi$ implies that f has the n-arc property at p for some integer n $(n \geq 2)$.

THEOREM 7. *Let f be meromorphic in D. If $\Pi(f, p) = \phi$, then f has the 3-arc property at p and there exist arcs σ_1 and σ_2 at p for which*

$$\textit{card}\ [C(f, p, \sigma_1) \cap C(f, p, \sigma_2)] < \aleph_0.$$

5. Open Questions

1. Does there exist a meromorphic function in D which has the 3-arc property at a point $p \in C$ but does not have the 3-separated-arc property at p?

2. Does the modular function m have the 3-separated-arc property at each point of C?

3. If the answer to Question 2 is in the negative, does there exist a meromorphic function in D having the 3-separated-arc property at each point of C?

REFERENCES

[1] Bagemihl, F.: Curvilinear cluster sets of arbitrary functions. Proc. Nat. Acad. Sci. U.S.A. **41**, 379–382 (1955).
[2] ————, G. Piranian, and G.S. Young: Intersections of cluster sets. Bul. Inst. Politehn. Iasi (N.S.) **5**, 29–34 (1959).
[3] Gresser, J.T.: On uniform approximation by rational functions with an application to chordal cluster sets. Nagoya Math. J. **34**, 143–148 (1969).
[4] Whyburn, G.T.: Topological analysis. Princeton (1958).

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