LATTICE ISOMORPHISMS OF LIE ALGEBRAS

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1. Introduction

Let L be a finite dimensional Lie algebra over the field F. We denote by $\mathscr{L}(L)$ the lattice of all subalgebras of L. By a lattice isomorphism (which we abbreviate to $\mathscr{L}$-isomorphism) of L onto a Lie algebra M over the same field F, we mean an isomorphism of $\mathscr{L}(L)$ onto $\mathscr{L}(M)$. It is possible for non-isomorphic Lie algebras to be $\mathscr{L}$-isomorphic, for example, the algebra of real vectors with product the vector product is $\mathscr{L}$-isomorphic to any 2-dimensional Lie algebra over the field of real numbers. Even when the field F is algebraically closed of characteristic 0, the non-nilpotent Lie algebra $L = \langle a, b_1, \cdots, b_r \rangle$ with product defined by $ab_i = b_i$, $b_i b_j = 0$ $(i, j = 1, 2, \cdots, r)$ is $\mathscr{L}$-isomorphic to the abelian algebra of the same dimension[1]. In this paper, we assume throughout that F is algebraically closed of characteristic 0 and are principally concerned with semi-simple algebras. We show that semi-simplicity is preserved under $\mathscr{L}$-isomorphism, and that $\mathscr{L}$-isomorphic semi-simple Lie algebras are isomorphic.

We write mappings exponentially, thus the image of A under the map φ will be denoted by A^φ. If $a_1, \cdots, a_n$ are elements of the Lie algebra L, we denote by $\langle a_1, \cdots, a_n \rangle$ the subspace of L spanned by $a_1, \cdots, a_n$, and denote by $\langle\langle a_1, \cdots, a_n \rangle\rangle$ the subalgebra generated by $a_1, \cdots, a_n$. For a single element a, $\langle a \rangle = \langle\langle a \rangle\rangle$. The product of two elements $a, b \in L$ will be denoted by ab. We use brackets only for products of more than two elements. Put

$$\ell(L) = \text{length of longest chain in } \mathscr{L}(L)$$
$$d(L) = \text{dimension of } L.$$

Then clearly $d(L) \geqq \ell(L)$. If L is soluble, then

$$\ell(L) = d(L) = \text{length of a composition series of } L.$$

We remark that, if L is insoluble (over an algebraically closed field of characteristic 0) then L has a subalgebra isomorphic to the simple algebra

[1] For some theorems on $\mathscr{L}$-isomorphisms between Lie algebras L, M both assumed nilpotent, see Barnes and Wall [1].

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$\mathfrak{A}_1$. If R is the radical of L, then by Levi's theorem [2], L has a subalgebra A isomorphic to L/R. A is semi-simple and (since L is insoluble) non-trivial. If α is a root of A, $e_\alpha, e_{-\alpha}$, eigenvectors for α and $-\alpha$ and $h_\alpha = e_\alpha e_{-\alpha}$, then $\langle h_\alpha, e_\alpha, e_{-\alpha}\rangle$ is a subalgebra isomorphic to $\mathfrak{A}_1$.

If $\langle h\rangle$ is a Cartan subalgebra of $\mathfrak{A}_1$ and e is a corresponding eigenvector, then

$$0 < \langle h\rangle < \langle h, e\rangle < \mathfrak{A}_1$$

is a chain of length 3 in $\mathscr{L}(\mathfrak{A}_1)$. It follows that, if $\ell(L) \leqq 3$, we have $\ell(L) \quad d(L)$ since L must either be soluble or isomorphic to $\mathfrak{A}_1$.

2. The radical

Lemma 1. *L is isomorphic to $\mathfrak{A}_1$ if and only if L has the properties:*

(i) $\ell(L) = 3$

(ii) *There exists $H < L$, $\ell(H) = 1$ such that there are exactly two subalgebras $A, B < L$ containing H.*

(iii) *For every $U < A$, $U \neq 0, H$, there exists $V < B$ such that $U \cup V = L$. For every $V < B$, $V \neq 0, H$, there exists $U < A$ such that $U \cup V = L$.*

The subalgebras H with the above properties are the Cartan subalgebras. The subalgebra $A > H$ has precisely one subalgebra $E \neq 0, A$ which is not a Cartan subalgebra of L. E is a weight space for the representation of H on L.

Proof. It is easily verified that if L is isomorphic to $\mathfrak{A}_1$, then the Cartan subalgebras H of L have the properties (ii) and (iii), and that if E is a 1-dimensional subalgebra of L which is not a Cartan subalgebra, then E is contained in exactly one 2-dimensional subalgebra A, E is the only 1-dimensional subalgebra of A which is not a Cartan subalgebra of L, and E is a weight space for each Cartan subalgebra $H < A$. Thus to prove the lemma, it is sufficient to prove that (i), (ii), (iii) imply that L is isomorphic to $\mathfrak{A}_1$.

Since $\ell(L) = 3$, we have $d(L) = 3$. It is sufficient to prove that $L' = L$ as $\mathfrak{A}_1$ is the only 3-dimensional algebra with this property. L can have no 1-dimensional ideal J since, if such a J existed, we would have either $H = J$ contrary to (ii) or we could take $A = H+J$, $U = J$ contrary to (iii). Thus $d(L') \geqq 2$. Suppose $d(L') = 2$. Then L is soluble and, since it has no 1-dimensional ideal, $L'' = 0$. Since $A \neq B$, we can suppose $A \neq L'$. But then $A \cap L'$ is 1-dimensional and is an ideal since it is an ideal in both A and L'. Therefore $L = L'$.

Corollary 1. *If L is $\mathscr{L}$-isomorphic to $\mathfrak{A}_1$, then L is isomorphic to $\mathfrak{A}_1$.*

Proof. The properties (i), (ii), (iii) are all properties of $\mathscr{L}(L)$.

[2] See Jacobson [2], p. 91.

COROLLARY 2. *Let $\varphi : \mathscr{L}(L) \to \mathscr{L}(M)$ be an $\mathscr{L}$-isomorphism. If L is soluble, then so is M.*

PROOF. L has a subalgebra isomorphic to $\mathfrak{A}_1$ if and only if M has. Thus L is insoluble if and only if M is insoluble.

LEMMA 2. *The radical R of L is the intersection of the maximal soluble subalgebras of L.*

PROOF. Every maximal soluble subalgebra of L contains R. We may therefore work in the algebra L/R and so need only consider the case $R = 0$.

Let H be a Cartan subalgebra of the semi-simple algebra L. Let e_α be an eigenvector for the root α. We suppose that the roots have been ordered in the usual manner [3]. Put

$$M = \langle H, e_\alpha | \alpha > 0 \rangle,$$
$$N = \langle H, e_\alpha | \alpha < 0 \rangle.$$

Then M, N are maximal soluble subalgebras of L (the Borel subalgebras) and $M \cap N = H$. It is therefore sufficient to prove that the intersection of the Cartan subalgebras of L is 0.

Suppose $u \in \cap \{H | H \text{ Cartan subalgebra of } L\}$. If x is a regular element of L, then the Fitting null component $L_{0,x}$ of the representation of $\langle x \rangle$ on L is a Cartan subalgebra of L [4]. Since $x \in L_{0,x}$ and the Cartan subalgebras of a semi-simple algebra are abelian, $ux = 0$ for all regular x. But the regular elements x are dense in L in the Zariski topology, and so span L. Thus u is in the centre of L and so $u = 0$.

THEOREM 1. *Let L, M be finite dimensional Lie algebras over the algebraically closed field F of characteristic 0. Let $\phi : \mathscr{L}(L) \to \mathscr{L}(M)$ be an $\mathscr{L}$-isomorphism of L onto M, and let R be the radical of L. Then R^φ is the radical of M.*

PROOF. From Lemma 1 Corollary 2, it follows that ϕ maps maximal soluble subalgebras of L to maximal soluble subalgebras of M. By Lemma 2, this implies that R^φ is the radical of M.

3. Semi-simple algebras

We investigate semi-simple algebras by studying the subalgebras generated by the weight spaces for some Cartan subalgebra.

LEMMA 3. *Let L be an insoluble algebra of dimension 4. Then $L = R \oplus S$ (algebra direct sum) where $R = \langle r \rangle$ is the radical of L and S is isomorphic to $\mathfrak{A}_1$.*

[3] See Jacobson [2] p. 119.
[4] See Jacobson [2], p. 59 Theorem 1.

PROOF. L/R is semi-simple of dimension at most 4 and thus must be isomorphic to $\mathfrak{A}_1$. Thus the radical R is 1-dimensional and $R = \langle r \rangle$ for some $r \in L$. By Levi's theorem, there exists a subalgebra $S < L$ such that $L = R+S$ and $R \cap S = 0$. To prove that $L = R \oplus S$, we have to prove that S is an ideal of L.

We can choose a basis h, e, f of S such that $he = e$, $hf = -f$, $ef = h$ since S is isomorphic to $\mathfrak{A}_1$. Since R is an ideal, $hr = \alpha r$, $er = \beta r$, $fr = \gamma r$ for some $\alpha, \beta, \gamma \in F$. By the Jacobi identity,

$$0 = (re)f + (fr)e + (ef)r = \alpha r$$
$$0 = (rh)e + (er)h + (he)r = \beta r$$
$$0 = (rf)h + (hr)f + (fh)r = \gamma r$$

and therefore $\alpha = \beta = \gamma = 0$.

LEMMA 4. *Let L be a semi-simple algebra and let $\phi : \mathscr{L}(L) \to \mathscr{L}(M)$ be an $\mathscr{L}$-isomorphism. Let H be a Cartan subalgebra of L and let L_α be the weight space of the root α. Then H^φ is a Cartan subalgebra of M and L_α^φ is the weight space of a root α^φ of M.*

PROOF. Since L is semi-simple, L_α is a 1-dimensional and so is a subalgebra. Thus L_α^ϕ is defined. There exist e_α, $e_{-\alpha}$, h_α such that $L_\alpha = \langle e_\alpha \rangle$, $L_{-\alpha} = \langle e_{-\alpha} \rangle$, $h_\alpha = e_\alpha e_{-\alpha} \in H$ and $h_\alpha e_\alpha = e_\alpha$, $h_\alpha e_{-\alpha} = -e_{-\alpha}$. By Lemma 1, we need only consider the case $d(H) > 1$. There exist $h_1, \cdots, h_s$ such that $h_\alpha, h_1, \cdots, h_s$ is a basis of H and $\alpha(h_i) = 0$.

Put $K = H^\varphi$, $\langle k_i \rangle = \langle h_i \rangle^\phi$. By Lemma 1, we can choose $k_{\alpha^\phi}, f_{\alpha^\phi}, f_{-\alpha^\phi}$ such that

$$\langle k_{\alpha^\phi} \rangle = \langle h_\alpha \rangle^\phi, \langle f_{\alpha^\phi} \rangle = \langle e_\alpha \rangle^\phi, \langle f_{-\alpha^\phi} \rangle = \langle e_{-\alpha} \rangle^\phi,$$

and

$$k_{\alpha^\phi} f_{\alpha^\phi} = f_{\alpha^\phi}, \quad k_{\alpha^\phi} f_{-\alpha^\phi} = -f_{-\alpha^\phi}, f_{\alpha^\phi} f_{-\alpha^\phi} = k_{\alpha^\phi}.$$

Since $\langle h_\alpha, e_\alpha, e_{-\alpha}, h_i \rangle$ is an insoluble algebra of dimension 4 with radical $\langle h_i \rangle$, $\langle k_{\alpha^\phi}, f_{\alpha^\phi}, f_{-\alpha^\phi}, k_i \rangle$ is insoluble of dimension 4 with radical $\langle k_i \rangle$. By Lemma 3, $k_i k_{\alpha^\phi} = k_i f_{\alpha^\phi} = k_i f_{-\alpha^\phi} = 0$. Thus k_{α^ϕ} is in the centre of $K = \langle k_{\alpha^\phi}, k_1, \cdots, k_s \rangle$. But the k_{α^ϕ} span K and so K is abelian. For all $k \in K$, $kf_{\alpha^\phi} \in \langle f_{\alpha^\phi} \rangle$. Thus f_{α^ϕ} is an eigenvector for the representation of K on M. Put $kf_{\alpha^\phi} = \alpha^\phi(k) f_{\alpha^\phi}$. Then $\alpha^\phi(k)$ is a weight of the representation of K on M.

Suppose $y \in N(K) = \{m | m \in M,\ mK \subseteqq K\}$. Put $\langle x \rangle = \langle y \rangle^{\phi^{-1}}$. Then $x = h + \sum_\alpha \lambda_\alpha e_\alpha$, $h \in H$, $\lambda_\alpha \in F$. For all $k \in K$, $ky \in K$ and so $\langle K, y \rangle$ is a subalgebra. Therefore, for all $h' \in H$,

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$$h'x = \sum_{\alpha} \lambda_\alpha \alpha(h')e_\alpha \in \langle H, x\rangle$$
$$= \mu x + h'' \quad (\mu \in F, h'' \in H).$$

Therefore $\sum \alpha \lambda_\alpha \alpha(h')e_\alpha = \mu(h') \sum_\alpha \lambda_\alpha e_\alpha$ for all $h' \in H$. Suppose $\lambda_{\alpha_1}, \lambda_{\alpha_2} \neq 0$. Then $\alpha_1(h') = \alpha_2(h')$ for all $h' \in H$, that is, $\alpha_1 = \alpha_2$. Therefore $x = h + \lambda e_\alpha$ and $y = k + \rho f_{\alpha^\phi}$ for some $k \in K$, $\rho \in F$ since

$$y \in \langle x\rangle^\phi \leqq \langle x, e_\alpha\rangle^\phi = \langle h\rangle^\phi \cup \langle e_\alpha\rangle^\phi.$$

But $k_{\alpha^\phi}(k + \rho f_{\alpha^\phi}) = \rho f_{\alpha^\phi}$. Since $k_{\alpha^\phi} y \in K$, we must have $\rho = 0$. Therefore $N(K) = K$ and K is a Cartan subalgebra of M, the α^φ are roots. Since M is semi-simple, the weight spaces M_{ϕ_α} corresponding to the roots α^φ are 1-dimensional. But $f_{\alpha^\phi} \in M_{\alpha^\phi}$ and therefore $L_\alpha^\phi = M_{\alpha^\phi}$.

Corollary. *Let L, M be $\mathscr{L}$-isomorphic Lie algebras over the algebraically closed field F of characteristic 0. Then $d(L) = d(M)$.*

Proof. Let $\phi : \mathscr{L}(L) \to \mathscr{L}(M)$ be an $\mathscr{L}$-isomorphism. Let R be the radical of L. Then R^ϕ is the radical of M and $d(R) = d(R^\phi)$ since R, R^ϕ are soluble. Thus we need only consider the case $R = 0$. Let H be a Cartan subalgebra of L. Then H^ϕ is a Cartan subalgebra of M and $d(H) = d(H^\phi)$. To every root α of L, there corresponds a root α^ϕ of M, and the α^ϕ are all the roots of M by Lemma 4 applied to ϕ and ϕ^{-1}. This correspondence is one-to-one. Since $d(L) = d(H) + 2s$ where $2s$ is the number of roots, we have $d(L) = d(M)$.

Theorem 2. *Let L, M be $\mathscr{L}$-isomorphic Lie algebras over the algebraically closed field F of characteristic 0. Suppose L is semi-simple. Then L and M are isomorphic.*

Proof. Let $\varphi : \mathscr{L}(L) \to \mathscr{L}(M)$ be an $\mathscr{L}$-isomorphism. We use the notation of the proof of Lemma 4 for Cartan subalgebras, weight spaces, etc. We have the one-to-one correspondence $\alpha \leftrightarrow \alpha^\phi$ between the roots of L and M by Lemma 4 applied to φ and φ^{-1}. By a well-known result[5], it is sufficient to prove for all roots α, β of L that $(-\alpha)^\phi = -(\alpha^\phi)$, that $\alpha+\beta$ is a root of L if and only if $\alpha^\phi + \beta^\phi$ is a root of M, and that if $\alpha+\beta$ is a root of L, then $(\alpha+\beta)^\phi = \alpha^\phi + \beta^\phi$.

$\alpha+\beta = 0$ if and only if $\langle\langle e_\alpha, e_\beta\rangle\rangle \cap H \neq 0$. This property is preserved by $\mathscr{L}$-isomorphisms, so $(-\alpha)^\phi = -(\alpha^\phi)$. If $\alpha+\beta \neq 0$, then $\langle\langle e_\alpha, e_\beta\rangle\rangle \subsetneqq \langle e_\gamma | \gamma = r\alpha + s\beta$ root of L; r, s non-negative integers$\rangle$.

$\alpha+\beta$ is a root if and only if $\langle\langle e_\alpha, e_\beta\rangle\rangle \supset \langle e_\gamma\rangle$ for some $\gamma \neq \alpha, \beta$. Therefore $\alpha+\beta$ is a root if and only if $\alpha^\phi + \beta^\phi$ is a root.

[5] This is essentially the assertion of [3] p. 11—06, Corollary 2.

Suppose $\alpha+\beta$ is a root. $\langle e_{\alpha+\beta}\rangle$ is characterised by

(i) $\langle e_{\alpha+\beta}\rangle \subset \langle e_\alpha\rangle \cup \langle e_\beta\rangle$ and

(ii) $\langle e_{\alpha+\beta}\rangle \subset \langle e_\gamma\rangle \cup \langle e_\delta\rangle \subseteqq \langle e_\alpha\rangle \cup \langle e_\beta\rangle, \gamma, \delta \neq \alpha+\beta$

implies either $\gamma = \alpha, \delta = \beta$ or $\gamma = \beta, \delta = \alpha$.
Therefore $(\alpha+\beta)^\phi = \alpha^\phi+\beta^\phi$.

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