BULL. AUSTRAL. MATH. SOC. 30C50
VOL. 43 (1991) [317–324]

SOME INEQUALITIES ON QUASI-SUBORDINATE FUNCTIONS

FUYAO REN, SHIGEYOSHI OWA AND SEIICHI FUKUI

The object of the present paper is to derive some interesting coefficient estimates for quasi-subordinate functions. Furthermore, a conjecture for quasi-subordinate functions is shown.

1. INTRODUCTION

Let $g(z)$ and $f(z)$ be analytic in the unit disk $U = \{z\colon |z| < 1\}$. A function $g(z)$ is said to be subordinate to $f(z)$ if there exists an analytic function $w(z)$ in the unit disk U with $w(0) = 0$ and $|w(z)| < 1$ $(z \in U)$, such that $g(z) = f(w(z))$. We denote this subordination by $g(z) \prec f(z)$. The concept of subordination can be traced to Lindelöf [1], but Littlewood [2, 3] and Rogosinski [5, 6] introduced the term and discovered the basic properties.

Further, a function $g(z)$ is said to be quasi-subordinate to $f(z)$ in the unit disk U if there exist the functions $\phi(z)$ and $w(z)$ (with constant coefficient zero) which are analytic and bounded by one in the unit disk U, such that

$$g(z) = \phi(z) f(w(z)). \tag{1.1}$$

Also, we denote this quasi-subordination by $g(z) \prec_q f(z)$. It is clear that the quasi-subordination is a generalisation of the subordination. The quasi-subordination was introduced by Robertson [4].

2. COEFFICIENT ESTIMATES

We begin with the statement of the following lemma due to Xia and Chang [7].

LEMMA. *Let* $w(z) = \sum_{n=1}^{\infty} c_n z^n$ *be analytic in the unit disk* U *and* $|w(z)| < 1$ $(z \in U)$. *Then*

$$|c_1| \leqslant 1, \tag{2.1}$$

$$|c_2| \leqslant 1 - |c_1|^2, \tag{2.2}$$

$$\left|c_3\left(1 - |c_1|^2\right) + \bar{c}_1 c_2^2\right| \leqslant \left(1 - |c_1|^2\right)^2 - |c_2|^2. \tag{2.3}$$

Applying the above lemma, we prove

Received 14th May 1990
This research of the authors was completed at Kinki University while the first author was visiting from Fudan University, Shanghai, People's Republic of China.

Theorem 1. *Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ and $g(z) = \sum_{n=0}^{\infty} b_n z^n$ be analytic in the unit disk U and let $g(z)$ be quasi-subordinate to $f(z)$. Then*

(i) $|b_0| \leqslant |a_0|$ *with equality only if* $g(z) = e^{i\theta} f(w(z))$ *(θ is real) in which* $w(z)$ *is analytic with* $w(0) = 0$ *and* $|w(z)| < 1$ $(z \in U)$.

(ii) $|b_1| \leqslant (5/4) \max(|a_0|, |a_1|)$ *with equality only if* $g(z) = \phi(z) f(w(z))$,

$$\phi(z) = e^{i\beta} \frac{z - \alpha}{1 - \overline{\alpha} z} \qquad (|\alpha| = 1/2;\ \beta \text{ is real}).$$

In particular, $g(z) = ((1 + 2z)/(2 + z)) f(z)$ *with* $a_0 = a_1$ *is an extremal function.*

(iii) $|b_2| \leqslant (7/27)(1 + 2\sqrt{7}) \max(|a_0|, |a_1|, |a_2|)$ *with equality only if*

$$g(z) = e^{i\beta} \frac{z - \alpha}{1 - \overline{\alpha} z} f(e^{i\theta} z) \qquad \left(|\alpha| = t = \left(\sqrt{7} - 1\right)/3\right),$$

where β and θ are real. In particular, $g(z) = ((z + t)/(1 + tz)) f(z)$, *with* $-a_0 = a_1 = a_2$ *is an extremal function.*

Proof: Note that there exist the functions $\phi(z) = \sum_{n=0}^{\infty} \alpha_n z^n$ and $w(z) = \sum_{n=1}^{\infty} c_n z^n$ such that $g(z) = \phi(z) f(w(z))$. Comparing the coefficients of both sides, we have

$$b_0 = \alpha_0 a_0, \tag{2.4}$$

$$b_1 = \alpha_0 c_1 a_1 + \alpha_1 a_0, \tag{2.5}$$

$$b_2 = \alpha_0 \left(c_1^2 a_2 + c_2 a_1\right) + \alpha_1 c_1 a_1 + \alpha_2 a_0. \tag{2.6}$$

It follows from (2.4), (2.5), and (2.6) that

$$|b_0| = |\alpha_0 a_0|, \tag{2.7}$$

$$|b_1| \leqslant (|\alpha_0 c_1| + |\alpha_1|) \max(|a_0|, |a_1|), \tag{2.8}$$

$$|b_2| \leqslant \left(\left|\alpha_0 c_1^2\right| + |\alpha_0 c_2| + |\alpha_1 c_1| + |\alpha_2|\right) \max(|a_0|, |a_1|, |a_2|), \tag{2.9}$$

respectively. Since $|\phi(z)| < 1$ $(z \in U)$, we see that the function $\phi_1(z)$ defined by

$$\phi_1(z) = \frac{\phi(z) - \alpha_0}{1 - \overline{\alpha_0} \phi(z)} = \sum_{n=1}^{\infty} d_n z^n \tag{2.10}$$

is analytic and bounded by one in the unit disk U. Note that

$$d_1 = \frac{\alpha_1}{1 - |\alpha_0|^2} \tag{2.11}$$

and

$$d_2 = \frac{1}{1-|\alpha_0|^2}\left(\alpha_2 + \frac{\alpha_1^2\overline{\alpha_0}}{1-|\alpha_0|^2}\right). \tag{2.12}$$

Therefore, using the Lemma, we have

$$|\alpha_1| \leqslant 1 - |\alpha_0|^2 \tag{2.13}$$

and

$$|d_2| \leqslant 1 - |d_1|^2. \tag{2.14}$$

With the aid of (2.11), (2.12) and (2.14), we obtain

$$|\alpha_2| \leqslant \left(1-|\alpha_0|^2\right) - \frac{|\alpha_1|^2}{1+|\alpha_0|}. \tag{2.15}$$

Thus, it follows from (2.13) that $|b_0| \leqslant |a_0|$ with equality only if $g(z) = e^{i\theta} f(w(z))$.

Further, using (2.1) and (2.13), we obtain

$$\begin{aligned} |b_1| &\leqslant \left(1+|\alpha_0|-|\alpha_0|^2\right)\max(|a_0|, |a_1|) \\ &\leqslant \frac{5}{4}\max(|a_0|, |a_1|) \end{aligned} \tag{2.16}$$

with equality only if

$$g(z) = e^{i\beta}\frac{z-\alpha}{1-\overline{\alpha}z} f(e^{i\theta}z) \qquad (|\alpha| = t = 1/2).$$

In particular, $g(z) = ((z+t)/(1+tz))f(z)$ is an extremal function.

Next, let

$$U = |\alpha_0 c_1^2| + |\alpha_0 c_2| + |\alpha_1 c_1| + |\alpha_2|. \tag{2.17}$$

Then, using (2.2) and (2.15), we have

$$U \leqslant |\alpha_0| + |\alpha_1 c_1| + \left(1-|\alpha_0|^2\right) - \frac{|\alpha_1|^2}{1+|\alpha_0|}. \tag{2.18}$$

Letting $t = |\alpha_0|$, $x = |c_1|$ and $y = |\alpha_1|$, (2.18) can be written in the form

$$U \leqslant 1 + t - t^2 + xy - y^2/(1+t) \equiv \psi(y). \tag{2.19}$$

https://doi.org/10.1017/S0004972700029117 Published online by Cambridge University Press

Clearly, $y_0 = x(1+t)/2$ is the maximum point of $\psi(y)$. If $y_0 \geqslant 1-t^2$, that is, if $1 - x/2 \leqslant t \leqslant 1$, $0 \leqslant x \leqslant 1$, then $y \leqslant 1 - t^2$, so

$$\begin{aligned} U &\leqslant \psi(1-t^2) \\ &= t + (x+t)(1-t^2) \\ &\leqslant 1 + 2t - t^2 - t^3 \equiv \Lambda(t). \end{aligned} \tag{2.20}$$

It is easy to see that $t_0 = (\sqrt{7}-1)/3$ is the maximum point of $\Lambda(t)$ of which the maximum is

$$\Lambda(t_0) = \frac{7(1+2\sqrt{7})}{27} = 1.631\,130\,309. \tag{2.21}$$

Hence, when $y_0 \geqslant 1 - t^2$, we obtain

$$U \leqslant \Lambda(t_0) = \frac{7(1+2\sqrt{7})}{27}. \tag{2.22}$$

If $y_0 = x(1+t)/2 < 1 - t^2$, that is, if $0 \leqslant t < 1 - x/2$, then

$$U \leqslant \psi(y) = \left(1 + \frac{x^2}{4}\right)(1+t) - t^2 \equiv \Omega(t). \tag{2.23}$$

Since $t_0 = (1 + x^2/4)/2$ is the maximum point of $\Omega(t)$, if $t_0 < 1 - x/2$, or, if $0 \leqslant x < 2(\sqrt{2}-1)$, then

$$U \leqslant \Omega(t_0) = 1 + \frac{x^2}{4} + \frac{1}{4}\left(1 + \frac{x^2}{4}\right)^2 < 10 - 6\sqrt{2}, \tag{2.24}$$

and, if $t_0 \geqslant 1 - x/2$, or, if $2(\sqrt{2}-1) \leqslant x \leqslant 1$, then

$$U \leqslant \Omega(1 - x/2) = \frac{8 + 4x + 2x^2 - x^3}{8} \leqslant \frac{13}{8}. \tag{2.25}$$

Therefore, it follows from (2.22), (2.24) and (2.25) that

$$|b_2| \leqslant \frac{7(1+2\sqrt{7})}{27} \max(|a_0|, |a_1|, |a_2|).$$

Finally, from the above process of the proof, we know that the extremal function $g(z) = \phi(z) f(w(z))$ occurs only if

$$|\alpha_0| = t = \frac{\sqrt{7}-1}{3}, \; |\alpha_1| = 1 - t^2 = \frac{1+2\sqrt{7}}{9},$$
$$|\alpha_2| = 1 - t^2 - \frac{|\alpha_1|^2}{1+t} = \frac{13-\sqrt{7}}{27}, \; |c_1| = 1, \; c_2 = 0.$$

Hence, $w(z) = e^{i\theta}z$, $\phi(z) = e^{i\beta}((z-\alpha)/(1-\overline{\alpha}z))$, $|\alpha| = t$. In particular, $g(z) = ((z+t)/(1+tz))f(z)$ with $-a_0 = a_1 = a_2$ is an extremal function. This completes the proof of Theorem 1. ∎

Next we derive

THEOREM 2. *Let $g(z) = \sum_{n=0}^{\infty} b_n z^n$ be quasi-subordinate to $f(z) = \sum_{n=0}^{\infty} a_n z^n$, and let $G(z) = \sum_{n=1}^{\infty} B_n z^n$ be quasi-subordinate to $F(z) = \sum_{n=1}^{\infty} A_n z^n$. Further, let*

$$U_n = \inf \{u_n : |b_n| \leqslant u_n \max(|a_0|, |a_1|, \dots, |a_n|) \text{ for all } f(z) \text{ and } g(z) \text{ such that } g(z) \prec_q f(z)\}$$

and let

$$V_n = \inf \{v_n : |B_n| \leqslant v_n \max(|A_1|, |A_2|, \dots, |A_n|) \text{ for all } F(z) \text{ and } G(z) \text{ such that } G(z) \prec_q F(z)\}.$$

Then

$$|b_n| \leqslant U_n \max(|a_0|, |a_1|, \dots, |a_n|), \tag{2.26}$$

$$|B_n| \leqslant V_n \max(|A_1|, |A_2|, \dots, |A_n|), \tag{2.27}$$

and

$$V_n \leqslant U_{n-1}. \tag{2.28}$$

Furthermore, for an arbitrary given natural number n, if the extremal function is $g(z) = \phi(z)f(w(z))$ which makes

$$|b_{n-1}| = U_{n-1} \max(|a_0|, |a_1|, \dots, |a_{n-1}|)$$

and $z\phi(z)/w(z)$ is analytic in U, then

$$V_n = U_{n-1}. \tag{2.29}$$

PROOF: The inequalities (2.26) and (2.27) are clear from the definitions of U_n and V_n. In order to prove (2.28), we first show that if $G(z) \prec_q F(z)$, then $G(z)/z \prec_q F(z)/z$. Indeed, since $G(z) \prec_q F(z)$, there exist the analytic functions $\Phi(z)$ and $w(z)$ (with constant coefficient zero) bounded by one in the unit disk U such that $G(z) = \Phi(z)F(w(z))$. Thus we have

$$\frac{G(z)}{z} = \phi(z)H(w(z)),$$

where $H(z) = F(z)/z$ and $\phi(z) = \Phi(z)w(z)/z$. This shows that $G(z)/z$ is quasi-subordinate to $F(z)/z$.

If $G(z) = \Phi(z)F(w(z))$ is an extremal function of (2.27), then

$$|B_n| = V_n \max(|A_1|, |A_2|, \ldots, |A_n|). \tag{2.30}$$

Since $G(z)/z$ is quasi-subordinate to $F(z)/z$, by (2.27), we get

$$|B_n| \leqslant U_{n-1} \max(|A_1|, |A_2|, \ldots, |A_n|). \tag{2.31}$$

Consequently, we have $V_n \leqslant U_{n-1}$ for all n.

Finally, in order to show (2.29), let $G(z) = zg(z)$, $F(z) = zf(z)$ and $\Phi(z) = z\phi(z)/w(z)$. Since $g(z) = \phi(z)f(w(z))$, we see that $G(z) = \Phi(z)F(w(z))$. Noting that $\Phi(z)$ is analytic in U and bounded by one in U, we conclude that $G(z) \prec_q F(z)$. Since $B_n = b_{n-1}$, $A_1 = a_0, \ldots, A_n = a_{n-1}$, (2.27) gives

$$|b_{n-1}| \leqslant V_n \max(|a_0|, |a_1|, \ldots, |a_{n-1}|). \tag{2.32}$$

But, by the hypothesis

$$|b_{n-1}| = U_{n-1} \max(|a_0|, |a_1|, \ldots, |a_{n-1}|),$$

we have $U_{n-1} \leqslant V_n$. Thus we complete the assertion of Theorem 2. □

COROLLARY. *Let $G(z) = \sum_{n=1}^{\infty} B_n z^n$ be quasi-subordinate to $F(z) = \sum_{n=1}^{\infty} A_n z^n$ in U. Then*

$$|A_1| \leqslant |B_1| \tag{2.33}$$

with equality only if $G(z) = e^{i\beta}F(e^{i\theta}z)$,

$$|B_2| \leqslant \frac{5}{4} \max(|A_1|, |A_2|) \tag{2.34}$$

with equality only if $G(z) = \Phi(z)F(w(z))$ in which $\Phi(z) = e^{i\beta}((z-\alpha)/(1-\overline{\alpha}z))$ $(|\alpha| = 1/2)$, $w(z) = e^{i\theta}z$. In particular, $G(z) = ((1+2z)/(2+z))F(z)$ with $A_1 = A_2$ is an extremal function, and

$$|B_3| \leqslant \frac{7(1+2\sqrt{7})}{27} \max(|A_1|, |A_2|, |A_3|), \tag{2.35}$$

with equality only if

$$G(z) = e^{i\beta}\frac{z-\alpha}{1-\overline{\alpha}z}F(e^{i\theta}z) \qquad \left(|\alpha| = t = \left(\sqrt{7}-1\right)/3\right).$$

In particular, $G(z) = ((z+t)/(1+tz))F(z)$ with $-A_1 = A_2 = A_3$ is an extremal function.

REMARK. The inequalities (2.33) and (2.34) in the Corollary were verified by Robertson [4].

3. Conjecture

In view of the last assertion of Theorem 2, it is enough to study the extremal problem (2.26) if we can prove that the extremal function of the extremal problem (2.26) for n has the property of Theorem 2, because we can immediately obtain the solution of the extremal problem (2.27) for $n+1$.

What can we say on the problem (2.26)? For this problem, it is reasonable to make the following conjecture.

Conjecture: If $g(z) = \sum_{n=0}^{\infty} b_n z^n$ is quasi-subordinate to $f(z) = \sum_{n=0}^{\infty} a_n z^n$, then

$$|b_n| \leqslant U_n \max(|a_0|, |a_1|, \dots, |a_n|), \tag{3.1}$$

where

$$|U_n| = \max U_n(t) = \max_{0 \leqslant t < 1} \left(1 + 2t - t^n - t^{n-1}\right). \tag{3.2}$$

The equality in (3.1) holds only if $g(z) = \phi(z) f(w(z))$ in which $\phi(z) = e^{i\beta}((z-\alpha)/(1-\overline{\alpha}z))$, $w(z) = e^{i\theta}z$, and $t = |\alpha|$ is a maximum point of $U_n(t)$ on $[0, 1]$.

We know that it is true for $n = 1, 2$, but, for $n \geqslant 3$, it is also an open problem.

References

[1] E. Lindelöf, 'Mémoire sur certaines inégalités dans la théorie des fonctions monogénes et sur quelques propriétés nouvelles de ces fonctions dans le voisinage d'un point singulier essentiel', *Acta Soc. Sci. Fenn.* **35** (1909), 1–35.

[2] J.E. Littlewood, 'On inequalities in the theory of functions', *Proc. London Math. Soc.* **23** (1925), 481–519.

[3] J.E. Littlewood, *Lectures on the theory of functions* (Oxford University Press, 1944).

[4] M.S. Robertson, 'Quasi-subordination and coefficient conjecture', *Bull. Amer. Math. Soc.* **76** (1970), 1–9.

[5] W. Rogosinski, 'On subordinate functions', *Proc. Cambridge Philos. Soc.* **35** (1939), 1–26.

[6] W. Rogosinski, 'On the coefficients of subordinate functions', *Proc. London Math. Soc.* **48** (1943), 48–82.

[7] D. Xia and K. Chang, 'Some inequalities on subordinate functions', *Acta. Math.* **8** (1958), 408–412.

Department of Mathematics
Fudan University
Shanghai
People's Republic of China

Department of Mathematics
Kinki University
Higashi-Osaka, Osaka 577
Japan

Department of Mathematics
Faculty of Education
Wakayama University
Wakayama 640
Japan