Bull. Aust. Math. Soc. **111** (2025), 177–179
doi:10.1017/S0004972724000856

ON MINIMAL DEGREES OF QUOTIENT GROUPS

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(Received 16 August 2024; first published online 14 October 2024)

2020 *Mathematics subject classification*: primary 20C99; secondary 05E10, 20B05, 20C30.

Keywords and phrases: permutation groups, quotient groups, minimal degrees, exceptional groups, almost exceptional groups, semidirect products, wreath product.

For a finite group G, the *minimal faithful permutation representation degree*, denoted by $\mu(G)$, is defined as the smallest $n \in \{0, 1, 2, \ldots\}$ such that G embeds in $\mathrm{Sym}(n)$. The task of determining $\mu(G)$ for an arbitrary G is a complex undertaking, and can be linked to addressing a difficult minimisation problem concerning the lattice of subgroups of G (see [11, 17]). It is interesting to note that the relationship between the minimal degrees of quotient groups and their parent groups is quite uncertain. Despite the fact that the quotient group may be simpler than the parent group, its lattice of subgroups may be more restrictive so that, when solving the minimisation problem, the minimal degree of the quotient group can actually be greater than the minimal degree of the parent group [23]. In such cases, the parent group is called exceptional. Though exceptional groups are not particularly rare, this terminology, introduced in the 1980s in [11], has persisted and this class of groups has been explored by a number of researchers (see [7, 11, 21, 22]).

In the dissertation, we study the delicate relationship between the minimal degrees of finite groups and their respective quotient groups. We address some gaps in the current literature, rectify some existing flaws, and introduce new terminologies and directions for future research. The thesis is a blend of mathematical argument and concrete examples, supported by the use of computer algebra software (in particular, [4, 9, 28]).

In Chapter 1, we provide an overview of the historical context of the concepts presented within the thesis, and highlight the contributions of researchers to this evolving field (in particular, [8–16, 18–20, 24–27, 29]), establishing the framework for the study that follows. In Chapter 2, we provide some essential background information for the subsequent chapters. We also revisit the work of Lemieux [21, 22] and provide a more accurate classification of the minimal degrees of groups of order p^4, where

Thesis submitted to the University of Sydney in December 2023; degree awarded on 17 June 2024; supervisor David Easdown.

p is an odd prime. We conclude this chapter by establishing an explicit isomorphism between some groups from Burnside's 1911 classification of groups of order p^4 [6] and some groups appearing in a 2017 paper by Britnel *et al.* [5], in the context of the classification of exceptional groups of order p^5. In Chapter 3, we reproduce a class of exceptional groups of order p^5, described in the aforementioned 2017 paper, but providing more direct and simplified proofs. This is followed by Chapter 4, where we present a comprehensive classification of all exceptional groups of order $243 = 3^5$, providing detailed explanations and proofs. This study serves as an adjustment to and amplification of a table that was previously published in [5]. In Chapter 5, we investigate minimal degrees of groups associated with certain wreath products. We also construct sequences of groups that have the property where some proper quotients are isomorphic to subgroups that have the same minimal degree, thus possessing what is known as the *almost exceptional property* which arises in the context of the abelian quotients conjecture [19, 20]. In addition, we demonstrate the possibility of having an almost exceptional group with an arbitrarily long chain of normal subgroups such that all of their respective quotients have the same minimal degree. Furthermore, it is possible to have an unlimited number of pairwise incomparable subgroups with the same minimal degree. The results depend on a theory of semidirect products where the base group is a vector space of dimension k over a field with p elements, with p a prime and k a positive integer. The base group is extended by a cyclic group of order p, which is represented by a $k \times k$ matrix, adapting a technique from [10, 15]. A final application is made to construct sequences of groups with the property that the direct products have minimal degrees that grow linearly with the number n of factors, while their respective quotients, realised as central products, have minimal degrees that grow exponentially with n. This generalises a result of Neumann [23].

The thesis is supported by three appendices. In Appendix A, we provide detailed information about minimal degrees of groups of order at most 63, extending a table in [12]. In Appendix B, we provide a comprehensive analysis of the minimal degrees of groups of order 243 and their corresponding quotient groups of order 81. In Appendix C, we present a table of wreath product groups of various orders, up to 500, including their minimal degrees.

Some of this research is available in [1–3].

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https://doi.org/10.1017/S0004972724000856 Published online by Cambridge University Press