

A TEST FOR COMMUTATORS

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1. Introduction. In the course of a study of commutator subgroups I. D. Macdonald [1] presented the free nilpotent group G_4 of class 2 on 4 generators as an example of a nilpotent group whose commutator subgroup has elements that are not commutators. To demonstrate this he proceeded as follows: let $G_4 = \langle a_1, a_2, a_3, a_4 \rangle$ and put $c_{ij} = [a_i, a_j]$ for $1 \leq i < j \leq 4$. Then the relations in G_4 are $[c_{ij}, a_k] = 1$ for $1 \leq i < j \leq 4$ and $1 \leq k \leq 4$, and their consequences. Macdonald observed that an arbitrary commutator may be written as

$$[a_1^{\alpha_1} a_2^{\alpha_2} a_3^{\alpha_3} a_4^{\alpha_4}, a_1^{\beta_1} a_2^{\beta_2} a_3^{\beta_3} a_4^{\beta_4}],$$

which simplifies to

$$\prod_{1 \leq i < j \leq 4} c_{ij}^{\delta_{ij}},$$

where $\delta_{ij} = \alpha_i \beta_j - \alpha_j \beta_i$. The indices δ_{ij} satisfy the relation

$$\delta_{12} \delta_{34} - \delta_{13} \delta_{24} + \delta_{14} \delta_{23} = 0. \tag{1}$$

It follows that the element $c_{13} c_{24}$ in G_4 (for which $\delta_{12} = \delta_{14} = \delta_{23} = \delta_{34} = 0$ and $\delta_{13} = \delta_{24} = 1$) is not a commutator.

This note was prompted by the discovery that relation (1) is not only necessary but also sufficient for the element $\prod c_{ij}^{\delta_{ij}}$ to be a commutator in the group G_4 . We shall prove a generalization to finitely generated nilpotent groups of class 2. As an application we prove that if the commutator subgroup G' of a group G is central in G and can be generated by one or two elements, then every element of G' is a commutator. The case of cyclic commutator subgroups was considered by Macdonald [2] by a different method. (He proved that G' is generated by a commutator, but it is an immediate consequence that G' consists entirely of commutators.)

2. The main theorem. Throughout this section let G denote a nilpotent group of class 2 that is generated by elements a_1, \dots, a_n , which need not be independent. Put $c_{ij} = [a_i, a_j]$ for $1 \leq i < j \leq n$. Then $c_{ji} = c_{ij}^{-1}$. An element c of the commutator subgroup G' is expressible in the form

$$c = \prod_{1 \leq i < j \leq n} c_{ij}^{\delta_{ij}}. \tag{2}$$

It is convenient to set $\delta_{ji} = -\delta_{ij}$ (and in particular $\delta_{ii} = 0$) for all $1 \leq i, j \leq n$. We associate with the expression (2) for the element c the $n \times n$ skew-symmetric matrix $\Delta = (\delta_{ij})$. Put

$$\Delta_{qrst} = \delta_{qr} \delta_{st} - \delta_{qs} \delta_{rt} + \delta_{qt} \delta_{rs} \tag{3}$$

whenever $1 \leq q, r, s, t \leq n$. The following rules are easily verified:

(a) If τ is a transposition interchanging two of the symbols q, r, s, t , then

$$\Delta_{q\tau r s t \tau} = -\Delta_{qrst}.$$

(b) If two of q, r, s, t are equal then $\Delta_{qrst} = 0$.

We have the following result.

THEOREM 1. *The element c of G' as given by (2) is a commutator if the associated matrix Δ satisfies*

$$\Delta_{qrst} = 0 \quad \text{for all } 1 \leq q, r, s, t \leq n. \tag{4}$$

REMARK. In view of properties (a) and (b), conditions (4) are equivalent to the conditions $\Delta_{qrst} = 0$ whenever $1 \leq q < r < s < t \leq n$. These conditions hold trivially when $n \leq 3$, so a special case of Theorem 1 is that C' consists entirely of commutators whenever G can be generated by not more than three elements.

Before proving Theorem 1 we shall establish the following lemma concerning the effect that certain changes of generating set of G have on the representation of c as a product of commutators. This will allow us to prove the existence of a presentation relative to which c is clearly displayed as a commutator.

LEMMA 1. *Consider the following types of ‘‘elementary’’ changes of generating sets of G :*

Type I (Transposition): $a_i^ = a_{i\tau}$ for all i , where τ is a transposition (kl) , $1 \leq k, l \leq n$.*

Type II (Inversion): $a_k^ = a_k^{-1}$, $a_i^* = a_i$ for all $i \neq k$.*

Type III (Combination): $a_k^ = a_k a_l^\alpha$ ($k \neq l, \alpha$ an integer), $a_i^* = a_i$ for all $i \neq k$.*

Put $c_{ij}^* = [a_i^*, a_j^*]$. Let the element c of G' be expressible as

$$c = \prod_{1 \leq i < j \leq n} c_{ij}^{\delta_{ij}}$$

relative to the generating set $\{c_{ij}, 1 \leq i < j \leq n\}$ of G' . Then there exists an expression for c of the form

$$c = \prod_{1 \leq i < j \leq n} c_{ij}^{*\delta_{ij}^*},$$

where the entries of the corresponding skew-symmetric matrices $\Delta = (\delta_{ij})$ and $\Delta^* = (\delta_{ij}^*)$ are related as follows:

Type I: $\delta_{ij}^* = \delta_{i\tau j\tau}$; in particular $\delta_{kl}^* = \delta_{lk} = -\delta_{kl}$.
 $\Delta_{qrst}^* = \Delta_{q\tau r\tau s\tau t\tau}$.

Type II: $\delta_{ik}^* = -\delta_{ik}$ for all i ; $\delta_{ij}^* = \delta_{ij}$ otherwise.

$$\Delta_{qrst}^* = (-1)^\lambda \Delta_{qrst}, \quad \text{where } \lambda = \begin{cases} 1 & \text{if } k \in \{q, r, s, t\} \\ 0 & \text{if } k \notin \{q, r, s, t\}. \end{cases}$$

Type III: $\delta_{il}^* = \delta_{il} - \alpha \delta_{ik}$ for all i , $\delta_{ij}^* = \delta_{ij}$ when $i, j \neq l$.

$$\Delta_{qrst}^* = \Delta_{qrst} \quad \text{if } l \notin \{q, r, s, t\}$$

$$\Delta_{lrst}^* = \Delta_{lrst} - \alpha \Delta_{krst}.$$

In particular, if $\Delta_{qrst} = 0$ always, then $\Delta_{qrst}^* = 0$ always.

Proof. Routine calculations, using well known relations such as $[uv, w] = [u, w][v, w]$ that hold in nilpotent groups of class 2.

Let us say that the matrix $\bar{\Delta}$ is *equivalent* to Δ if $\bar{\Delta}$ is related to Δ through a finite sequence of transformations of Types I, II and III. These transformations are clearly invertible, so we are indeed dealing with an equivalence relation.

Proof of Theorem 1. The proof is by induction on n , the number of generators in the generating set a_1, \dots, a_n of G . The theorem is trivially true when $n = 1$ or 2 . So take G on n generators, $n \geq 3$, and assume the validity of Theorem 1 for all nilpotent groups of class 2 defined on fewer than n generators. Let c be an element of G' as given by (2) and such that $\Delta_{qrst} = 0$ for all $1 \leq q, r, s, t \leq n$. We shall prove that c is a commutator. In view of the induction hypothesis we may assume that c has the form

$$c = \prod_{1 \leq k \leq n-1} c_{kn}^{\delta_{kn}} \prod_{1 \leq i < j \leq n-1} c_{ij}^{\delta_{ij}},$$

where some δ_{kn} is non-zero. Let d be the highest common factor of $\delta_{1n}, \delta_{2n}, \dots, \delta_{n-1n}$, and put $\delta_{kn} = d\alpha_k$ for all $1 \leq k \leq n-1$. Then there is a generating set $\bar{a}_1, \bar{a}_2, \dots, \bar{a}_n$ of G_n where in particular

$$\bar{a}_1 = a_1^{\alpha_1} a_2^{\alpha_2} \dots a_{n-1}^{\alpha_{n-1}} \quad \text{and} \quad \bar{a}_n = a_n,$$

and this set is obtainable from a_1, a_2, \dots, a_n by a sequence of elementary changes of Types I, II and III. Hence there exists a matrix $\bar{\Delta} = (\bar{\delta}_{ij})$ equivalent to Δ and a corresponding expression for c of the form

$$c = \prod_{1 \leq i < j \leq n} \bar{c}_{ij}^{\bar{\delta}_{ij}},$$

where $\bar{\delta}_{1n} = d$ and $\bar{\delta}_{kn} = 0$ for all $2 \leq k \leq n-1$. So

$$c = \bar{c}_{1n}^d \prod_{1 \leq i < j \leq n-1} \bar{c}_{ij}^{\bar{\delta}_{ij}}.$$

For the case $n = 3$ this gives $c = \bar{c}_{13}^d \bar{c}_{12}^{\bar{\delta}_{12}} = [\bar{a}_1, \bar{a}_2^{\bar{\delta}_{12}} \bar{a}_3^d]$, proving that c is a commutator. Suppose then that $n \geq 4$. By Lemma 1, $\bar{\Delta}_{qrst} = 0$ for all $1 \leq q, r, s, t \leq n$. In particular, let us choose q, r, s and t such that $q = 1 < r < s < n = t$. Since $\bar{\delta}_{kn} = 0$ for $1 < k < n$, we obtain $\bar{\delta}_{1n} \bar{\delta}_{rs} = 0$, and hence $\bar{\delta}_{rs} = 0$. Therefore

$$c = \bar{c}_{1n}^d \prod_{j=2}^{n-1} \bar{c}_{1j}^{\bar{\delta}_{1j}}$$

which is clearly a commutator of the form $[a_1, x]$. This completes the proof of Theorem 1.

3. Applications. We shall apply our test for commutators to prove the following result.

THEOREM 2. *Suppose the commutator subgroup G' of a group G is central in G and generated by not more than two generators. Then every element of G' is a commutator.*

Proof. The case G' a cyclic group. It is sufficient to consider G' of infinite order, for it is easy to see that if G' is finite and cyclic then G is a homomorphic image of a nilpotent group of class 2 with an infinite cyclic commutator subgroup. An alternative short proof is given by Macdonald [2].

Suppose then that $G' = \langle c \rangle$, an infinite cycle. On the assumption that G' does not consist entirely of commutators, there exists a minimal positive integer $\gamma > 1$ such that c^γ is a commutator. Macdonald [2] showed that G' is then a two-generator group, $G' = \langle c^\gamma, c^\delta \rangle$, where $\delta > \gamma$, $(\delta, \gamma) = 1$ and there exist elements a_1, a_2, a_3, a_4 in G such that

$$c^\gamma = [a_1, a_2] = c_{12}, \quad c^\delta = [a_3, a_4] = c_{34}. \tag{5}$$

Put $c_{ij} = [a_i, a_j]$ for $1 \leq i, j \leq 4$. Since G' is cyclic there exist relations

$$c_{ij} = c^{\gamma_{ij}}, \quad ij = 13, 14, 23 \text{ and } 24. \tag{6}$$

Moreover, from the minimality of γ it follows that γ divides γ_{ij} in each case. To see this, suppose for example that $\gamma_{13} = q\gamma + r$, where $0 < r < \gamma$. Then $c^r = [a_1, a_2^{-q} a_3]$, which, being a commutator, gives a contradiction. So we may write the relations (6) in the form

$$c_{13} = c_{12}^{\alpha_1}, \quad c_{14} = c_{12}^{\alpha_2}, \quad c_{23} = c_{12}^{\alpha_3}, \quad c_{24} = c_{12}^{\alpha_4}. \tag{7}$$

We now regard G' as the commutator subgroup of the subgroup $H = \langle a_1, a_2, a_3, a_4 \rangle$ of G . Consider the change of generators of H given by

$$a_1^* = a_1, \quad a_2^* = a_2, \quad a_3^* = a_3 a_2^{-\alpha_1}, \quad a_4^* = a_4 a_2^{-\alpha_2}.$$

Putting $c_{ij}^* = [a_i^*, a_j^*]$, we have

$$\begin{cases} c_{12}^* = c_{12} = c^\gamma, & c_{13}^* = c_{14}^* = 1, & c_{23}^* = c_{24}^* = c_{12}^{*\alpha_3}, \\ c_{24}^* = c_{24} = c_{12}^{*\alpha_4}, & c_{34}^* = c^{\delta^*}, \end{cases} \tag{8}$$

where $\delta^* = \delta + \gamma(\alpha_2 \alpha_3 - \alpha_1 \alpha_4)$. Since $(\delta, \gamma) = 1$ it follows that $(\delta^*, \gamma) = 1$. Let h and k be integers such that $h\gamma + k\delta^* = 1$. Therefore

$$c = c^{h\gamma + k\delta^*} = c_{12}^{*h} c_{34}^{*k} (c_{13}^*)^\lambda (c_{24}^* c_{12}^{*-\alpha_4})^\mu, \tag{9}$$

and, by virtue of relations (8), the expression (9) for c is true for all integers λ and μ . To complete the proof we shall demonstrate the existence of integers λ and μ that make the right-hand side of (9) a commutator. We note that the matrix Δ^* associated with the expression (9) for c is such that

$$\Delta_{1234}^* = (h - \alpha_4 \mu)k - \lambda \mu.$$

This can be made zero by choosing $\mu = 1, \lambda = (h - \alpha_4)k$. Indeed, we observe that

$$c = c_{12}^{*h} c_{34}^{*k} c_{13}^{*(h - \alpha_4)k} c_{24}^* c_{12}^{*-\alpha_4} = [a_1^{*h - \alpha_4} a_4^{*-1}, a_2^* a_3^{*k}].$$

The case $G' = \langle c, d \rangle$. We shall first prove that G' is generated by two commutators. Put $D = \langle d \rangle$. Then G/D has cyclic commutator subgroup $\langle cD \rangle$ consisting of commutators, as has

just been proved. So there exists an integer γ such that cd^γ is a commutator in G . Now put $N = \langle cd^\gamma \rangle$. Then G/N has cyclic commutator subgroup $\langle dN \rangle$ and so there exists an integer μ such that $d(cd^\gamma)^\mu$ is a commutator in G . The commutators $cd, d(cd^\gamma)^\mu$ generate G' .

Let us simplify the notation and take c and d to be commutators generating G' . We put

$$c = [a_1, a_2] = c_{12}, \quad d = [a_3, a_4] = c_{34},$$

and proceed to show that every element of G' is a commutator in the subgroup H of G generated by a_1, a_2, a_3, a_4 . As usual, put $c_{ij} = [a_i, a_j]$. There exist relations

$$c_{13} = c_{12}^{\alpha_1} c_{34}^{\beta_1}, \quad c_{14} = c_{12}^{\alpha_2} c_{34}^{\beta_2}, \quad c_{23} = c_{12}^{\alpha_3} c_{34}^{\beta_3}, \quad c_{24} = c_{12}^{\alpha_4} c_{34}^{\beta_4}. \tag{10}$$

Suppose that not all β_i are zero. Suppose also that of all presentations of H by 4 generators subject to relations of type (10) our presentation involves the numerically smallest possible non-zero index of c_{34} . By a relabelling of generators, if necessary, we can ensure that this index is β_1 .

We claim that β_1 divides β_2 . For if not, there exist q and r such that $\beta_2 = q\beta_1 + r$, $0 < r < |\beta_1|$. The Type III transformation $a_4^* = a_4 a_3^{-q}$, $a_i^* = a_i$, $i = 1, 2, 3$, yields relations of type (10) and including $c_{14}^* = c_{12}^{\alpha_2} c_{34}^{*r}$, and this contradicts the minimality of β_1 . By performing a Type III transformation just described we may arrange that $\beta_2 = 0$. Similar considerations apply to β_3 , which may be set to zero without affecting the values of β_1 or β_2 .

Thus the relations (10) may be taken to be of the form

$$c_{13} = c_{12}^{\alpha_1} c_{34}^{\beta_1}, \quad c_{14} = c_{12}^{\alpha_2}, \quad c_{23} = c_{12}^{\alpha_3}, \quad c_{24} = c_{12}^{\alpha_4} c_{34}^{\beta_4}. \tag{11}$$

This form also applies (with $\beta_1 = \beta_4 = 0$) to the case that in (10) all β_i are zero.

Now let c be an arbitrary element of G' . Then there exist integers δ, ϵ such that, for all integers λ and μ ,

$$c = c_{12}^{\delta} c_{34}^{\epsilon} (c_{14} c_{12}^{-\alpha_2})^\lambda (c_{23} c_{12}^{-\alpha_3})^\mu. \tag{12}$$

Let Δ be the 4×4 matrix associated with the expression (12) for c . Then

$$\Delta_{1234} = (\delta - \alpha_2 \lambda - \alpha_3 \mu) \epsilon + \lambda \mu = \mu(\lambda - \alpha_3 \epsilon) - \epsilon(\alpha_2 \lambda - \delta),$$

and this can be made zero by choosing

$$\lambda = \alpha_3 \epsilon + 1, \quad \mu = \epsilon(\alpha_2 \lambda - \delta).$$

Therefore the element c of G' is a commutator. (Using the above values it will be found that

$$c = [a_1^{\alpha_3 \epsilon + 1} a_3^\epsilon, a_4 a_2^{\delta - \alpha_2 - \alpha_2 \alpha_3 \epsilon}],$$

an expression which would hardly have been discovered without an algorithmic approach.) This completes the proof of Theorem 2.

Concerning generalizations of Theorem 2, we do not know whether the result extends to rank 3 central commutator subgroups. In this direction Rodney [3] has shown that a *finite* commutator subgroup that is central and is generated by not more than 3 generators contains commutators only. Finally, Theorem 2 does not extend to rank 4 commutator subgroups. Consider the quotient group

$$G_4/R, \quad \text{where } R = \langle c_{13}, c_{14} \rangle.$$

The commutator subgroup $(G_4/R)'$ is torsion free and of rank 4. The element $c_{12}c_{34}R$ is not a commutator, for no element $c_{12}c_{34}c_{13}^\lambda c_{14}^\mu$ of G_4' is a commutator. This follows from Macdonald's argument.

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