J. Austral. Math. Soc. (*Series A*) **28** (1979), 362–368

ON THE RELATIONSHIP BETWEEN A SUMMABILITY MATRIX AND ITS TRANSPOSE

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(Received 21 June 1977; revised 5 February 1979)

Communicated by A. P. Robertson

Abstract

Let E, F be sequence spaces and A an infinite matrix that maps E to F. Sufficient conditions are given so that the transposed matrix maps F^β to E^β.

1980 Mathematics subject classification (*Amer. Math. Soc.*): primary 40 C 05, 40 D 25; secondary 40 H 05.

1. Introduction

Let A be an infinite matrix of complex numbers and A' its transpose. Vermes (1957) considered the relationships between A, as a regular sequence to sequence summability method, and A', as a regular series to series method. Jakimovski and Russell (1972) obtained some additional results on the relationships between A and A', when A is a mapping between BK spaces.

In this note we consider A as a mapping between two sequence spaces, E and F, and determine when A' maps F^β to E^β. The range of corollaries includes some of the results of Jakimovski and Russell (1972), a result of Skerry (1974), and a result related to one announced by Dawson (1976).

2. Preliminaries

A sequence space is a vector subspace of the space ω of all complex sequences. A sequence space E with a locally convex topology, τ, is a K space if the linear functionals

$$x \to x_j, \quad j = 0, 1, 2, \ldots,$$

https://doi.org/10.1017/S1446788700012325 Published online by Cambridge University Press

are continuous. In addition, if (E, τ) is complete and metrizable, then E is an FK space. A normed FK space is a BK space.

If E is a sequence space, we write

$$E^{\beta} = \left\{ y \in \omega : \sum_{j=0}^{\infty} x_j y_j \text{ converges for all } x \in E \right\},$$

$$E^{\alpha} = \left\{ y \in \omega : \sum_{j=0}^{\infty} |x_j y_j| < \infty \text{ for all } x \in E \right\},$$

$$E^{\gamma} = \left\{ y \in \omega : \sup_n \left| \sum_{j=0}^{n} x_j y_j \right| < \infty \text{ for all } x \in E \right\}.$$

Let φ be the space of sequences with only finitely many non-zero terms. In this paper, it will be assumed that all sequence spaces contain φ.

If F is a subspace of E^{β}, then E and F form a dual pair under the bilinear form

$$\langle x, y \rangle = \sum_{j=0}^{\infty} x_j y_j.$$

The weak topology on E by F, $\sigma(E, F)$, is a K space topology. Topologies for dual pairings of the type described above have been considered by Garling (1967a).

If $x \in \omega$, let $P_n x = \{x_0, x_1, \ldots, x_n, 0, 0, \ldots\}$. If (E, τ) is a K space such that $P_n x \to x$ for each $x \in E$, then E is called an AK space.

If $A = (a_{nk})$ is an infinite matrix of complex numbers the sequence $Ax = \{(Ax)_n\}$ is defined by

$$(Ax)_n = \sum_{k=0}^{\infty} a_{nk} x_k, \quad n = 0, 1, 2, \ldots.$$

$E_A = \{x \in \omega : Ax \in E\}$, where E is a sequence space. Also A' denotes the transpose of A.

The following spaces will be used in the sequel:

$$m = \left\{ x \in \omega : \sup_n |x_n| < \infty \right\};$$

$$c_0 = \left\{ x \in \omega : \lim_{n \to \infty} x_n = 0 \right\};$$

$$l^p \, (1 \leqslant p < \infty) = \left\{ x \in \omega : \sum_{n=0}^{\infty} |x_n|^p < \infty \right\};$$

$$bs = \left\{ x \in \omega : \sup_n \left| \sum_{k=0}^{n} x_k \right| < \infty \right\};$$

$$bv = \left\{ x \in \omega : \sum_{n=0}^{\infty} |x_n - x_{n+1}| < \infty \right\};$$

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$$bv_0 = bv \cap c_0;$$

$$cs = \left\{ x \in \omega \colon \sum_{n=0}^{\infty} x_n \text{ converges} \right\}.$$

Each of the above is a BK space when topologized in the usual way. In addition, all except bs, m and bv are AK spaces.

It is well known that $(l^p)^\beta = l^q$, $(1/p)+(1/q) = 1$ and $p \neq 1$; $l^\beta = m$; $m^\beta = l$; $bv_0^\beta = bs$; $bs^\beta = bv_0$; $bv^\beta = cs$; $cs^\beta = bv$; $cs^\gamma = bv$; $bv^\gamma = bs$ and $c_0^\beta = l$.

3. Main results

Let E be a sequence space containing φ such that $(E^\beta, \sigma(E^\beta, E))$ is sequentially complete. Let $B = (b_{nk})$ be an infinite matrix such that $\{(Bx)_n\}$ is convergent for every $x \in E$. For each $n = 0, 1, 2, \ldots$, let $b^{(n)} = \{b_{nk}\}_{k=0}^{\infty}$. Then $\{b^{(n)}\}$ is a Cauchy sequence in $(E^\beta, \sigma(E^\beta, E))$. Thus, there exists $b = \{b_k\} \in E^\beta$ such that

$$\lim_{n\to\infty} (Bx)_n = \sum_{k=0}^{\infty} b_k x_k$$

for every $x \in E$. Since E contains φ it follows that, for each $k = 0, 1, 2, \ldots$,

$$\lim_{n\to\infty} (Be^k)_n = b_k,$$

where e^k denotes the sequence with a one in the kth coordinate and zeroes elsewhere.

These considerations provide the key to the following theorem. The complete proof may be found in Swetits (1978), Theorem 2.1.

THEOREM 3.1. *Let E and F be sequence spaces, each containing φ, such that $(E^\beta, \sigma(E^\beta, E))$ and $(F, \sigma(F, F^\beta))$ are sequentially complete. If $A = (a_{nk})$ is an infinite matrix then the following are equivalent:*

(i) *F_A contains E;*
(ii) *$E_{A'}^\beta$ contains F^β;*
(iii) *F_A contains $(E^\beta)^\beta$.*

If the hypotheses in Theorem 3.1 are omitted, then the conclusions can fail. Define $A = (a_{nk})$ by

$$a_{nk} = \begin{cases} 1, & k = n, \\ -1, & k = n+1, \\ 0, & \text{otherwise.} \end{cases}$$

Then l_A contains bv. However, cs is not $\sigma(cs, bv)$ sequentially complete and $cs_{A'}$ does not contain m. Thus, (i) $\to$ (ii) fails.

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Let $B = A'$ where A is the matrix defined above. Then cs_B contains c_0, $l_{B'}$ contains bv, but cs_B does not contain m. Thus, (ii) $\Rightarrow$ (ii) fails.

Examples of spaces, E, that satisfy the conditions of Theorem 3.1 are monotone spaces (that is, the coordinatewise product $xy \in E$ if $x \in E$ and y is a sequence of zeroes and ones (Bennett (1974), p. 55)), *FK–AK* spaces, and Garling's class of B_0 invariant spaces (Garling (1967b)).

Examples of spaces, F, that satisfy the conditions of Theorem 3.1 are perfect spaces ($F = (F^{\alpha})^{\alpha}$), bs, bv and bv_0. Each of the spaces mentioned in Section 2 is in one of the above categories.

The first corollary to Theorem 3.1 is well known. For each p, $l \leqslant p \leqslant \infty$, l^p is a perfect space.

COROLLARY 3.2. *$(l^p)_A$ contains l^q if and only if $(l^{q'})_{A'}$ contains $l^{p'}$, where* $(1/p)+(1/p') = 1$ *and* $(1/q)+(1/q') = 1$.

A sequence x is said to be entire if $\sum_{n=0}^{\infty} |x_n| p^n < \infty$ for all $p > 0$. x is analytic if $\sum_{n=0}^{\infty} |x_n| p^n < \infty$ for some $p > 0$. Let $\mathscr{E}$ be the space of entire sequences and $\mathscr{A}$ the space of analytic sequences. Then $\mathscr{E}^{\beta} = \mathscr{A}$ and $\mathscr{A}^{\beta} = \mathscr{E}$, and both $\mathscr{E}$ and $\mathscr{A}$ are perfect spaces. The following result has been obtained by Skerry (1974), Theorem 4.5.

COROLLARY 3.3. *$\mathscr{E}_A$ contains $\mathscr{E}$ if and only if $\mathscr{A}'_A$ contains $\mathscr{A}$.*

Macphail (1951), Theorem 2, established necessary and sufficient conditions for a matrix $A = (a_{nk})$ to transform every analytic sequence into l. His result, combined with Theorem 3.1, yields

COROLLARY 3.4. *$\mathscr{E}_A$ contains m if and only if, for every $r > 0$, there is a constant $M(r)$ such that*

$$\sum_{k=0}^{\infty} |a_{nk}| < M(r) r^n, \quad n = 0, 1, 2, \ldots.$$

The next two corollaries are stated in Jakimovski and Russell (1972), p. 352. They are consequences of Theorem 3.1, (i) $\Rightarrow$ (ii).

COROLLARY 3.5. *If c_A contains c_0, then $l_{A'}$ contains l.*

COROLLARY 3.6. *If c_A contains bv_0, then $bs_{A'}$ contains l.*

The next result enlarges the class of spaces, F, for which the equivalence between (i) and (iii) of Theorem 3.1 is valid.

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THEOREM 3.7. *Let E, F be sequence spaces, each containing φ, such that $(E^{\beta}, \sigma(E^{\beta}, E))$ is sequentially complete and $F = (F^{\gamma})^{\gamma}$. If F_A contains E, then F_A contains $(E^{\beta})^{\beta}$.*

PROOF. Let $\{t_k\} \in F^{\gamma}$ and $\{x_k\} \in E$. Then

$$\sup_j \left| \sum_{n=0}^{j} t_n \sum_{k=0}^{\infty} a_{nk} x_k \right| < \infty.$$

This means

$$\sup_j |(Bx)_j| < \infty,$$

where $B = (b_{jk})$ is defined by

$$b_{jk} = \sum_{n=0}^{j} t_n a_{nk}.$$

Thus m_B contains E. Since $(m, \sigma(m, l))$ is sequentially complete, Theorem 3.1 implies that m_B contains $(E^{\beta})^{\beta}$. Thus, for any $x \in (E^{\beta})^{\beta}$,

$$\sup_j \left| \sum_{n=0}^{j} t_n \sum_{k=0}^{\infty} a_{nk} x_k \right| < \infty.$$

It follows that $Ax \in (F^{\gamma}) = F$. Hence F_A contains $(E^{\beta})^{\beta}$.

The following corollary is immediate.

COROLLARY 3.8. *Let F be as in Theorem 3.1 or Theorem 3.7. If F_A contains c_0, then F_A contains m.*

The space of convergent quasiconvex sequences of order r, c.q.s.(r) is defined as follows: $x \in$ c.q.s.(r) if

$$\sum_{k=0}^{\infty} \binom{k+r-1}{k} |\Delta^r x_k| < \infty$$

where

$$\Delta^r x_k = \sum_{n=0}^{r} (-1)^n \binom{r}{n} x_{k+n}.$$

Jakimovski and Livne (1972), Theorem 4.2, have characterized those matrices, A, such that c_A contains c.q.s.(r). Using their result, it is an easy matter to verify that $((\text{c.s.q.}(r))^{\gamma})^{\gamma} = \text{c.q.s.}(r)$. With $F = \text{c.q.s.}(r)$, Corollary 3.8 is closely related to a result recently announced by Dawson (1976).

For any BK space E, define

$$\| y \|_{E^{\gamma}} = \sup_n \sup_{\|x\|_E \leq 1} \left| \sum_{k=0}^{n} x_k y_k \right| < \infty.$$

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If E, T are BK spaces and A is a matrix, let

$$\| A \|_{(E,F)} = \sup_p \sup_q \sup_{\|x\|_E \leq 1} \sup_{\|y\|_{F^\gamma} \leq 1} \left| \sum_{j=0}^{q} y_j \sum_{k=0}^{j} a_{jk} x_k \right|.$$

Jakimovski and Livne (1971), Theorem 5.2, have shown that, if E is a BK–AK space and $F = G^\gamma$ where G is a BK space, then F_A contains E if and only if $\| A \|_{(E,F)} < \infty$. This result, combined with Theorem 3.7, yields

COROLLARY 3.9. *Let E be a BK–AK space and $F = G^\gamma$ where G is a BK space. Then F_A contains $(E^\beta)^\beta$ if and only if $\| A \|_{(E,F)} < \infty$.*

In Corollary 3.9, $(E^\beta)^\beta$ cannot be replaced by $(E^\gamma)^\gamma$. Let $E = cs$ and $F = l$. Then $(cs^\gamma)^\gamma = bs$. Let A be the matrix whose first row consists entirely of ones and all of whose other entries are zero. Then $l_A = cs$.

A special case of Corollary 3.9 is the well-known equivalence of the following:

(i) m_A contains c_0;

(ii) m_A contains m;

(iii) $\sup_n \sum_{k=0}^{\infty} | a_{nk} | < \infty$.

A BK space E has the property FAK if $\{f(P_n x)\}$ converges for every $x \in E$ and every continuous linear functional, f, on E. E has the property AB if $\{\| P_n x \|\}$ is bounded for each $x \in E$ (see Zeller (1951); Sargent 1964)). It is known that FAK implies AB.

Let E_0 be the closure in E of φ. If E has AB, then E_0 is a BK–AK space with the norm of E (Sargent (1964), Theorem 2). Sargent (1964), Theorem 3, has shown that E has FAK if and only if $E_0^\beta = E^\beta$. Combining these results with Corollary 3.9 we have

COROLLARY 3.10. *Let E be a BK–FAK space, E_0 the closure in E of φ, and $F = G^\gamma$ where G is a BK space. Then F_A contains E if and only if $\| A \|_{(E_0, F)} < \infty$.*

Corollary 3.10 cannot be extended to BK–AB spaces. Let $E = bs$, $E_0 = cs$, $F = l$, and use the example following Corollary 3.9.

Finally, it is noted that Theorem 3.1 proved useful in characterizing dense barrelled subspaces of an FK–AK space (Swetits (1978)).

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https://doi.org/10.1017/S1446788700012325 Published online by Cambridge University Press