EXAMPLES OF WEAK BOUNDARY COMPONENTS

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1. Let D be a plane domain and Γ be a component of the boundary of D *consisting of a single point.* According to Sario [5] we shall call Γ a ***weak*** boundary component of D if its image under any conformal mapping of D consists of a single point. A weak boundary component has been introduced by Grötzsch [2], who called it "vollkommen punktförmig". If Γ is not weak we shall say that it is *unstable* (Sario [5]). We know that the weakness depends merely on the configuration of D in a neighborhood of Γ (see [4], p. 274).

Let E be a compact set on the non-negative real axis such that $0 \in E$, $E \subset [0, 1]$, and that the component of 0 contains no other point. Let $h(\xi)$ be a real (finite) valued function which is defined on E, upper semi-continuous, non-negative, and such that $h(0) = 0$. For any $\xi \in E$, let

$$S_{\xi, h} = \{z ; \operatorname{Re} z = \xi, \ |\operatorname{Im} z| \leqq h(\xi)\}.$$

Then $D_{E,h} = \{z ; \ |z| \leqq \infty\} - \cup_{\xi \in E} S_{\xi, h}$ is a domain and $\Gamma_{E,h} = \{0\}$ is its boundary component consisting of a single point.

It would be useful to give convenient condition on E and $h(\xi)$ to determine when $\Gamma_{E,h}$ is weak or unstable.

2. We remark first that the following "comparison theorem" would enlarge the range of applicability of criteria given in the sequel: *If Γ_{E,h_1} is weak and*

$$\varlimsup_{\xi \in E, \ \xi \to 0} \frac{h_2(\xi)}{h_1(\xi)} < \infty$$

then Γ_{E,h_2} is also weak. The proof is immediate from the local property and the quasi-conformal invariance of weakness (see [4], p. 274).

3. The former author has shown that, if $E = \{a_n\}_{n=1}^{\infty} (a_n > a_{n+1} > 0, \lim_{n \to \infty} a_n = 0)$ and $h(\xi) \leqq c\xi$ $(c > 0)$, then $\Gamma_{E,h}$ is weak (see [1]). It is generalized as follows (cf. the comparison theorem):

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THEOREM 1. *If* $D_{F,h}$ *and* $\Gamma_{F,h}$ *are given by the restriction onto* E *of a function* $h(\xi)$ *defined on* $0 \leqq \xi \leqq 1$ *such that*

(i) $\sqrt{\xi^2 + h(\xi)^2}$ *is a non-decreasing function of* ξ *with the derivative* (existing almost every where) *bounded away from zero,*

(ii) $$\int_{[0,1]-E} \frac{d\xi}{\sqrt{\xi^2 + h(\xi)^2}} = \infty,$$

then $\Gamma_{E,h}$ *is a weak boundary component.*

Proof. Take $b > 0$ such that $\{z;\ b \leqq |z| \leqq \infty\} \subset D_{E,h}$. Let $\{\gamma\}$ be the family of all the closed rectifiable curves in $D_{F,h} \cap \{z;\ |z| < b\}$ separating $\Gamma_{E,h}$ from $|z| = b$. It has been shown by Jurchescu [3] that $\Gamma_{E,h}$ is weak if and only if the extremal length $\lambda\{\gamma\}$ of the family $\{\gamma\}$ vanishes (see also [4], Theorems 2, 3).

For a $\xi \in [0, 1] - E$, let γ_ξ be the union of $\gamma'_\xi = \{z;\ \mathrm{Re}\, z = \xi, |\mathrm{Im}\, z| \leqq h(\xi)\}$ and $\gamma''_\xi = \{z;\ |z|^2 = \xi^2 + h(\xi)^2,\ \arctan(h(\xi)/\xi) \leqq |\arg z| \leqq \pi\}$. Evidently $\{\gamma_\xi\} = \{\gamma_\xi;\ \xi \in [0, 1] - E\}$ is contained in $\{\gamma\}$ and, therefore, it is sufficient to show that $\lambda\{\gamma_\xi\} = 0$. On making use of usual notations, we have

$$L_\rho\{\gamma_\xi\}^2 \leqq \left(\int_{\gamma_\xi} \rho ds\right)^2 \leqq \left(\int_{\gamma_\xi} ds\right)\left(\int_{\gamma_\xi} \rho^2 ds\right)^2 \leqq 2\pi\sqrt{\xi^2 + h(\xi)^2}\left(\int_{\gamma_{\xi'}} \rho^2 dy + \int_{\gamma_{\xi''}} \rho^2 r d\theta\right).$$

where $r = \sqrt{\xi^2 + h(\xi)^2}$. Divide it by r and integrate it with respect to ξ over $[0, 1] - E$. Since it is assumed that $dr/d\xi \geqq \alpha > 0$, we have, on putting $\Delta' = \cup_\xi \gamma'_\xi$ and $\Delta'' = \cup_\xi \gamma''_\xi$, that

$$L_\rho\{\gamma_\xi\}^2 \int_{[0,1]-E} \frac{d\xi}{\sqrt{\xi^2 + h(\xi)^2}} \leqq 2\pi \iint_{\Delta'} \rho^2 dxdy + \frac{2\pi}{\alpha} \iint_{\Delta''} \rho^2 r dr d\theta \leqq \mathrm{const} \iint_D \rho^2 dxdy.$$

Therefore, $L_\rho\{\gamma_\xi\}^2 = 0$ for any square integrable ρ, i.e., $\lambda\{\gamma_\xi\} = 0$.

4. A result of the former author [1] saying that $\Gamma_{F,h}$ is unstable when $E = \{1/n\}_{n=1}^\infty$ and $h(\xi) = \xi^p$ $(0 < p < 1)$ will be contained in the following:

THEOREM 2. *If* $D_{E,h}$ *and* $\Gamma_{E,h}$ *are given by the restriction onto* E *of a function* $h(\xi)$ *defined on* $0 \leqq \xi \leqq 1$ *such that*

(i) *monotone non-decreasing*

(ii) *there exists a constant* K *such that, for any* $\xi \in E - \{1\}$, *it is possible to find a* $\xi' \in E$ *with* $\xi < \xi'$ *and* $h(\xi') \leqq Kh(\xi)$.

(iii) $$\int_{[0,1]-E} \frac{d\xi}{h(\xi)} < \infty,$$

then $\Gamma_{E,h}$ is an unstable boundary component.

That the condition (ii) cannot be omitted for the case $\int_0^1 d\xi/h < \infty$ is seen from an easily constructed example. Whether or not we can omit it for the case $\int_0^1 d\xi/h = \infty$ is not clear, however, we can do so for $h(\xi) = \xi^p (p \geqq 1)$ as follows:

COROLLARY. *If $h(\xi) = \xi^p$ $(p \geqq 1)$, then $\Gamma_{E,h}$ is an unstable boundary component of $D_{E,h}$ provided*

$$\int_{[0,1]-E} \frac{d\xi}{\xi^p} < \infty.$$

Proof of Theorem 2. We shall apply the following criterion due to Grötzsch ([2]; see also [4], Theorem 3): Take b such that $\{z; b \leqq |z| \leqq \infty\} \subset D_{E,h}$; $\Gamma_{E,h}$ is unstable if and only if there exists a finite number M such that $\sum_{\nu=1}^{k} \operatorname{mod} A_\nu \leqq M$ holds for any finite set $\{A_1, A_2, \ldots, A_k\}$ of doubly connected domains A_ν with the following conditions:

(1) $A_\nu \subset D_{E,h} \cap \{z; |z| < b\}$,

(2) A_ν separates $\Gamma_{E,h}$ from $|z| = b$,

(3) A_ν separates $A_{\nu-1}$ from $A_{\nu+1}$.

On looking over the argument in [4] we understand that the result is true if every A_ν is so restricted that the boundary consists of closed analytic curves.

Let $(0, b) - E = \bigcup_{n=1}^{\infty} I_n$, where $I_n = (\xi_n, \xi_n')$ are mutually disjoint open intervals. Consider the quadrilaterals $Q_n = \{z; \operatorname{Re} z \in I_n, |\operatorname{Im} z| \leqq h(\xi_n)\}$ $(n = 1, 2, \ldots)$. By the condition (2) every A_ν passes through a Q_n vertically, i.e., every closed arc in A_ν separating its boundary components contains a subarc connecting in $Q_n \cap A_\nu$ the top and the bottom sides of Q_n. There may be more than one Q_n's; we then take the Q_n corresponding to the left most I_n (remember that the boundary of A_ν consists of analytic curves). For a Q_n, consider all the A_ν's with the above property. Then, by the condition (3), the sum of their moduli does not exceed $2\pi/\lambda\{\gamma\}_n$, where $\{\gamma\}_n$ is the family of all the closed curves in $D_{E,h} \cap \{z; |z| < b\}$ which separate $\Gamma_{E,h}$ from $|z| = b$ and pass through Q_n vertically.

We thus have a grouping of the set $\{A_1, A_2, \ldots, A_k\}$ in terms Q_n. Since an A_ν does not appear in different groups, $\sum_{\nu=1}^{k} \operatorname{mod} A_\nu \leqq 2\pi \sum_{n=1}^{\infty} 1/\lambda\{\gamma\}_n$.

Evidently $\lambda\{\gamma\}_n$ is not less than mod Q_n, the "vertical" modulus of the quadrilateral Q_n, which is equal to $h(\xi_n)/(\xi'_n-\xi_n)$. We conclude, on using the condition (ii) that

$$\sum_{\nu=1}^{k} \text{mod } A_\nu \leqq 2\pi\sum_{n=1}^{\infty}\frac{\xi'_n-\xi_n}{h(\xi_n)} \leqq 2\pi K\sum_{n=1}^{\infty}\frac{\xi'_n-\xi_n}{h(\xi'_n)} \leqq 2\pi K\int_{[0,b]-F}\frac{d\xi}{h(\xi)}<\infty$$

and that $\Gamma_{E,h}$ is unstable.

Proof of Corollary. Since

$$\sum_{n=1}^{\infty}\left(1-\frac{\xi_n}{\xi'_n}\right) \leqq \sum_{n=1}^{\infty}\int_{\xi_n}^{\xi_n'}\frac{d\xi}{\xi^p}=\int_{[0,b]-E}\frac{d\xi}{\xi^p}<\infty,$$

$(\xi'_n)^p/(\xi_n)^p$ is bounded. This fact plays the role of (ii) in the above proof.

5. In a paper of the latter author, we proved the following ([4], Theorem 8):

Consider in particular $E=\{0\}\cup\bigcup_{n=1}^{\infty}[u_n, u'_n]$, *where* $0<u_n<u'_n<u_{n-1}<1$ $(n=2, 3, \ldots)$ *and* $\lim_{n\to\infty}u_n=0$. *Then, under the assumption that* $\lim_{n\to\infty}(u_n/u'_{n+1})=1$ *and* $u_n/u_{n+1}\geqq 1+\delta>1$, *the* $\Gamma_{E,h}$ *for* $h\equiv 0$ *is weak if and only if*

$$\sum_{n=1}^{\infty}\frac{1}{\log\dfrac{u'_{n+1}}{u_n-u'_{n+1}}}=\infty.$$

Concerning such an E, Theorem 1 is merely saying that $\Gamma_{E,h}$ for $h\equiv 0$ is weak if $\sum_{n=1}^{\infty}((u_n/u'_{n+1})-1)=\infty$. Theorem 2 is not applicable to the case where $h\equiv 0$. We see that there is a wide room into which our Theorems 1 and 2 should be extended.

6. Our theorems, however, may be extended into a different direction. To show this, we first introduce the notations

$$S(\xi,\theta,c)=\{z;\ |z+c|=\xi+c,\ |\arg z|\leqq\theta\} \qquad (0\leqq c<\infty,\ 0\leqq\theta<\pi,\ 0<\xi)$$

and

$$S(\xi,\theta,\infty)=\{z;\ \text{Re } z=\xi,\ |\text{Im } z|\leqq\xi\tan\theta\} \qquad (0\leqq\theta<\pi/2,\ 0<\xi).$$

Let E be, as before, a compact set on the non-negative real axis such that $0\in E$, $E\subset[0,1]$, and that the component of 0 contains no other point. Let $\theta(\xi)$ and $c(\xi)$ be functions which are defined on E and satisfy the following conditions: $\theta(\xi)$ is upper semi-continuous and is such that $0\leqq\theta(\xi)<\pi$; $c(\xi)$

is continuous, non-decreasing, and is such that $0 \leqq c(\xi) \leqq \infty$. Suppose further that $\theta(\xi) < \pi/2$ whenever $c(\xi) = \infty$, and that

$$\overline{\lim_{\xi \in E,\, \xi \to 0}} \theta(\xi) \leqq \frac{\pi}{2} \qquad \text{if} \quad 0 < \lim_{\xi \in E,\, \xi \to 0} c(\xi) < \infty,$$

$$\overline{\lim_{\xi \in E,\, \xi \to 0}} \xi \tan \theta(\xi) = 0 \qquad \text{if} \quad c(\xi) \equiv \infty.$$

Then

$$D(E, \theta, c) = \{z\,;\ 0 < |z| \leqq \infty\} - \bigcup_{\xi \in E - \{0\}} S(\xi, \theta(\xi), c(\xi))$$

is a domain and $\Gamma(E, \theta, c) = \{0\}$ is its boundary component. The domain $D_{E,h}$ discussed in the previous sections is the $D(E, \theta, c)$ for $\theta(\xi) = \arctan(h(\xi)/\xi)$ and $c(\xi) \equiv \infty$.

THEOREM 1′. *Suppose that $D(E, \theta, c)$ and $\Gamma(E, \theta, c)$ described above are given by restrictions onto E of $\theta(\xi)$ and $c(\xi)$ defined on $0 \leqq \xi \leqq 1$, where $c(\xi)$ is non-decreasing on $0 \leqq \xi \leqq 1$. If either*

(I) $\lim_{\xi \to 0} c(\xi) = 0$ *and*

$$\int_{[0,1]-E} \frac{d\xi}{\xi + c(\xi)} = \infty$$

or

(II) *the distance $r(\xi)$ between 0 and the endpoints of $S(\xi, \theta(\xi), c(\xi))$ is a non-decreasing function of ξ with the derivative bounded away from zero and*

$$\int_{[0,1]-E} \frac{d\xi}{r(\xi)} = \infty,$$

then $\Gamma(E, \theta, c)$ is weak.

Proof. If (I) is assumed, we may suppose without loss of generality that $c(\xi)$ is finite. The weakness of $\Gamma(E, \xi, c)$ follows from the vanishing of the extremal length of the family $\{\gamma_\xi\,;\ \xi \in [0, 1] - E\}$, where $\gamma_\xi = \{z\,;\ |z + c(\xi)| = \xi + c(\xi)\}$. Under the supposition of (II), we similarly consider $\{\gamma'_\xi \cup \gamma''_\xi\,;\ \xi \in [0, 1] - E\}$, where $\gamma'_\xi = S(\xi, \theta(\xi), c(\xi))$ and $\gamma''_\xi = \{z\,;\ |z| = r(\xi),\ \theta(\xi) \leqq |\arg z| \leqq \pi\}$. The proof in detail will be omitted since it is completely analogous to that of Theorem 1.

THEOREM 2′. *Suppose that $D(E, \theta, c)$ and $\Gamma(E, \theta, c)$ described above are given by restrictions onto E of $\theta(\xi)$ and $c(\xi)$ defined on $0 \leqq \xi \leqq 1$, where $c(\xi)$ is non-decreasing on $0 \leqq \xi \leqq 1$. If*

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(i) *the length* $l(\xi)$ *of* $S(\xi, \theta(\xi), c(\xi))$ *is a non-decreasing function of* ξ *provided that* $c(\xi) \equiv \infty$, *and* $l(\xi)/(\xi + c(\xi))$ *is non-decreasing otherwise,*

(ii) *there exists a constant* K *such that, for any* $\xi \in E - \{1\}$, *it is possible to find a* $\xi' \in E$ *with* $\xi < \xi'$ *and* $l(\xi') \leqq Kl(\xi)$,

(iii)

$$\int_{[0,1]-E} \frac{d(\xi + c(\xi))}{l(\xi)} < \infty,$$

where it is regarded that $dc(\xi) \equiv 0$ *on the interval on which* $c(\xi) \equiv \infty$, *then* $\Gamma(E, \theta, c)$ *is unstable.*

Proof is completely similar to that of Theorem 2. We shall just indicate the estimation of the modulus of the quadrilateral Q defined by the domain bounded by $C_\xi = \{z;\ |z + c(\xi)| = \xi + c(\xi)\}$, $C_{\xi'} = \{z;\ |z + c(\xi')| = \xi' + c(\xi')\}$, $\{z;\ \arg z = l(\xi)/(\xi + c(\xi))\}$, and $\{z;\ \arg z = -l(\xi)/(\xi + c(\xi))\}$ where $\xi < \xi'$ and $c(\xi') < \infty$. Map the interior of $C_{\xi'}$ onto $|\zeta| < 1$ by a linear transformation which maps C_ξ onto the circle $|\zeta| = a < 1$. An elementary estimation of non-euclidean quantities shows that $a > (\xi + c(\xi))/(\xi' + c(\xi'))$ and that the image of Q contains $\{\zeta;\ 1/a < |\zeta| < 1,\ |\arg \zeta| < l(\xi)/(\xi + c(\xi))\}$. We conclude that

$$\operatorname{mod} Q \leqq \frac{\xi + c(\xi)}{l(\xi)} \log \frac{\xi' + c(\xi')}{\xi + c(\xi)} \leqq \frac{\xi' - \xi + c(\xi') - c(\xi)}{l(\xi)} \leqq K \int_{\xi}^{\xi'} \frac{d(\xi + c(\xi))}{l(\xi)}$$

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