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SELF-ADJOINT FREDHOLM OPERATORS AND SPECTRAL FLOW

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ABSTRACT. We study the topology of the nontrivial component, $\mathcal{F}_*^{sa}$, of self-adjoint Fredholm operators on a separable Hilbert space. In particular, if $\{B_t\}$ is a path of such operators, we can associate to $\{B_t\}$ an integer, $\mathrm{sf}(\{B_t\})$, called the *spectral flow* of the path. This notion, due to M. Atiyah and G. Lusztig, assigns to the path $\{B_t\}$ the net number of eigenvalues (counted with multiplicities) which pass through 0 in the positive direction. There are difficulties in making this precise — the usual argument involves looking at the graph of the spectrum of the family (after a suitable perturbation) and then counting intersection numbers with $y = 0$.

We present a completely different approach using the functional calculus to obtain *continuous* paths of eigenprojections (at least locally) of the form $\chi_{[-a,a]}(B_t)$. The spectral flow is then defined as the dimension of the nonnegative eigenspace at the end of this path minus the dimension of the nonnegative eigenspace at the beginning. This leads to an easy proof that spectral flow is a well-defined homomorphism from the homotopy groupoid of $\mathcal{F}_*^{sa}$ onto Z. For the sake of completeness we also outline the seldom-mentioned proof that the restriction of spectral flow to $\pi_1(\mathcal{F}_*^{sa})$ is an isomorphism onto Z.

Introduction. Most workers in the field of (self-adjoint) operator algebras are quite conversant with the ideas and techniques of K-theory. While most (if not all) of us are aware of the Atiyah-Janich theorem [A] which states that the space, $\mathcal{F}$, of Fredholm operators is a classifying space for K^0 (that is, $K_0\big(C(X)\big) = K^0(X) \cong [X, \mathcal{F}]$) and that, by definition, $U(\infty) = \lim_n U(n)$ is a classifying space for K^1 (that is, $K_1\big(C(X)\big) = K^1(X) = [X, U(\infty)]$), some of us are less aware of the Atiyah-Singer result that the nontrivial component, $\mathcal{F}_*^{sa}$, of the self-adjoint Fredholm operators is also a classifying space for K^1. This is the central result of their paper on Index Theory for Skew-Adjoint Fredholm Operators [AS]. There are, of course, other realizations of a classifying space for K^1 but $\mathcal{F}_*^{sa}$ is central to the Index Theorem for families of self-adjoint elliptic operators. In particular, $\pi_1(\mathcal{F}_*^{sa}) = [S^1, \mathcal{F}_*^{sa}] \cong K^1(S^1) = \mathbf{Z}$ arises from the notion of *spectral flow* which is an important concept in index theory [W]. Heuristically, the spectral flow of a one-parameter family of self-adjoint Fredholm operators is just the net number of eigenvalues (counting multiplicities) which pass through zero in the positive direction from the start of the path to its end. The usual way of making this idea rigorous involves looking at the graph of the spectrum of the family after a suitable perturbation, and counting intersection numbers with $y = 0$ (taking into account multiplicities) [APS, BW]. This

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approach is appealing to topologists, but leaves some analysts feeling uneasy. We present here a completely different approach using the functional calculus for self-adjoint operators. The definition then involves counting dimensions of finite-rank projections and is slightly combinatorial in nature. The resulting map on paths is easily seen to be a well-defined, homotopy invariant homomorphism onto **Z**. This definition should appeal to specialists in operator algebras while also being accessible to topologists and geometers. For the sake of completeness we also provide the seldom-mentioned proof that the map is also one-to-one.

Spectral Flow. Let $\mathcal{H}$ be a separable Hilbert space and let $\mathcal{B} = \mathcal{B}(\mathcal{H})$, $\mathcal{K} = \mathcal{K}(\mathcal{H})$ denote the algebras of bounded and compact operators on $\mathcal{H}$, respectively. Let $\mathcal{Q} = \mathcal{Q}(\mathcal{H}) := \mathcal{B}/\mathcal{K}$ be the Calkin algebra and $\pi\colon \mathcal{B} \longrightarrow \mathcal{Q}$ the canonical homomorphism. Let $\mathcal{F} = \{T \in \mathcal{B} \mid \pi(T) \text{ is invertible in } \mathcal{Q}\}$ denote the set of Fredholm operators on $\mathcal{H}$ and $\mathcal{F}^{sa}$ the set of self-adjoint Fredholm operators on $\mathcal{H}$. The space $\mathcal{F}^{sa}$ has three components determined by the essential spectrum:

$$\mathcal{F}^{sa}_{+} := \{T \in \mathcal{F}^{sa} \mid \pi(T) > 0\}$$
$$\mathcal{F}^{sa}_{-} := \{T \in \mathcal{F}^{sa} \mid \pi(T) < 0\}$$
$$\mathcal{F}^{sa}_{*} := \big\{T \in \mathcal{F}^{sa} \mid \mathrm{sp}\big(\pi(T)\big) \not\subseteq \mathbf{R}^{+} \text{ and } \mathrm{sp}\big(\pi(T)\big) \not\subseteq \mathbf{R}^{-}\big\}.$$

PROPOSITION 1. *$\mathcal{F}^{sa}_{+}$ and $\mathcal{F}^{sa}_{-}$ are contractible [AS].*

PROOF. For $(T, \lambda) \in \mathcal{F}^{sa}_{+} \times [0, 1]$ let $\phi(T, \lambda) = (1 - \lambda)T + \lambda 1$. Then, $\pi\big(\phi(T, \lambda)\big) = [(1 - \lambda)\pi(T) + \lambda 1] > 0$ so $\phi(T, \lambda) \in \mathcal{F}^{sa}_{+}$. Clearly, $\phi(T, 0) = T$ and $\phi(T, 1) = 1$. Similarly, $\mathcal{F}^{sa}_{-}$ is contractible to -1.

DISCUSSION. We give a brief motivation for our definition of *spectral flow*. While our definition is, of course, logically equivalent to that of [APS] and [BW], we feel that it is technically easier to work with and easier to make rigorous. If $t \longrightarrow B_t$ for $t \in [0, 1]$ is a continuous path in $\mathcal{F}^{sa}_{*}$ then, heuristically, the spectral flow of $\{B_t\}_{t\in[0,1]}$ is the net number of eigenvalues (counted with multiplicities) which pass through 0 in the positive direction as t goes from 0 to 1. *If* we could choose a continuous family $\{E_t\}$ of finite rank projections so that each E_t were a spectral projection of B_t corresponding to some (say fixed) interval $[-a, +a]$ then the spectral flow should be the dimension of the nonnegative eigenspace of B_1E_1 (in $E_1(\mathcal{H})$) minus the dimension of the nonnegative eigenspace of B_0E_0 (in $E_0(\mathcal{H})$). We observe that the assumed continuity of $\{E_t\}$ precludes any "leakage" of eigenvalues through the boundary, $\pm a$. While, in general, it is not possible to find such a family $\{E_t\}$ on all of $[0, 1]$, one can construct them locally on subintervals: to obtain the spectral flow over $[0, 1]$ one merely adds up the flows over the subintervals. One advantage of this approach is that we do not need to restrict ourselves to any particular homotopy equivalent subsets of $\mathcal{F}^{sa}_{*}$ (see [BW] for this approach).

We use $\mathrm{sp}(\cdot)$ to denote spectrum of an element in a C^*-algebra.

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LEMMA. *Given $B \in \mathcal{F}_*^{sa}$, there is a positive number a and a neighbourhood N of B in $\mathcal{F}_*^{sa}$ so that $S \longmapsto \chi_{[-a,a]}(S)$ is a norm-continuous, finite-rank projection-valued function on N. Where $\chi_{[-a,a]}$ denotes the characteristic function of $[-a, a]$.*

PROOF. Since B is a self-adjoint Fredholm operator, there is an $a > 0$ so that $\pm a$ are not in $\mathrm{sp}(B)$ and $\chi_{[-a,a]}(B)$ is a finite-rank projection. Since $\pm a$ are not in $\mathrm{sp}(B)$, there exists $\epsilon > 0$ so that $[-a-\epsilon, -a+\epsilon]$ and $[a-\epsilon, a+\epsilon]$ are disjoint from $\mathrm{sp}(B)$. The set

$$N_1 = \{S \in \mathcal{F}_*^{sa} \mid \mathrm{sp}(S) \text{ is disjoint from } [-a-\epsilon, \; -a+\epsilon] \cup [a-\epsilon, a+\epsilon]\}$$

is open and on this set the function $S \longmapsto \chi_{[-a,a]}(S)$ is norm-continuous, as $\chi_{[-a,a]}$ agrees with the function f defined below in Figure 1 on the spectrum of any S in N_1:

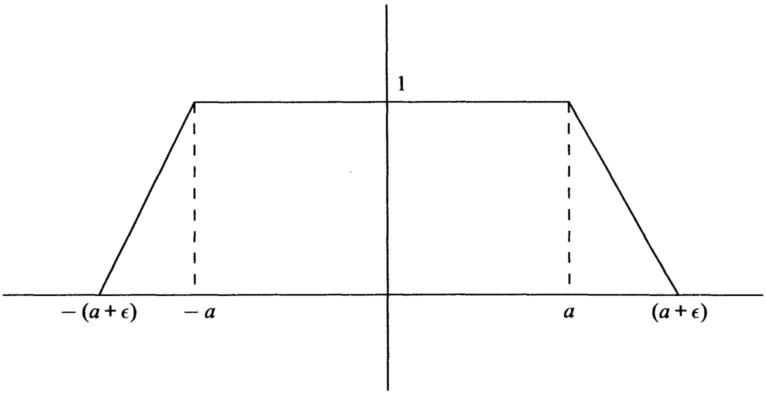

FIGURE 1: Graph of f

Let $N = \{S \in N_1 \mid \|\chi_{[-a,a]}(S) - \chi_{[-a,a]}(B)\| < 1\}$. Thus for all $S \in N$, $\dim\bigl(\chi_{[-a,a]}(S)\bigr)$ equals $\dim\bigl(\chi_{[-a,a]}(B)\bigr)$ which is finite.

NOTATION. If E is a finite-rank spectral projection for the self-adjoint operator S, let E^+ denote the projection on the subspace of $E(\mathcal{H})$ spanned by those eigenvectors for S in $E(\mathcal{H})$ having nonnegative eigenvalues.

DEFINITION. Let $B\colon [0, 1] \longrightarrow \mathcal{F}_*^{sa}$ be a continuous path. By compactness and the previous lemma, choose a partition, $0 = x_0 < x_1 < \cdots < x_n = 1$ of $[0, 1]$ and positive numbers $a_1, a_2, \ldots, a_n$ so that for each $i = 1, 2, \ldots, n$ the function $t \longmapsto E_i(t) := \chi_{[-a_i,a_i]}(B_t)$ is continuous and finite-rank on $[x_{i-1}, x_i]$. We define the spectral flow of B, $\mathrm{sf}(B)$ to be

$$\sum_{k=1}^{n} \Bigl(\dim\bigl(E_k^+(x_k)\bigr) - \dim\bigl(E_k^+(x_{k-1})\bigr)\Bigr).$$

PROPOSITION 2. *Spectral flow is well-defined, that is, it depends only on the continuous function $B\colon [0, 1] \longrightarrow \mathcal{F}_*^{sa}$.*

PROOF. As usual this breaks into two problems:

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(1) Refining the partition using the same projection-valued functions on the refinement,
(2) Keeping the same partition, but changing the projection-valued functions.

(1) For each new point, x_*, added to the partition, the same number, $\dim\big(E_k^+(x_*)\big)$, will be both added and subtracted so that $\mathrm{sf}(B)$ will not change.

(2) Let $[c, d]$ be a fixed subinterval of $[0, 1]$ and

$$E_1(t) = \chi_{[-a_1,a_1]}(B_t) \quad \text{and} \quad E_2(t) = \chi_{[-a_2,a_2]}(B_t)$$

be two continuous functions on $[c, d]$ as in the definition. Without loss of generality $a_1 \geq a_2$ so that $E_1(t) \geq E_2(t)$ for all t in $[c, d]$. Moreover, a_1 and a_2 are not in $\mathrm{sp}(B_t)$ for any t in $[c, d]$ and so $E_1^+(t) - E_2^+(t) = \chi_{(a_2,a_1]}(B_t)$ is a *continuous* function of t on $[c, d]$, and hence of constant dimension, say n. Thus

$$\begin{aligned}\Big[\dim\big(E_1^+(d)\big) - \dim\big(E_1^+(c)\big)\Big] &= \Big[\Big(\dim\big(E_2^+(d)\big) + n\Big) - \Big(\dim\big(E_2^+(c)\big) + n\Big)\Big] \\ &= \Big[\dim\big(E_2^+(d)\big) - \dim\big(E_2^+(c)\big)\Big]\end{aligned}$$

and so $\mathrm{sf}(B)$ does not depend on the choice of $a > 0$ on the interval $[c, d]$.

REMARK. For an explicit example of a loop with spectral flow +1, see the proof of Proposition 6.

PROPOSITION 3. *Spectral flow is homotopy invariant, that is, if $\{B_t\}$ and $\{B_t'\}$ are two continuous paths with $B_0 = B_0'$ and $B_1 = B_1'$ which are homotopic via a homotopy leaving the endpoints fixed, then* $\mathrm{sf}(B) = \mathrm{sf}(B')$.

PROOF. We first observe that if S and T in $\mathcal{F}_*^{sa}$ both belong to a neighbourhood N of the type given in the Lemma, then any two paths from S to T lying entirely in N have the same spectral flow. This is a trivial but crucial observation.

Now, if $H: I \times I \longrightarrow \mathcal{F}_*^{sa}$ is a homotopy from $\{B_t\}$ to $\{B_t'\}$, that is, H is continuous, $H(t, 0) = B_t$ for all t, $H(t, 1) = B_t'$ for all t, $H(0, s) = B_0 = B_0'$ for all s, and $H(1, s) = B_1 = B_1'$ for all s then, by compactness we can cover the image of H by a finite number of neighbourhoods $\{N_1, \ldots, N_k\}$ as in the Lemma. The inverse images of these neighbourhoods $\{H^{-1}(N_1), \ldots, H^{-1}(N_k)\}$ is a finite cover of $I \times I$. Thus, there exists $\epsilon_0 > 0$ (the Lebesgue number of the cover) so that any subset of $I \times I$ of diameter $\leq \epsilon_0$ is contained in some element of this finite cover if $I \times I$. Thus, if we partition $I \times I$ into a grid of squares of diameter $\leq \epsilon_0$, then the image of each square will lie entirely within some N_i. Effectively, this breaks H up into a finite sequence of "short" homotopies by restricting H to $I \times J_i$ where J_i are subintervals of I (of length $\leq \frac{\epsilon_0}{\sqrt{2}}$). These short homotopies have the added property that for fixed J_i we can choose a single partition of I so that for each subinterval J_ℓ of the partition, $H(J_\ell \times J_i)$ is contained in one of

$\{N_1, \ldots, N_k\}$. By concentrating on the i-th "short homotopy" and relabelling $N_1, \ldots, N_k$ if necessary we can assume H has the form indicated in Figure 2:

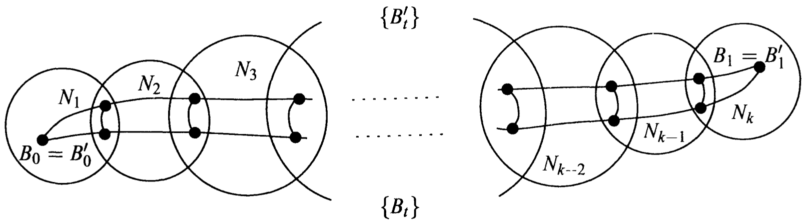

FIGURE 2.

By the observation in the first paragraph of this proof, in each of the following pairs of short paths in Figure 3, the spectral flow of the upper path equals the spectral flow of the lower path.

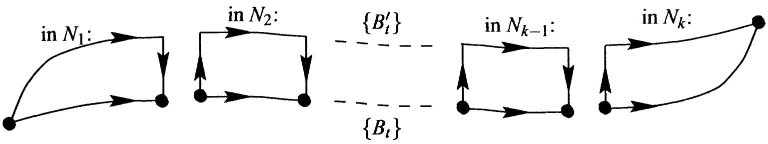

FIGURE 3.

By definition, the sum of the spectral flows of the lower paths is $\mathrm{sf}(B)$. Since the spectral flows of the vertical paths cancel in pairs, the sum of the spectral flows of the upper paths equals $\mathrm{sf}(B')$ and hence $\mathrm{sf}(B) = \mathrm{sf}(B')$.

REMARK. It is clear from our definition that spectral flow does not change under reparametrization of intervals (provided we keep the same orientation of the interval) and is additive when we combine two paths which are composable (*i.e.*, the end point of the first path is the initial point of the second path). Thus, spectral flow defines a groupoid homomorphism from the homotopy groupoid, $\mathrm{Hom}(\mathcal{F}_*^{sa})$ to $\mathbf{Z}$. By restricting to loops based at a point B_0 in $\mathcal{F}_*^{sa}$ we get a group homomorphism $\mathrm{sf}\colon \pi_1(\mathcal{F}_*^{sa}) \longrightarrow \mathbf{Z}$. Since it is easy to construct explicitly paths with spectral flow n for any $n \in \mathbf{Z}$ the map $\mathrm{sf}\colon \pi_1(\mathcal{F}_*^{sa}) \longrightarrow \mathbf{Z}$ is onto. The proof that this map is one-to-one is deeper as it depends on the homotopy equivalence $\mathcal{F}_*^{sa} \simeq U(\infty)$ of [AS] (see Step 3 below). We have almost nothing new to contribute here except a willingness to lay out the argument explicitly.

THEOREM. *Spectral flow,* $\mathrm{sf}\colon \pi_1(\mathcal{F}_*^{sa}) \longrightarrow \mathbf{Z}$ *is an isomorphism.*

PROOF. The proof breaks into three parts.
Step 1: $\mathcal{F}_*^{sa}$ is homotopy equivalent to the subspace

$$\hat{F}_*^{\infty} = \left\{B \in \mathcal{F}_*^{sa} \mid \|B\| = 1,\ \mathrm{sp}(B) \text{ is finite, and } \mathrm{sp}\big(\pi(B)\big) = \{1, -1\}\right\}.$$

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Step 2: The map $B \longmapsto \exp i\pi(B+1)$ is a homotopy equivalence from $\hat{F}_*^\infty$ to $U(\infty)$.
Step 3: As mentioned above, sf: $\pi_1(\mathcal{F}_*^{sa}) \longrightarrow \mathbf{Z}$ is onto. By Step 2, $\pi_1(\mathcal{F}_*^{sa}) \cong \pi_1\big(U(\infty)\big) \cong \mathbf{Z}$, and since any onto homomorphism from $\mathbf{Z}$ to $\mathbf{Z}$ is an isomorphism, we see that sf is an isomorphism.

Our proof of Step 1 is somewhat different from [AS] or [BW]. Both of these papers use the result of [AS] that the larger subspace

$$\hat{F}_* = \big\{B \in \mathcal{F}_*^{sa} \mid \|B\| = 1,\ \mathrm{sp}\big(\pi(B)\big) = \{1,\ -1\}\big\}$$

is a deformation retract of $\mathcal{F}_*^{sa}$. We do not use this (admittedly easy) result, but instead define a mapping $\mathcal{F}_*^{sa} \longrightarrow \hat{F}_*^\infty$ which we show directly is a homotopy equivalence. In Section 17B of [BW] they make a slight misstatement: $\hat{F}_*^\infty$ cannot be a deformation retract of $\mathcal{F}_*^{sa}$: since $\hat{F}_*^\infty$ is dense in $\hat{F}_*$, any continuous map $\mathcal{F}_*^{sa} \longrightarrow \mathcal{F}_*^{sa}$ which is the identity on $\hat{F}_*^\infty$ must also be the identity on $\hat{F}_*$ and thus does *not* map into $\hat{F}_*^\infty$. We show that our mapping $\mathcal{F}_*^{sa} \longrightarrow \hat{F}_*^\infty$ is a weak deformation retraction (whose restriction to $\hat{F}_*$ is also a weak deformation retraction): this is probably what [BW] meant.

PROPOSITION 4. *There is a continuous map ϕ: $\mathcal{F}_*^{sa} \longrightarrow \hat{F}_*^\infty$ which is a homotopy inverse to the injection i: $\hat{F}_*^\infty \longrightarrow \mathcal{F}_*^{sa}$. The restriction of ϕ to $\hat{F}_*$ is a homotopy inverse to the injection $\hat{F}_*^\infty \longrightarrow \hat{F}_*$.*

PROOF. We first observe that if f: $\mathbf{R} \longrightarrow \mathbf{R}$ is a continuous monotone function, $f(0) = 0$, and f is strictly increasing in a neighbourhood of 0 then f: $\mathcal{F}_*^{sa} \longrightarrow \mathcal{F}_*^{sa}$ by elementary spectral theory. Moreover, any finite convex combination of such functions is another such function. Furthermore, if f is such a function satisfying $f(t) = 1$ for all $t \geq \delta$ and $f(t) = -1$ for all $t \leq -\delta$ and $B \in \mathcal{F}_*^{sa}$ satisfies $\mathrm{sp}\big(\pi(B)\big) \cap [-\delta, \delta] = \emptyset$ then $f(B) \in \hat{F}_*^\infty$. With these observations in mind we construct our map ϕ: $\mathcal{F}_*^{sa} \longrightarrow \hat{F}_*^\infty$.

As observed in the proof of Lemma 3, for each B in $\mathcal{F}_*^{sa}$ there is a neighbourhood N of B and a $\delta > 0$ so that for all $S \in N$, $\mathrm{sp}(S) \cap [-\delta, \delta]$ is a finite set of eigenvalues for S with finite multiplicities. Let f be the unique monotone continuous function on $\mathbf{R}$ which is +1 on $[\delta, \infty)$, -1 on $(-\infty, -\delta]$ and linear on $[-\delta, \delta]$. Then $S \longmapsto f(S)$: $N \longrightarrow \hat{F}_*^\infty$ is continuous. Since $\mathcal{F}_*^{sa}$ is a metric space, it is paracompact and so we can find a neighbourhood-finite cover $\{N_\alpha\}$ of $\mathcal{F}_*^{sa}$ and monotone continuous functions f_α: $N_\alpha \longrightarrow \hat{F}_*^\infty$ of the type described in the first paragraph of this proof. Let $\{p_\alpha\}$ be a partition of unity subordinate to this cover. For B in $\mathcal{F}_*^{sa}$ define $\phi(B) = \Sigma_\alpha p_\alpha(B) f_\alpha(B)$. Then ϕ: $\mathcal{F}_*^{sa} \longrightarrow \hat{F}_*^\infty$ is continuous.

One easily checks that the linear homotopy, $H(B, \lambda) = (1-\lambda)B + \lambda\phi(B)$ for $\lambda \in [0, 1]$ can be considered as a mapping: $\mathcal{F}_*^{sa} \times I \longrightarrow \mathcal{F}_*^{sa}$, or $\hat{F}_*^\infty \times I \longrightarrow \hat{F}_*^\infty$, or $\hat{F}_* \times I \longrightarrow \hat{F}_*$, and that in each case this linear homotopy does the required job.

PROPOSITION 5. *The map $B \longmapsto \exp i\pi(B+1)$ is a homotopy equivalence from $\hat{F}_*^\infty$ to $U(\infty)$.*

REMARK. This is essentially Proposition 3.3 of [AS] which is really the central difficult point of that paper and whose proof comprises Propositions 3.4, 3.5, 3.6 and 3.7 of

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[AS]. The heuristic "reason" the proposition holds is that the fibre of the mapping over a point u in $U(\infty)$ is the space of nontrivial symmetries (*i.e.*, unitaries of the form $2P-1$ for P an infinite, co-infinite projection) on $\mathcal{H}_1 = \{\xi \in \mathcal{H} \mid u\xi = \xi\}$. This space is easily seen to be contractible using Kuiper's theorem and so an application of the long exact sequence of homotopy would show that the map is a weak homotopy equivalence. The argument would be finished off by applying general results on infinite dimensional manifolds (à la Palais) to conclude that the map is a homotopy equivalence. The difficult point is that the map $\hat{F}^{\infty}_{*} \longrightarrow U(\infty)$ is *not* a fibration! Nevertheless, Atiyah and Singer are able to show that the long exact sequence of homotopy does apply and so the argument can be completed. At the time of their paper they probably had the idea of quasifibrations in mind.

This completes our discussion of Step 2, and as noted above, Step 3 follows immediately so that sf is an isomorphism.

We conclude with an interesting proposition which identifies spectral flow as the composition, $\pi_1(\mathcal{F}^{sa}_{*}) \cong \pi_1\big(U(\infty)\big) \cong \mathbf{Z}$.

PROPOSITION 6. The following diagram is a commuting square of isomorphisms:

$$\begin{array}{ccc} \pi_1(\mathcal{F}^{sa}_{*}) & \xrightarrow{\text{sf}} & \mathbf{Z} \\ {\scriptstyle i_*}\uparrow & & \uparrow \\ \pi_1(\hat{F}^{\infty}_{*}) & \longrightarrow & \pi_1\big(U(\infty)\big), \end{array}$$

where as usual, the map $\pi_1\big(U(\infty)\big) \longrightarrow \mathbf{Z}$ is the winding number of the determinant.

PROOF. Since all of the maps are known to be isomorphisms and the only automorphisms of $\mathbf{Z}$ are $\pm id$, it suffices to see the diagram commutes on a single nonzero element. To this end let $1 = P_+ + P_- + P_0$ where P_+, P_- are infinite projections and P_0 is a rank 1 projection. Then $(P_+ + P_0) - P_-$ and $P_+ - (P_0 + P_-)$ are unitarily equivalent via a shift operator, u_1 which can be (explicitly) connected by a path $\{u_t\}$ to $1 = u_0$. Then $t \longrightarrow u_t[(P_+ + P_0) - P_-]u_t^*$ is a path from $(P_+ + P_0) - P_-$ to $P_+ - (P_0 + P_-)$ with spectral flow 0. On the other hand, $\{t \longrightarrow (P_+ - P_- + tP_0)\}_{t\in[-1,1]}$ is a path from $P_+ - (P_- + P_0)$ to $(P_+ + P_0) - P_-$ with spectral flow 1 and so justaposing them yields a loop based at $P_+ - (P_- + P_0)$ with spectral flow 1. The second half of this loop (with spectral flow 0) is of the form $P_t - (1 - P_t)$ and so we get:

$$\exp \pi i\Big[\big(P_t - (1 - P_t)\big) + 1\Big] = \exp 2\pi i P_t = 1,$$

the constant loop at 1. The first half of the loop yields:

$$\begin{aligned} \exp \pi i[(P_+ - P_- + tP_0) + 1] &= \exp \pi i[2P_+ + (1+t)P_0] \\ &= \exp\big(\pi i(1+t)P_0\big) \\ &= P_+ + P_- + \big(\exp \pi i(1+t)\big)P_0 \end{aligned}$$

which is a loop based at 1 with determinant $\exp \pi i(1+t) = -\exp \pi i t$ for t in $[-1, 1]$. This clearly has winding number +1. So, we have constructed a loop in $\hat{F}^{\infty}_{*}$ with spectral flow +1 whose image in $U(\infty)$ has winding number +1, as required.

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