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DOUBLE-BARRIER PARISIAN OPTIONS

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Abstract

In this paper we study the excursion time of a Brownian motion with drift outside a corridor by using a four-state semi-Markov model. In mathematical finance, these results have an important application in the valuation of double-barrier Parisian options. We subsequently obtain an explicit expression for the Laplace transform of its price.

Keywords: Excursion time; four-state semi-Markov model; double-barrier Parisian option; Laplace transform

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Secondary 60J65; 60G44; 60J25

1. Introduction

The concept of Parisian options was first introduced by Chesney *et al*. [4]. It is a special case of path-dependent options. The owner of a Parisian option will either gain the right or lose the right to exercise the option upon the price reaching a predetermined barrier level L and staying above or below the level for a predetermined time d before the maturity date T.

More precisely, the owner of a *Parisian down-and-out option* loses the option if the underlying asset price S reaches the level L and remains constantly below this level for a time interval longer than d. For a *Parisian down-and-in option*, the same event gives the owner the right to exercise the option. For details on the pricing of Parisian options, see [4], [8], and [13].

Double-barrier Parisian options are a two-barrier version of the standard Parisian options introduced by Chesney *et al*. [4]. In contrast to the Parisian options mentioned above, we consider the excursions both below the lower barrier and above the upper barrier, i.e. outside a corridor formed by these two barriers. Let us look at two examples, depending on whether the condition is that the required excursions above the upper barrier and below the lower barrier have to both happen before the maturity date or that either one of them happens before the maturity. In the first example, the owner of a *double-barrier Parisian max-out option* loses the option if the underlying asset price process S has both an excursion above the upper barrier for longer than a continuous period d_1 and below the lower barrier for longer than d_2 before the maturity of the option. In the second example, the owner of a *double-barrier Parisian min-out option* loses the right to exercise the option if either one of these two events happens before the maturity. Later on, we will derive the Laplace transforms which can be used to price options of this type.

In this paper we are going to use the same definition for the excursion as in [4] and [5]. Let S be a stochastic process, and let l_1 and l_2, $l_1 > l_2$, be the levels of these two barriers. As in [4],

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we define

$$g^S_{l_i,t} := \sup\{s \le t \mid S_s = l_i\}, \qquad d^S_{l_i,t} := \inf\{s \ge t \mid S_s = l_i\}, \qquad i = 1, 2,$$

with the usual conventions that $\sup\{\varnothing\} = 0$ and $\inf\{\varnothing\} = \infty$. Assuming that $d_1 > 0$ and $d_2 > 0$, we now define

$$\tau^S_1 := \inf\{t > 0 \mid \mathbf{1}_{\{S_t > l_1\}}(t - g^S_{l_1,t}) \ge d_1\}, \tag{1}$$

$$\tau^S_2 := \inf\{t > 0 \mid \mathbf{1}_{\{l_2 < S_t < l_1\}}\mathbf{1}_{\{g^S_{l_1,t} > g^S_{l_2,t}\}}(t - g^S_{l_1,t}) \ge d_2\}, \tag{2}$$

$$\tau^S_3 := \inf\{t > 0 \mid \mathbf{1}_{\{l_2 < S_t < l_1\}}\mathbf{1}_{\{g^S_{l_1,t} < g^S_{l_2,t}\}}(t - g^S_{l_2,t}) \ge d_3\}, \tag{3}$$

$$\tau^S_4 := \inf\{t > 0 \mid \mathbf{1}_{\{S_t < l_2\}}(t - g^S_{l_2,t}) \ge d_4\}, \tag{4}$$

$$\tau^S := \tau^S_1 \wedge \tau^S_4. \tag{5}$$

We can see that τ^S_1 is the first time that the length of the excursion of the process S above the barrier l_1 reaches a given level d_1, τ^S_4 corresponds to the one below l_2 with required length d_4, and τ^S is the smaller of τ^S_1 and τ^S_4. We also see that τ^S_2 is the first time that the length of the excursion in the corridor reaches given level d_2, given that the excursion starts from the upper barrier l_1; and τ^S_3 corresponds to the one in the corridor starting from the lower barrier l_2. Our aim is to study the excursion outside the corridor; therefore, τ^S_2 and τ^S_3 are not of interest here. However, we need to use these two stopping times to define our four-state semi-Markov model that will be the main tool used for calculation.

Now assume that r is the risk-free rate, T is the term of the option, S_t is the price of its underlying asset, K is the strike price, and Q is the risk neutral measure. If we have a double-barrier Parisian min-out call option with barriers l_1 and l_2, its price can be expressed as

$$\mathrm{DP}_{\text{min-out call}} = \mathrm{e}^{-rT}\,\mathrm{E}_Q(\mathbf{1}_{\{\tau^S > T\}}(S_T - K)^+),$$

and the price of a double-barrier Parisian min-in put option is expressed as

$$\mathrm{DP}_{\text{min-in put}} = \mathrm{e}^{-rT}\,\mathrm{E}_Q(\mathbf{1}_{\{\tau^S < T\}}(K - S_T)^+).$$

In this paper we study the excursion time outside the corridor using a semi-Markov model consisting of four states. By applying the model to a Brownian motion we can obtain the explicit form of the Laplace transform for the price of double-barrier options. We can then invert using techniques given in [8].

In Section 2 we introduce the four-state semi-Markov model as well as a new process, the doubly perturbed Brownian motion, which has the same behaviour as a Brownian motion except that each time it hits one of the two barriers, it moves towards the other side of the barrier by a jump of size ε. In Section 3 we obtain the martingale to which we can apply the optional sampling theorem and obtain the Laplace transform that we can use for pricing later. We give our main results applied to Brownian motion in Section 4, including the Laplace transforms for the stopping times we defined in (1)–(5) for both a Brownian motion with drift, i.e. $S = W^\mu$, and a standard Brownian motion, i.e. $S = W$. In Section 5 we focus on pricing the double-barrier Parisian options.

2. Definitions

From the description above, it is clear that we are actually considering four states: the state when the stochastic process is above the barrier l_1; the state when it is below l_2; and two states when it is between l_1 and l_2, depending on whether it comes into the corridor through l_1 or l_2. For each state, we are interested in the time the process spends in it. We introduce a new process:

$$Z_t^S := \begin{cases} 1 & \text{if } S_t > l_1, \\ 2 & \text{if } l_1 > S_t > l_2 \text{ and } g_{l_1,t}^S > g_{l_2,t}^S, \\ 3 & \text{if } l_1 > S_t > l_2 \text{ and } g_{l_1,t}^S < g_{l_2,t}^S, \\ 4 & \text{if } S_t < l_2. \end{cases}$$

We can now express the variables defined above in terms of Z_t:

$$\begin{aligned} g_{l_i,t}^S &= \sup\{s \le t \mid Z_s^S \ne Z_t\}, \\ d_{l_i,t}^S &= \inf\{s \ge t \mid Z_s^S \ne Z_t\}, \\ \tau_1^S &= \inf\{t > 0 \mid \mathbf{1}_{\{Z_t^S=1\}}(t - g_{l_1,t}^S) \ge d_1\}, \\ \tau_2^S &= \inf\{t > 0 \mid \mathbf{1}_{\{Z_t^S=2\}}(t - g_{l_1,t}^S) \ge d_2\}, \\ \tau_3^S &= \inf\{t > 0 \mid \mathbf{1}_{\{Z_t^S=3\}}(t - g_{l_2,t}^S) \ge d_3\}, \\ \tau_4^S &= \inf\{t > 0 \mid \mathbf{1}_{\{Z_t^S=4\}}(t - g_{l_2,t}^S) \ge d_4\}. \end{aligned}$$

We then define

$$V_t^S := t - \max(g_{l_1,t}^S, g_{l_2,t}^S),$$

the time Z_t^S has spent in the current state. It is easy to see that (Z_t^S, V_t^S) is a Markov process. Therefore, Z_t^S is a semi-Markov process with state space $\{1, 2, 3, 4\}$, where 1 stands for the state when the stochastic process S is above the barrier l_1, 4 corresponds to the state below the barrier l_2, and 2 and 3 represent the states when S is in the corridor given that it comes in through l_1 and l_2, respectively.

For Z_t^S, the transition intensities $\lambda_{ij}(u)$ satisfy

$$\begin{aligned} \mathrm{P}(Z_{t+\Delta t}^S = j,\, i \ne j \mid Z_t^S = i,\ V_t^S = u) &= \lambda_{ij}(u)\Delta t + o(\Delta t), \\ \mathrm{P}(Z_{t+\Delta t}^S = i \mid Z_t^S = i,\ V_t^S = u) &= 1 - \sum_{i\ne j} \lambda_{ij}(u)\Delta t + o(\Delta t). \end{aligned}$$

Define

$$\bar{P}_i(\mu) := \exp\bigg\{-\int_0^\mu \sum_{i\ne j} \lambda_{ij}(v)\,\mathrm{d}v\bigg\}, \qquad p_{ij}(\mu) = \lambda_{ij}(\mu)\bar{P}_i(\mu).$$

Note that

$$P_i(\mu) = 1 - \bar{P}_i(\mu)$$

is the distribution function of the excursion time in state i, which is a random variable U_i defined as

$$U_i := \inf_{s>0}\{Z_s^S \ne i \mid Z_0^S = i,\ V_0^S = 0\}.$$

Note that because the process is time homogeneous, this has the same distribution as

$$\inf_{s>0}\{Z_{t+s}^S \ne i \mid Z_t^S = i,\ V_t^S = 0\} \quad \text{for any time } t.$$

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We therefore have

$$p_{ij}(\mu) = \lim_{\triangle\mu\to 0} \frac{\mathrm{P}(U_i \in (\mu, \mu+\triangle\mu),\ Z^S_{U_i} = j)}{\triangle\mu}.$$

Moreover, in the definition of Z^S, we deliberately ignore the situation when $S_t = l_i,\ i = 1, 2$. The reason is that we consider only the processes for which $\int_0^t \mathbf{1}_{\{S_u = l_i\}}\,\mathrm{d}u = 0,\ i = 1, 2$.

Also, when l_1 and l_2 are regular points of the process (see [1] for a definition), we have to deal with the degeneration of p_{ij}. Let us take a Brownian motion as an example. Assume that $W^\mu_t = \mu t + W_t$ with $\mu \geq 0$, where W_t is a standard Brownian motion. Setting x_0 to be its starting point, we know that its density for the first hitting time of level $l_i,\ i = 1, 2$, is

$$p_{x_0} = \frac{|l_i - x_0|}{\sqrt{2\pi t^3}} \exp\biggl\{-\frac{(l_i - x_0 - \mu t)^2}{2t}\biggr\}$$

(see [12, Sections I.9 and I.13]). According to the definition of the transition density, $p_{12}(t) = p_{21}(t) = p_{l_1}(t) = 0$ and $p_{34}(t) = p_{43}(t) = p_{l_2}(t) = 0$ for $t > 0$.

The problem is not regularity in itself, but the fact that there are infinitely many excursions outside and also inside the barriers. In [6], in order to solve the single-barrier problem, we introduced the perturbed Brownian motion $X^{(\varepsilon)}_t$ with respect to the barrier we are interested in. We will extend this idea here, and construct a new process, *double-perturbed Brownian motion*. The anonymous referee pointed out that maybe regularity itself should be exploited in an attempt to considerably simplify our proofs (using perhaps an approach as in [9, Section III.2]). Moreover, the referee also suggested the use of excursion theory as in [10] and [11, Section VI.8]. This approach seems suitable for simplifying the arguments in [6]; a similar promising line can be found in [14, Chapter 15], where the excursion time is formulated as a Markov process whose generator is provided. However, there are two reasons why we will not adopt these ideas here. One reason is that our method can also be used to generalise some of our results for Lévy processes that can have jumps. The most important reason is that we make use of excursions between the two barriers. These are not discussed in the references mentioned and so our method seems the most appropriate one at this stage.

We now construct the new process, *double-perturbed Brownian motion*, $Y^{(\varepsilon)}_t$, $\varepsilon > 0$, with respect to barriers l_1 and l_2. Assume that $W^\mu_0 = l_1 + \varepsilon$. Define the sequence of stopping times

$$\delta_0 = 0, \qquad \sigma_n = \inf\{t > \delta_n \mid W^\mu_t = l_1\}, \qquad \delta_{n+1} = \inf\{t > \sigma_n \mid W^\mu_t = l_1 + \varepsilon\},$$

where $n = 0, 1, \ldots$ (see Figure 1). Now define

$$X^{(\varepsilon)}_t := \begin{cases} W^\mu_t & \text{if } \delta_n \leq t < \sigma_n, \\ W^\mu_t - \varepsilon & \text{if } \sigma_n \leq t < \delta_{n+1}. \end{cases}$$

Similarly, we now define another sequence of stopping times with respect to the process $X^{(\varepsilon)}_t$ and barrier l_2:

$$\zeta_0 := 0, \qquad \eta_n := \inf\{t > \zeta_n \mid X^{(\varepsilon)}_t = l_2\}, \qquad \zeta_{n+1} := \inf\{t > \eta_n \mid X^{(\varepsilon)}_t = l_2 + \varepsilon\},$$

where $n = 0, 1, \ldots$ (see Figure 2). Now define

$$Y^{(\varepsilon)}_t := \begin{cases} X^{(\varepsilon)}_t & \text{if } \zeta_n \leq t < \eta_n, \\ X^{(\varepsilon)}_t - \varepsilon & \text{if } \eta_n \leq t < \zeta_{n+1}. \end{cases}$$

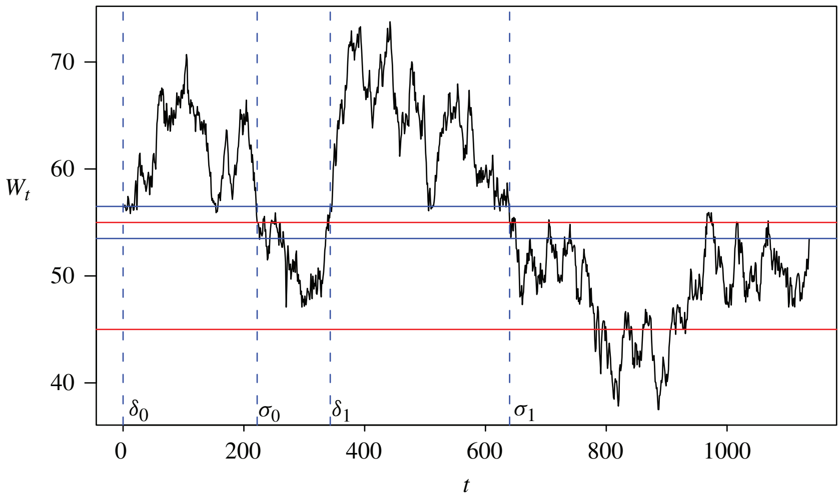

FIGURE 1: A sample path of the original Brownian motion, $W_t^{(\varepsilon)}$.

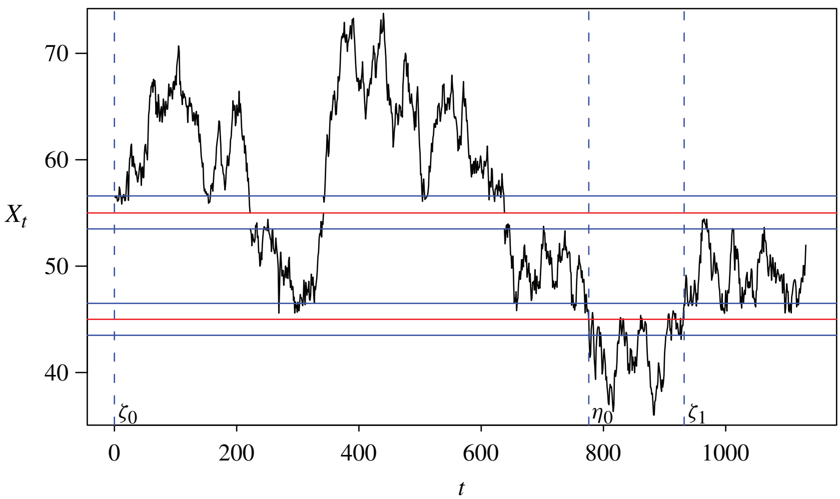

FIGURE 2: A sample path of the process $X_t^{(\varepsilon)}$.

It is actually a process which starts from $l_1 + \varepsilon$ and has the same behaviour as the related Brownian motion, except that each time it hits the barrier l_1 or l_2, it will jump towards the opposite side of the barrier with size ε (see Figure 3).

From the definition, it is clear that l_1 and l_2 become irregular points for $Y_t^{(\varepsilon)}$. Also, $Y_t^{(\varepsilon)}$ converges to W_t^{μ} with $W_0^{\mu} = l_1$ almost surely for all t. Therefore, as we prove in Appendix A, the Laplace transforms of the variables defined based on $Y_t^{(\varepsilon)}$ converge to those based on W_t^{μ}. As a result, we can obtain the results for the Brownian motion by carrying out the calculation for $Y_t^{(\varepsilon)}$ and taking the limit $\varepsilon \to 0$.

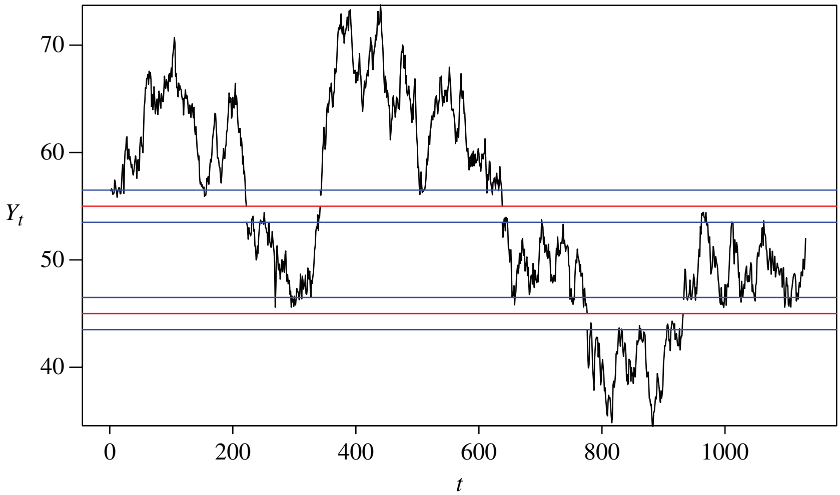

FIGURE 3: A sample path of the process $Y_t^{(\varepsilon)}$.

For $Y_t^{(\varepsilon)}$, we can define Z^Y, τ_1^Y, τ_2^Y, and τ^Y as above (we suppress the (ε) superscript). For Z^Y, we have the transition densities (see [2, Equations 2.0.2 and 3.0.6])

$$p_{12}(t) = \frac{\varepsilon}{\sqrt{2\pi t^3}} \exp\left\{-\frac{(\varepsilon + \mu t)^2}{2t}\right\}, \tag{6}$$

$$p_{21}(t) = \exp\left\{\mu\varepsilon - \frac{\mu^2 t}{2}\right\} ss_t(l_1 - l_2 - \varepsilon, l_1 - l_2), \tag{7}$$

$$p_{24}(t) = \exp\left\{-\mu(l_1 - l_2 - \varepsilon) - \frac{\mu^2 t}{2}\right\} ss_t(\varepsilon, l_1 - l_2), \tag{8}$$

$$p_{31}(t) = \exp\left\{\mu(l_1 - l_2 - \varepsilon) - \frac{\mu^2 t}{2}\right\} ss_t(\varepsilon, l_1 - l_2), \tag{9}$$

$$p_{34}(t) = \exp\left\{-\mu\varepsilon - \frac{\mu^2 t}{2}\right\} ss_t(l_1 - l_2 - \varepsilon, l_1 - l_2), \tag{10}$$

$$p_{43}(t) = \frac{\varepsilon}{\sqrt{2\pi t^3}} \exp\left\{-\frac{(\varepsilon - \mu t)^2}{2t}\right\}, \tag{11}$$

where

$$ss_t(x, y) = \sum_{k=-\infty}^{\infty} \frac{(2k+1)y - x}{\sqrt{2\pi t^3}} \exp\left\{-\frac{((2k+1)y - x)^2}{2t}\right\}.$$

Also, we know that

$$p_{23}(t) = p_{32}(t) = p_{14}(t) = p_{41}(t) = 0. \tag{12}$$

Clearly, all the arguments above apply to the standard Brownian motion, which is a special case of W_t^{μ} when $\mu = 0$.

3. Results for the semi-Markov model

In Section 2 we introduced the Markov process (Z_t^S, V_t^S). Now we apply the same definition to the doubly perturbed Brownian motion $Y_t^{(\varepsilon)}$; therefore, we have (Z_t^Y, V_t^Y), where Z_t^Y is the current state of $Y_t^{(\varepsilon)}$, taking values from the state space $\{1, 2, 3, 4\}$, and V_t^Y is the time $Y_t^{(\varepsilon)}$ has spent in the current state. The time V_t^Y is also a stochastic process. Now we consider a function of the form

$$f(V_t^Y, Z_t^Y, t) = f_{Z_t^Y}(V_t^Y, t),$$

where the f_i, $i = 1, 2, 3, 4$, are functions from $\mathbb{R}^2$ to $\mathbb{R}$. The generator $\mathcal{A}$ is defined as an operator such that

$$f(V_t^Y, Z_t^Y, t) - \int_0^s \mathcal{A} f(V_s^Y, Z_s^Y, s)\,\mathrm{d}s$$

is a martingale (see [7, Chapter 2]). Therefore, solving

$$\mathcal{A} f = 0$$

subject to certain conditions will provide us with martingales of the form $f(V_t^Y, Z_t^Y, t)$, to which we can apply the optional stopping theorem to obtain the Laplace transform we are interested in. More precisely, we will have

$$\begin{aligned}
\mathcal{A} f_1(u,t) &= \frac{\partial f_1(u,t)}{\partial t} + \frac{\partial f_1(u,t)}{\partial u} + \lambda_{12}(u)(f_2(0,t) - f_1(u,t)),\\
\mathcal{A} f_2(u,t) &= \frac{\partial f_2(u,t)}{\partial t} + \frac{\partial f_2(u,t)}{\partial u} + \lambda_{21}(u)(f_1(0,t) - f_2(u,t))\\
&\quad + \lambda_{24}(u)(f_4(0,t) - f_2(u,t)),\\
\mathcal{A} f_3(u,t) &= \frac{\partial f_3(u,t)}{\partial t} + \frac{\partial f_3(u,t)}{\partial u} + \lambda_{31}(u)(f_1(0,t) - f_3(u,t))\\
&\quad + \lambda_{34}(u)(f_4(0,t) - f_3(u,t)),\\
\mathcal{A} f_4(u,t) &= \frac{\partial f_4(u,t)}{\partial t} + \frac{\partial f_4(u,t)}{\partial u} + \lambda_{43}(u)(f_4(0,t) - f_3(u,t)).
\end{aligned}$$

Assume that f_i has the form

$$f_i(u,t) = \mathrm{e}^{-\beta t} g_i(u).$$

By solving the equation $\mathcal{A} f = 0$, i.e.

$$\mathcal{A} f_1 = 0, \qquad \mathcal{A} f_2 = 0, \qquad \mathcal{A} f_3 = 0, \qquad \mathcal{A} f_4 = 0$$

subject to

$$g_1(d_1) = \alpha_1, \qquad g_2(d_2) = \alpha_2, \qquad g_3(d_2) = \alpha_3, \qquad g_4(d_2) = \alpha_4,$$

we obtain

$$\begin{aligned}
g_i(u) &= \alpha_i \exp\biggl\{-\int_u^{d_i} \biggl(\beta + \sum_{j\neq i} \lambda_{ij}(v)\biggr)\mathrm{d}v\biggr\}\\
&\quad + \sum_{j\neq i} g_j(0) \int_u^{d_i} \lambda_{ij}(s) \exp\biggl\{-\int_u^{s} \biggl(\beta + \sum_{k\neq i} \lambda_{ik}(v)\biggr)\mathrm{d}v\biggr\}\mathrm{d}s. \qquad (13)
\end{aligned}$$

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In our case, we are interested only in the excursion outside the corridor. Hence, we set d_2 and d_3 to be ∞. Also, $\lim_{d_2\to\infty} g_2(d_2) = \lim_{d_3\to\infty} g_3(d_3) = 0$ gives $\alpha_2 = \alpha_3 = 0$. Therefore, we have

$$g_1(0) = \alpha_1 \mathrm{e}^{-\beta d_1} \bar{P}_1(d_1) + \{g_1(0)\hat{P}_{21}(\beta) + g_4(0)\hat{P}_{24}(\beta)\}\tilde{P}_{12}(\beta), \tag{14}$$

$$g_4(0) = \alpha_4 \mathrm{e}^{-\beta d_4} \bar{P}_4(d_4) + \{g_1(0)\hat{P}_{31}(\beta) + g_4(0)\hat{P}_{34}(\beta)\}\tilde{P}_{43}(\beta). \tag{15}$$

Solving (14) and (15) gives

$$\begin{aligned} g_1(0) = {} & [\alpha_1 \exp\{\beta d_1\}\bar{P}_1(d_1)(1 - \hat{P}_{34}(\beta)\tilde{P}_{43}(\beta)) + \alpha_4 \exp\{\beta d_4\}\bar{P}_4(d_4)\hat{P}_{24}(\beta)\tilde{P}_{12}(\beta)] \\ & \times [1 - \hat{P}_{21}(\beta)\tilde{P}_{12}(\beta) - \hat{P}_{34}(\beta)\tilde{P}_{43}(\beta) + \hat{P}_{21}(\beta)\tilde{P}_{12}(\beta)\hat{P}_{34}(\beta)\tilde{P}_{43}(\beta) \\ & \quad - \hat{P}_{31}(\beta)\tilde{P}_{43}(\beta)\hat{P}_{24}(\beta)\tilde{P}_{12}(\beta)]^{-1}, \\ g_4(0) = {} & [\alpha_4 \exp\{\beta d_4\}\bar{P}_4(d_4)(1 - \hat{P}_{21}(\beta)\tilde{P}_{12}(\beta)) + \alpha_1 \exp\{\beta d_1\}\bar{P}_1(d_1)\hat{P}_{31}(\beta)\tilde{P}_{43}(\beta)] \\ & \times [1 - \hat{P}_{21}(\beta)\tilde{P}_{12}(\beta) - \hat{P}_{34}(\beta)\tilde{P}_{43}(\beta) + \hat{P}_{21}(\beta)\tilde{P}_{12}(\beta)\hat{P}_{34}(\beta)\tilde{P}_{43}(\beta) \\ & \quad - \hat{P}_{31}(\beta)\tilde{P}_{43}(\beta)\hat{P}_{24}(\beta)\tilde{P}_{12}(\beta)]^{-1}, \end{aligned}$$

where

$$\hat{P}_{ij}(\beta) = \int_0^\infty \mathrm{e}^{-\beta s} p_{ij}(s)\,\mathrm{d}s,$$

$$\tilde{P}_{ij}(\beta) = \int_0^{d_i} \mathrm{e}^{-\beta s} p_{ij}(s)\,\mathrm{d}s.$$

As a result, we have obtained the martingale

$$M_t = f(V_t^Y, t) = \mathrm{e}^{-\beta t} g_{Z_t^Y}(V_t^Y), \qquad i = 1, 2, 3, 4.$$

We now can apply the optional stopping theorem to M_t with the stopping time $\tau^Y \wedge t$, where τ^Y is the stopping time defined by (5):

$$\mathrm{E}(M_{\tau^Y \wedge t}) = \mathrm{E}(M_0). \tag{16}$$

The right-hand side of (16) is

$$\mathrm{E}(M_{\tau^Y \wedge t}) = \mathrm{E}(M_{\tau^Y} \mathbf{1}_{\{\tau^Y < t\}}) + \mathrm{E}(M_t \mathbf{1}_{\{\tau^Y > t\}}).$$

Furthermore,

$$\begin{aligned} & \mathrm{E}(M_{\tau^Y} \mathbf{1}_{\{\tau^Y < t\}}) \\ & \quad = \mathrm{E}(M_{\tau^Y} \mathbf{1}_{\{\tau_1^Y < \tau_4^Y\}} \mathbf{1}_{\{\tau_1^Y < t\}}) + \mathrm{E}(M_{\tau^Y} \mathbf{1}_{\{\tau_1^Y > \tau_4^Y\}} \mathbf{1}_{\{\tau_4^Y < t\}}) \\ & \quad = \mathrm{E}(\exp\{-\beta\tau^Y\} g_1(d_1) \mathbf{1}_{\{\tau_1^Y < \tau_4^Y\}} \mathbf{1}_{\{\tau_1^Y < t\}}) \\ & \qquad + \mathrm{E}(\exp\{-\beta\tau^Y\} g_4(d_4) \mathbf{1}_{\{\tau_1^Y > \tau_4^Y\}} \mathbf{1}_{\{\tau_4^Y < t\}}) \\ & \quad = \alpha_1 \mathrm{E}(\exp\{-\beta\tau^Y\} \mathbf{1}_{\{\tau_1^Y < \tau_4^Y\}} \mathbf{1}_{\{\tau_1^Y < t\}}) + \alpha_4 \mathrm{E}(\exp\{-\beta\tau^Y\} \mathbf{1}_{\{\tau_1^Y > \tau_4^Y\}} \mathbf{1}_{\{\tau_4^Y < t\}}). \end{aligned}$$

We also have

$$\mathrm{E}(M_t \mathbf{1}_{\{\tau^Y > t\}}) = \mathrm{e}^{-\beta t} \mathrm{E}(g_{Z_t^Y}(V_t^Y) \mathbf{1}_{\{\tau^Y > t\}}),$$

where Z_t^Y can take the value 1, 2, 3, or 4.

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When $Z_t^Y = 1$ or 4, since $\tau^Y > t$, we have $0 \le V_t^Y < d_1 \wedge d_4$. According to the definition of $g_i(\mu)$ in (13), $g_1(V_t^Y)$ and $g_4(V_t^Y)$ are bounded. When $Z_t^Y = 2$ or 3, since $\lim_{d_2\to\infty} g_2(d_2) = \lim_{d_3\to\infty} g_3(d_3) = 0$, and looking at (13) with d_2 and d_3 replaced by ∞, $g_2(V_t^Y)$ and $g_3(V_t^Y)$ are bounded. Therefore,

$$\lim_{t\to\infty} \mathrm{E}(M_t \mathbf{1}_{\{\tau^Y > t\}}) = 0.$$

Hence, we have

$$\lim_{t\to\infty} \mathrm{E}(M_{\tau^Y \wedge t}) = \alpha_1 \mathrm{E}(\mathrm{e}^{-\beta\tau^Y} \mathbf{1}_{\{\tau_1^Y < \tau_4^Y\}}) + \alpha_4 \mathrm{E}(\mathrm{e}^{-\beta\tau^Y} \mathbf{1}_{\{\tau_1^Y > \tau_4^Y\}}).$$

The left-hand side of (16) gives

$$\lim_{t\to\infty} \mathrm{E}(M_0) = \mathrm{E}(M_0) = \begin{cases} g_1(0), & Y_0^{(\varepsilon)} = l_1 + \varepsilon, \\ g_4(0), & Y_0^{(\varepsilon)} = l_2 - \varepsilon. \end{cases}$$

By taking $\alpha_1 = 1$, $\alpha_4 = 0$ and $\alpha_1 = 0$, $\alpha_4 = 1$, then, when $Y_0^{(\varepsilon)} = l_1 + \varepsilon$,

$$\begin{aligned} &\mathrm{E}(\exp(-\beta\tau^Y)\, \mathbf{1}_{\{\tau_1^Y < \tau_4^Y\}}) \\ &\quad = [\exp\{-\beta d_1\} \bar{P}_{12}(d_1)(1 - \hat{P}_{34}(\beta)\tilde{P}_{43}(\beta))] \\ &\qquad \times [1 - \hat{P}_{21}(\beta)\tilde{P}_{12}(\beta) - \hat{P}_{34}(\beta)\tilde{P}_{43}(\beta) + \hat{P}_{21}(\beta)\tilde{P}_{12}(\beta)\hat{P}_{34}(\beta)\tilde{P}_{43}(\beta) \\ &\qquad\quad - \hat{P}_{31}(\beta)\tilde{P}_{43}(\beta)\hat{P}_{24}(\beta)\tilde{P}_{12}(\beta)]^{-1}, \end{aligned} \tag{17}$$

$$\begin{aligned} &\mathrm{E}(\exp\{-\beta\tau^Y\}\, \mathbf{1}_{\{\tau_1^Y > \tau_4^Y\}}) \\ &\quad = [\exp\{-\beta d_4\} \bar{P}_{43}(d_4)\hat{P}_{24}(\beta)\tilde{P}_{12}(\beta)] \\ &\qquad \times [1 - \hat{P}_{21}(\beta)\tilde{P}_{12}(\beta) - \hat{P}_{34}(\beta)\tilde{P}_{43}(\beta) + \hat{P}_{21}(\beta)\tilde{P}_{12}(\beta)\hat{P}_{34}(\beta)\tilde{P}_{43}(\beta) \\ &\qquad\quad - \hat{P}_{31}(\beta)\tilde{P}_{43}(\beta)\hat{P}_{24}(\beta)\tilde{P}_{12}(\beta)]^{-1}, \end{aligned} \tag{18}$$

and, when $Y_0^{(\varepsilon)} = l_2 - \varepsilon$,

$$\begin{aligned} &\mathrm{E}(\exp\{-\beta\tau^Y\}\, \mathbf{1}_{\{\tau_1^Y < \tau_4^Y\}}) \\ &\quad = [\exp\{-\beta d_1\} \bar{P}_{12}(d_1)\hat{P}_{31}(\beta)\tilde{P}_{43}(\beta)] \\ &\qquad \times [1 - \hat{P}_{21}(\beta)\tilde{P}_{12}(\beta) - \hat{P}_{34}(\beta)\tilde{P}_{43}(\beta) + \hat{P}_{21}(\beta)\tilde{P}_{12}(\beta)\hat{P}_{34}(\beta)\tilde{P}_{43}(\beta) \\ &\qquad\quad - \hat{P}_{31}(\beta)\tilde{P}_{43}(\beta)\hat{P}_{24}(\beta)\tilde{P}_{12}(\beta)]^{-1}, \end{aligned} \tag{19}$$

$$\begin{aligned} &\mathrm{E}(\exp\{-\beta\tau^Y\}\, \mathbf{1}_{\{\tau_1^Y > \tau_4^Y\}}) \\ &\quad = [\exp\{-\beta d_4\} \bar{P}_{43}(d_4)(1 - \hat{P}_{21}(\beta)\tilde{P}_{12}(\beta))] \\ &\qquad \times [1 - \hat{P}_{21}(\beta)\tilde{P}_{12}(\beta) - \hat{P}_{34}(\beta)\tilde{P}_{43}(\beta) + \hat{P}_{21}(\beta)\tilde{P}_{12}(\beta)\hat{P}_{34}(\beta)\tilde{P}_{43}(\beta) \\ &\qquad\quad - \hat{P}_{31}(\beta)\tilde{P}_{43}(\beta)\hat{P}_{24}(\beta)\tilde{P}_{12}(\beta)]^{-1}. \end{aligned} \tag{20}$$

4. Main results

In Section 2 we stated that the main difficulty with the Brownian motion is that the probability that W_t^μ will return to the origin at arbitrarily small times is 1, and there are infinitely many excursions outside but also inside the barriers. We therefore introduced the new processes $Y_t^{(\varepsilon)}$ and (Z_t^Y, V_t^Y) with transition densities for Z_t^Y defined in (6)–(12).

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In order to simplify the expressions, we define

$$\Psi(x) := 2\sqrt{\pi}x\mathcal{N}(\sqrt{2}x) - \sqrt{\pi}x + \mathrm{e}^{-x^2},$$

where $\mathcal{N}(\cdot)$ is the cumulative distribution function for the standard normal distribution.

Theorem 1. *For a Brownian motion W_t^{μ}, with $\tau_1^{W^\mu}$, $\tau_4^{W^\mu}$, and τ^{W^μ} defined as in (1), (4), and (5), and $S_t = W_t^{\mu}$, we have the following Laplace transforms. When $W_0^{\mu} = l_1$,*

$$\mathrm{E}(\exp\{-\beta\tau^{W^\mu}\}\mathbf{1}_{\{\tau_1^{W^\mu} < \tau_4^{W^\mu}\}}) = \frac{G_1(d_1, d_4, \mu)}{G(d_1, d_4, \mu)}, \tag{21}$$

$$\mathrm{E}(\exp\{-\beta\tau^{W^\mu}\}\mathbf{1}_{\{\tau_1^{W^\mu} > \tau_4^{W^\mu}\}}) = \frac{G_2(d_4, d_1, -\mu)}{G(d_1, d_4, \mu)}, \tag{22}$$

$$\mathrm{E}(\exp\{-\beta\tau^{W^\mu}\}) = \frac{G_1(d_1, d_4, \mu) + G_2(d_4, d_1, -\mu)}{G(d_1, d_4, \mu)}. \tag{23}$$

When $W_0^{\mu} = l_2$,

$$\mathrm{E}(\exp\{-\beta\tau^{W^\mu}\}\mathbf{1}_{\{\tau_1^{W^\mu} < \tau_4^{W^\mu}\}}) = \frac{G_2(d_1, d_4, \mu)}{G(d_1, d_4, \mu)}, \tag{24}$$

$$\mathrm{E}(\exp\{-\beta\tau^{W^\mu}\}\mathbf{1}_{\{\tau_1^{W^\mu} > \tau_4^{W^\mu}\}}) = \frac{G_1(d_4, d_1, -\mu)}{G(d_1, d_4, \mu)}, \tag{25}$$

$$\mathrm{E}(\exp\{-\beta\tau^{W^\mu}\}) = \frac{G_1(d_4, d_1, -\mu) + G_2(d_1, d_4, \mu)}{G(d_1, d_4, \mu)}. \tag{26}$$

Here

$$\begin{aligned}
G_1(x, y, z) = {} & \exp\{-2(l_1 - l_2)\sqrt{2\beta + z^2} - \beta x\}\bigg\{\sqrt{y}\Psi\bigg(|z|\sqrt{\frac{x}{2}}\bigg) + z\sqrt{\frac{\pi x y}{2}}\bigg\} \\
& + \frac{(1 - \exp\{-2(l_1 - l_2)\sqrt{2\beta + z^2}\})\mathrm{e}^{-\beta x}}{2\sqrt{2\beta + z^2}}\bigg\{\Psi\bigg(|z|\sqrt{\frac{x}{2}}\bigg) + z\sqrt{\frac{\pi x}{2}}\bigg\} \\
& \quad \times \bigg\{\sqrt{\frac{2}{\pi}}\Psi\bigg(\sqrt{\frac{(2\beta + z^2)y}{2}}\bigg) + \sqrt{(2\beta + z^2)y}\bigg\}, \\
G_2(x, y, z) = {} & \exp\{-(l_1 - l_2)(\sqrt{2\beta + z^2} - z) - \beta x\}\bigg\{\sqrt{y}\Psi\bigg(|z|\sqrt{\frac{x}{2}}\bigg) + z\sqrt{\frac{\pi x y}{2}}\bigg\}, \\
G(x, y, z) = {} & \exp\{-2(l_1 - l_2)\sqrt{2\beta + z^2}\} \\
& \times \bigg\{\sqrt{y}\Psi\bigg(\sqrt{\frac{(2\beta + z^2)x}{2}}\bigg) + \sqrt{x}\Psi\bigg(\sqrt{\frac{(2\beta + z^2)y}{2}}\bigg)\bigg\} \\
& + \frac{(1 - \exp\{-2(l_1 - l_2)\sqrt{2\beta + z^2}\})}{2\sqrt{2\beta + z^2}} \\
& \quad \times \bigg\{\Psi\bigg(\sqrt{\frac{(2\beta + z^2)x}{2}}\bigg) + \sqrt{\frac{(2\beta + z^2)\pi x}{2}}\bigg\} \\
& \quad \times \bigg\{\sqrt{\frac{2}{\pi}}\Psi\bigg(\sqrt{\frac{(2\beta + z^2)y}{2}}\bigg) + \sqrt{(2\beta + z^2)y}\bigg\}.
\end{aligned}$$

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Proof. We apply the transition densities in (6)–(12) to the results in (17)–(20) and take the limit as $\varepsilon \to 0$. According to the definition of $Y^{(\varepsilon)}$, we know that

$$Y_t^{(\varepsilon)} \to W_t^{\mu} \quad \text{almost surely for all } t.$$

As we saw in [6], since $Y_t^{(\varepsilon)} \to W_t^{\mu}$ almost surely for all t, by taking the limit $\varepsilon \to 0$, the quantities defined based on $Y_t^{(\varepsilon)}$ converge to those based on Brownian motion with drift. Therefore, we will obtain the results given in (21), (22), (24), and (25). We can thus obtain (23) and (26) by

$$\mathrm{E}(\exp\{-\beta\tau^{W^{\mu}}\}) = \mathrm{E}(\exp\{-\beta\tau^{W^{\mu}}\}\mathbf{1}_{\{\tau_1^{W^{\mu}} < \tau_4^{W^{\mu}}\}}) + \mathrm{E}(\exp\{-\beta\tau^{W^{\mu}}\}\mathbf{1}_{\{\tau_1^{W^{\mu}} > \tau_4^{W^{\mu}}\}}).$$

Corollary 1. *For a standard Brownian motion ($\mu = 0$), we have the following Laplace transforms. When $W_0 = l_1$,*

$$\begin{aligned}
\mathrm{E}(\exp\{-\beta\tau^{W^{\mu}}\}\mathbf{1}_{\{\tau_1^{W} < \tau_4^{W}\}}) &= \frac{G_1(d_1, d_4, 0)}{G(d_1, d_4, 0)},\\
\mathrm{E}(\exp\{-\beta\tau^{W^{\mu}}\}\mathbf{1}_{\{\tau_1^{W} > \tau_4^{W}\}}) &= \frac{G_2(d_4, d_1, 0)}{G(d_1, d_4, 0)},\\
\mathrm{E}(\exp\{-\beta\tau^{W^{\mu}}\}) &= \frac{G_1(d_1, d_4, 0) + G_2(d_4, d_1, 0)}{G(d_1, d_4, 0)}.
\end{aligned}$$

When $W_0 = l_2$,

$$\begin{aligned}
\mathrm{E}(\exp\{-\beta\tau^{W^{\mu}}\}\mathbf{1}_{\{\tau_1^{W} < \tau_4^{W}\}}) &= \frac{G_2(d_1, d_4, 0)}{G(d_1, d_4, 0)},\\
\mathrm{E}(\exp\{-\beta\tau^{W^{\mu}}\}\mathbf{1}_{\{\tau_1^{W} > \tau_4^{W}\}}) &= \frac{G_1(d_4, d_1, 0)}{G(d_1, d_4, 0)},\\
\mathrm{E}(\exp\{-\beta\tau^{W^{\mu}}\}) &= \frac{G_1(d_4, d_1, 0) + G_2(d_1, d_4, 0)}{G(d_1, d_4, 0)}.
\end{aligned}$$

Here

$$\begin{aligned}
G_1(x, y, 0) &= \exp\{-2(l_1 - l_2)\sqrt{2\beta} - \beta x\}\sqrt{y}\\
&\quad + \frac{(1 - \exp\{-2(l_1 - l_2)\sqrt{2\beta}\})\mathrm{e}^{-\beta x}}{2\sqrt{2\beta}}\left\{\sqrt{\frac{2}{\pi}}\Psi(\sqrt{\beta y}) + \sqrt{2\beta y}\right\},\\
G_2(x, y, 0) &= \exp\{-(l_1 - l_2)\sqrt{2\beta} - \beta x\}\sqrt{y},\\
G(x, y, 0) &= \exp\{-2(l_1 - l_2)\sqrt{2\beta}\}\{\sqrt{y}\Psi(\sqrt{\beta x}) + \sqrt{x}\Psi(\sqrt{\beta y})\}\\
&\quad + \frac{(1 - \exp\{-2(l_1 - l_2)\sqrt{2\beta}\})}{2\sqrt{2\beta}}\{\Psi(\sqrt{\beta x}) + \sqrt{\beta\pi x}\}\\
&\quad \times \left\{\sqrt{\frac{2}{\pi}}\Psi(\sqrt{\beta y}) + \sqrt{2\beta y}\right\}.
\end{aligned}$$

Remark 1. By taking the limit $l_1 - l_2 \to 0$, we obtain the result for the single-barrier two-sided excursion case, as in [6].

Remark 2. If we only want to consider the excursion above a barrier, we can let $l_2 \to -\infty$. Similarly, for the excursion below a barrier, we can let $l_1 \to +\infty$. These results have been shown in [6].

Corollary 2. *For a Brownian motion W_t^μ, with τ^{W^μ} defined as in (5) and $S_t = W_t^\mu$, we have the following Laplace transforms. When $W_0^\mu = x_0,\ x_0 > l_1$,*

$$\begin{aligned}
&\mathrm{E}(\exp\{-\beta\tau^{W^\mu}\})\\
&= \bigg\{\exp\{-(\mu+\sqrt{2\beta+\mu^2})(x_0-l_1)\}\mathcal{N}\bigg(\sqrt{(2\beta+\mu^2)d_1}-\frac{x_0-l_1}{\sqrt{d_1}}\bigg)\\
&\quad + \exp\{-(\mu-\sqrt{2\beta+\mu^2})(x_0-l_1)\}\mathcal{N}\bigg(-\sqrt{(2\beta+\mu^2)d_1}-\frac{x_0-l_1}{\sqrt{d_1}}\bigg)\bigg\}\\
&\quad \times \frac{G_1(d_1,d_4,\mu)+G_2(d_4,d_1,-\mu)}{G(d_1,d_4,\mu)}\\
&\quad + \mathrm{e}^{-\beta d_1}\bigg\{1-\exp\{-(\mu+|\mu|)(x_0-l_1)\}\mathcal{N}\bigg(|\mu|\sqrt{d_1}-\frac{x_0-l_1}{\sqrt{d_1}}\bigg)\\
&\qquad - \exp\{-(\mu-|\mu|)(x_0-l_1)\}\mathcal{N}\bigg(-|\mu|\sqrt{d_1}-\frac{x_0-l_1}{\sqrt{d_1}}\bigg)\bigg\}. \qquad (27)
\end{aligned}$$

When $W_0^\mu = x_0$, $l_2 \le x_0 \le l_1$,

$$\begin{aligned}
&\mathrm{E}(\exp\{-\beta\tau^{W^\mu}\})\\
&= \frac{\mathrm{e}^{(l_1-x_0)\mu}\{\mathrm{e}^{\sqrt{2\beta+\mu^2}(x_0-l_2)}-\mathrm{e}^{-\sqrt{2\beta+\mu^2}(x_0-l_2)}\}\{G_1(d_1,d_4,\mu)+G_2(d_4,d_1,-\mu)\}}{\{\mathrm{e}^{\sqrt{2\beta+\mu^2}(l_1-l_2)}-\mathrm{e}^{-\sqrt{2\beta+\mu^2}(l_1-l_2)}\}G(d_1,d_2,\mu)}\\
&\quad + \frac{\mathrm{e}^{(l_2-x_0)\mu}\{\mathrm{e}^{\sqrt{2\beta+\mu^2}(l_1-x_0)}-\mathrm{e}^{-\sqrt{2\beta+\mu^2}(l_1-x_0)}\}\{G_2(d_1,d_4,\mu)+G_1(d_4,d_1,-\mu)\}}{\{\mathrm{e}^{\sqrt{2\beta+\mu^2}(l_1-l_2)}-\mathrm{e}^{-\sqrt{2\beta+\mu^2}(l_1-l_2)}\}G(d_1,d_2,\mu)}. \qquad (28)
\end{aligned}$$

When $W_0^\mu = x_0$, $x_0 < l_2$,

$$\begin{aligned}
&\mathrm{E}(\exp(-\beta\tau^{W^\mu}))\\
&= \bigg\{\exp\{(\mu-\sqrt{2\beta+\mu^2})(l_2-x)\}\mathcal{N}\bigg(\sqrt{(2\beta+\mu^2)d_4}-\frac{l_2-x}{\sqrt{d_4}}\bigg)\\
&\quad + \exp\{(\mu+\sqrt{2\beta+\mu^2})(l_2-x)\}\mathcal{N}\bigg(-\sqrt{(2\beta+\mu^2)d_4}-\frac{l_2-x}{\sqrt{d_4}}\bigg)\bigg\}\\
&\quad \times \frac{G_1(d_4,d_1,-\mu)+G_2(d_1,d_4,\mu)}{G(d_1,d_4,\mu)}\\
&\quad + \mathrm{e}^{-\beta d_4}\bigg\{1-\exp\{(\mu-|\mu|)(l_2-x)\}\mathcal{N}\bigg(|\mu|\sqrt{d_4}-\frac{l_2-x}{\sqrt{d_4}}\bigg)\\
&\qquad - \exp\{(\mu+|\mu|)(l_2-x)\}\mathcal{N}\bigg(-|\mu|\sqrt{d_4}-\frac{l_2-x}{\sqrt{d_4}}\bigg)\bigg\}.
\end{aligned}$$

Proof. We will first prove the case when $x_0 > l_1$. Define $T = \inf\{t \mid W_t^\mu = l_1\}$, i.e. the first time W_t^μ hits l_1. By definition we have $\tau^{W^\mu} = d_1$ if $T \ge d_1$ and $\tau^{W^\mu} = T + \tau^{\tilde{W}^\mu}$ if $T < d_1$, where $\tilde{W}^\mu$ here stands for a Brownian motion with drift started from l_1. As a result,

$$\begin{aligned}
\mathrm{E}(\exp\{-\beta\tau^{W^\mu}\}) &= \mathrm{E}(\exp\{-\beta\tau^{W^\mu}\}\mathbf{1}_{\{T\ge d_1\}}) + \mathrm{E}(\exp\{-\beta\tau^{W^\mu}\}\mathbf{1}_{\{T<d_1\}})\\
&= \mathrm{e}^{-\beta d_1}\,\mathrm{P}(T\ge d_1) + \mathrm{E}(\mathrm{e}^{-\beta T}\mathbf{1}_{\{T<d_1\}})\,\mathrm{E}(\exp\{-\beta\tau^{\widetilde{W}^\mu}\}).
\end{aligned}$$

The term $\mathrm{E}(\exp\{-\beta\tau^{\widetilde{W}^\mu}\})$ has been calculated in Theorem 1 (see (23)). The density for T was given in [2, Equation 2.0.2] as

$$p_{x_0} = \frac{|l_1 - x_0|}{\sqrt{2\pi t^3}} \exp\left\{-\frac{(l_i - x_0 - \mu t)^2}{2t}\right\}.$$

We can therefore calculate

$$\begin{aligned}
\mathrm{P}(T \geq d_1) &= 1 - \exp\{-(\mu + |\mu|)(x_0 - l_1)\}\mathcal{N}\left(|\mu|\sqrt{d_1} - \frac{x_0 - l_1}{\sqrt{d_1}}\right)\\
&\quad - \exp\{-(\mu - |\mu|)(x_0 - l_1)\}\mathcal{N}\left(-|\mu|\sqrt{d_1} - \frac{x_0 - l_1}{\sqrt{d_1}}\right),\\
\mathrm{E}(\mathrm{e}^{-\beta T}\mathbf{1}_{\{T<d_1\}}) &= \exp\{-(\mu + \sqrt{2\beta + \mu^2})(x_0 - l_1)\}\mathcal{N}\left(\sqrt{(2\beta + \mu^2)d_1} - \frac{x_0 - l_1}{\sqrt{d_1}}\right)\\
&\quad + \exp\{-(\mu - \sqrt{2\beta + \mu^2})(x_0 - l_1)\}\mathcal{N}\left(-\sqrt{(2\beta + \mu^2)d_1} - \frac{x_0 - l_1}{\sqrt{d_1}}\right).
\end{aligned}$$

We therefore obtain the result in (27). For the case when $x_0 < l_2$, we can apply the same argument.

When $l_2 \leq x_0 \leq l_1$, we define $\widetilde{T} = \inf(t \mid W_t^\mu \notin (l_2, l_1))$. By definition we have $\tau^{W^\mu} = T + \tau^{\widetilde{W}^\mu}$ if $W_T^\mu = l_1$ and $\tau^{W^\mu} = T + \tau^{\underline{W}^\mu}$ if $W_T^\mu = l_2$, where $\underline{W}^\mu$ stands for a Brownian motion with drift started from l_2. Consequently,

$$\begin{aligned}
&\mathrm{E}(\exp\{-\beta\tau^{W^\mu}\})\\
&\quad = \mathrm{E}(\mathrm{e}^{-\beta T}\exp\{-\beta\tau^{\widetilde{W}^\mu}\}\mathbf{1}_{\{T=l_1\}}) + \mathrm{E}(\mathrm{e}^{-\beta T}\exp\{-\beta\tau^{\underline{W}^\mu}\}\mathbf{1}_{\{T=l_2\}})\\
&\quad = \mathrm{E}(\mathrm{e}^{-\beta T}\mathbf{1}_{\{T=l_1\}})\,\mathrm{E}(\exp\{-\beta\tau^{\widetilde{W}^\mu}\}) + \mathrm{E}(\mathrm{e}^{-\beta T}\mathbf{1}_{\{T=l_2\}})\,\mathrm{E}(\exp\{-\beta\tau^{\underline{W}^\mu}\}).
\end{aligned}$$

The terms $\mathrm{E}(\exp\{-\beta\tau^{\widetilde{W}^\mu}\})$ and $\mathrm{E}(\exp\{-\beta\tau^{\underline{W}^\mu}\})$ have been obtained in Theorem 1 (see (23) and (26)). According to [2, Equation 3.0.5], we have

$$\begin{aligned}
\mathrm{E}(\mathrm{e}^{-\beta T}\mathbf{1}_{\{T=l_1\}}) &= \frac{\mathrm{e}^{(l_1-x_0)\mu}\{\mathrm{e}^{\sqrt{2\beta+\mu^2}(x_0-l_2)} - \mathrm{e}^{-\sqrt{2\beta+\mu^2}(x_0-l_2)}\}}{\mathrm{e}^{\sqrt{2\beta+\mu^2}(l_1-l_2)} - \mathrm{e}^{-\sqrt{2\beta+\mu^2}(l_1-l_2)}},\\
\mathrm{E}(\mathrm{e}^{-\beta T}\mathbf{1}_{\{T=l_2\}}) &= \frac{\mathrm{e}^{(l_2-x_0)\mu}\{\mathrm{e}^{\sqrt{2\beta+\mu^2}(l_1-x_0)} - \mathrm{e}^{-\sqrt{2\beta+\mu^2}(l_1-x_0)}\}}{\mathrm{e}^{\sqrt{2\beta+\mu^2}(l_1-l_2)} - \mathrm{e}^{-\sqrt{2\beta+\mu^2}(l_1-l_2)}}.
\end{aligned}$$

We have therefore obtained (28).

Theorem 2. *The probability that W_t^μ with $W_0^\mu = x_0$, $l_2 \leq x_0 \leq l_1$, achieves an excursion above l_1 with length at least d_1 before it achieves an excursion below l_2 with length at least d_4 is*

$$\begin{aligned}
\mathrm{P}(\tau_1^{W^\mu} < \tau_4^{W^\mu}) &= \frac{\mathrm{e}^{(l_1-x_0)\mu}\{\mathrm{e}^{|\mu|(x_0-l_2)} - \mathrm{e}^{-|\mu|(x_0-l_2)}\}F_1(d_1, d_4, \mu)}{\{\mathrm{e}^{|\mu|(l_1-l_2)} - \mathrm{e}^{-|\mu|(l_1-l_2)}\}F(d_1, d_4, \mu)}\\
&\quad + \frac{\mathrm{e}^{(l_2-x_0)\mu}\{\mathrm{e}^{|\mu|(l_1-x_0)} - \mathrm{e}^{-|\mu|(l_1-x_0)}\}F_2(d_1, d_4, \mu)}{\{\mathrm{e}^{|\mu|(l_1-l_2)} - \mathrm{e}^{-|\mu|(l_1-l_2)}\}F(d_1, d_4, \mu)},\\
\mathrm{P}(\tau_1^{W^\mu} > \tau_4^{W^\mu}) &= \frac{\mathrm{e}^{(l_1-x_0)\mu}\{\mathrm{e}^{|\mu|(x_0-l_2)} - \mathrm{e}^{-|\mu|(x_0-l_2)}\}F_2(d_4, d_1, -\mu)}{\{\mathrm{e}^{|\mu|(l_1-l_2)} - \mathrm{e}^{-|\mu|(l_1-l_2)}\}F(d_1, d_4, \mu)}\\
&\quad + \frac{\mathrm{e}^{(l_2-x_0)\mu}\{\mathrm{e}^{|\mu|(l_1-x_0)} - \mathrm{e}^{-|\mu|(l_1-x_0)}\}F_1(d_4, d_1, -\mu)}{\{\mathrm{e}^{|\mu|(l_1-l_2)} - \mathrm{e}^{-|\mu|(l_1-l_2)}\}F(d_1, d_4, \mu)},
\end{aligned}$$

where

$$F_1(x,y,z) = \exp\{-2(l_1-l_2)|z|\}\left\{\sqrt{y}\Psi\left(|z|\sqrt{\frac{x}{2}}\right) + z\sqrt{\frac{\pi xy}{2}}\right\} + \frac{(1-\exp\{-2(l_1-l_2)|z|)}{2|z|}\}\left\{\Psi\left(|z|\sqrt{\frac{x}{2}}\right) + z\sqrt{\frac{\pi x}{2}}\right\} \times \left\{\sqrt{\frac{2}{\pi}}\Psi\left(|z|\sqrt{\frac{y}{2}}\right) + |z|\sqrt{y}\right\},$$

$$F_2(x,y,z) = \exp\{-(l_1-l_2)(|z|-z)\}\left\{\sqrt{y}\Psi\left(|z|\sqrt{\frac{x}{2}}\right) + z\sqrt{\frac{\pi xy}{2}}\right\},$$

$$F(x,y,z) = \exp\{-2(l_1-l_2)|z|\}\left\{\sqrt{y}\Psi\left(|z|\sqrt{\frac{x}{2}}\right) + \sqrt{x}\Psi\left(|z|\sqrt{\frac{y}{2}}\right)\right\} + \frac{(1-\exp\{-2(l_1-l_2)|z|)}{2|z|}\}\left\{\Psi\left(|z|\sqrt{\frac{x}{2}}\right) + |z|\sqrt{\frac{\pi x}{2}}\right\} \times \left\{\sqrt{\frac{2}{\pi}}\Psi\left(|z|\sqrt{\frac{y}{2}}\right) + |z|\sqrt{y}\right\}.$$

Proof. From Theorem 1 and (28), we actually know that, when $W_0^\mu = x_0,\ l_2 \le x_0 \le l_1$,

$$\mathrm{E}(\exp\{-\beta\tau^{W^\mu}\}\mathbf{1}_{\{\tau_1^{W^\mu}<\tau_4^{W^\mu}\}}) = \frac{\mathrm{e}^{(l_1-x_0)\mu}\{\mathrm{e}^{|\mu|(x_0-l_2)} - \mathrm{e}^{-|\mu|(x_0-l_2)}\}G_1(d_1,d_4,\mu)}{\{\mathrm{e}^{|\mu|(l_1-l_2)} - \mathrm{e}^{-|\mu|(l_1-l_2)}\}G(d_1,d_4,\mu)} + \frac{\mathrm{e}^{(l_2-x_0)\mu}\{\mathrm{e}^{|\mu|(l_1-x_0)} - \mathrm{e}^{-|\mu|(l_1-x_0)}\}G_2(d_1,d_4,\mu)}{\{\mathrm{e}^{|\mu|(l_1-l_2)} - \mathrm{e}^{-|\mu|(l_1-l_2)}\}G(d_1,d_4,\mu)}, \quad (29)$$

$$\mathrm{E}(\exp\{-\beta\tau^{W^\mu}\}\mathbf{1}_{\{\tau_1^{W^\mu}>\tau_4^{W^\mu}\}}) = \frac{\mathrm{e}^{(l_1-x_0)\mu}\{\mathrm{e}^{|\mu|(x_0-l_2)} - \mathrm{e}^{-|\mu|(x_0-l_2)}\}G_2(d_4,d_1,-\mu)}{\{\mathrm{e}^{|\mu|(l_1-l_2)} - \mathrm{e}^{-|\mu|(l_1-l_2)}\}G(d_1,d_4,\mu)} + \frac{\mathrm{e}^{(l_2-x_0)\mu}\{\mathrm{e}^{|\mu|(l_1-x_0)} - \mathrm{e}^{-|\mu|(l_1-x_0)}\}G_1(d_4,d_1,-\mu)}{\{\mathrm{e}^{|\mu|(l_1-l_2)} - \mathrm{e}^{-|\mu|(l_1-l_2)}\}G(d_1,d_4,\mu)}. \quad (30)$$

Setting $\beta = 0$ in (29) and (30) yields the results.

Theorem 2 leads to the following remarkable result.

Corollary 3. *For a standard Brownian motion W_t with $W_0 = x_0,\ l_2 \le x_0 \le l_1$, we have*

$$\mathrm{P}(\tau_1^W < \tau_4^W) = \frac{\sqrt{d_4} + (x_0-l_2)\sqrt{2/\pi}}{\sqrt{d_1}+\sqrt{d_4}+(l_1-l_2)\sqrt{2/\pi}},$$

$$\mathrm{P}(\tau_1^W > \tau_4^W) = \frac{\sqrt{d_1} + (l_1-x_0)\sqrt{2/\pi}}{\sqrt{d_1}+\sqrt{d_4}+(l_1-l_2)\sqrt{2/\pi}}.$$

Remark 3. When we take $l_1 \to 0$, $l_2 \to 0$, and $x_0 \to 0$, we can obtain the results for the one-barrier case, as in [6].

Remark 4. We observe that the formulae in Corollary 3 are linear in the starting point x_0, as is also the case for the exit probabilities of a standard Brownian motion or, more generally, a diffusion in its natural state (see [3, Section 16.5]) . If we set $d_1 \to 0$ and $d_4 \to 0$, we recover $(x_0-l_2)/(l_1-l_2)$ and $(l_1-x_0)/(l_1-l_2)$, the exit probabilities.

We will now extend Corollary 2 to obtain the joint distribution of W_t and τ^W at an exponential time. This is an application of (28) and Girsanov's theorem.

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Theorem 3. *For a standard Brownian motion W_t with $W_0 = x_0$, $l_2 \le x_0 \le l_1$, and τ^W defined as in (3) with $S_t = W_t$, we have the following results. For the case in which $x > l_1$,*

$$\mathrm{P}(W_{\tilde{T}} \in \mathrm{d}x,\ \tau^W < \tilde{T}) = (a_1(x_0) f(x - l_1, d_1) + a_2(x_0) f(x - l_2, d_4) + a_1(x_0) h(x - l_1, d_1))\, \mathrm{d}x.$$

For the case in which $l_2 \le x \le l_1$,

$$\mathrm{P}(W_{\tilde{T}} \in \mathrm{d}x,\ \tau^W < \tilde{T}) = (a_1(x_0) f(x - l_1, d_1) + a_2(x_0) f(x - l_2, d_4))\, \mathrm{d}x. \tag{31}$$

For the case in which $x < l_2$,

$$\mathrm{P}(W_{\tilde{T}} \in \mathrm{d}x,\ \tau^W < \tilde{T}) = (a_1(x_0) f(x - l_1, d_1) + a_2(x_0) f(x - l_2, d_4) + a_2(x_0) h(x - l_2, d_4))\, \mathrm{d}x.$$

Here $\tilde{T}$ is a random variable with an exponential distribution of parameter γ that is independent of W_t and

$$f(x, y) = \frac{\mathrm{e}^{-\sqrt{2\gamma}|x|}}{\sqrt{2\gamma}} - \mathrm{e}^{\gamma y - \sqrt{2\gamma}|x|} \sqrt{2\pi y}\, \mathcal{N}(-\sqrt{2\gamma y}),$$

$$h(x, y) = \sqrt{2\pi y}\, \mathrm{e}^{\gamma y} \bigg\{ \mathrm{e}^{-\sqrt{2\gamma}|x|} \mathcal{N}\bigg(\frac{|x|}{\sqrt{y}} - \sqrt{2\gamma y} \bigg) - \mathrm{e}^{\sqrt{2\gamma}|x|} \mathcal{N}\bigg(-\frac{|x|}{\sqrt{y}} - \sqrt{2\gamma y} \bigg) \bigg\},$$

$$a_1(x_0) = \frac{\gamma \{\mathrm{e}^{\sqrt{2\gamma}(x_0 - l_2)} - \mathrm{e}^{-\sqrt{2\gamma}(x_0 - l_2)}\} b_1(d_1, d_4)}{G\{\mathrm{e}^{\sqrt{2\gamma}(l_1 - l_2)} - \mathrm{e}^{-\sqrt{2\gamma}(l_1 - l_2)}\}} + \frac{\gamma \{\mathrm{e}^{\sqrt{2\gamma}(l_1 - x_0)} - \mathrm{e}^{-\sqrt{2\gamma}(l_1 - x_0)}\} b_2(d_1, d_4)}{G\{\mathrm{e}^{\sqrt{2\gamma}(l_1 - l_2)} - \mathrm{e}^{-\sqrt{2\gamma}(l_1 - l_2)}\}},$$

$$a_2(x_0) = \frac{\gamma \{\mathrm{e}^{\sqrt{2\gamma}(x_0 - l_2)} - \mathrm{e}^{-\sqrt{2\gamma}(x_0 - l_2)}\} b_2(d_4, d_1)}{G\{\mathrm{e}^{\sqrt{2\gamma}(l_1 - l_2)} - \mathrm{e}^{-\sqrt{2\gamma}(l_1 - l_2)}\}} + \frac{\gamma \{\mathrm{e}^{\sqrt{2\gamma}(l_1 - x_0)} - \mathrm{e}^{-\sqrt{2\gamma}(l_1 - x_0)}\} b_1(d_4, d_1)}{G\{\mathrm{e}^{\sqrt{2\gamma}(l_1 - l_2)} - \mathrm{e}^{-\sqrt{2\gamma}(l_1 - l_2)}\}},$$

$$b_1(x, y) = \mathrm{e}^{-2(l_1 - l_2)\sqrt{2\gamma} - \gamma x} \sqrt{y} + \frac{1 - \mathrm{e}^{-2\gamma\sqrt{2\gamma}}}{2\sqrt{2\gamma}} \mathrm{e}^{-\gamma x} \bigg\{ \sqrt{\frac{2}{\pi}} \Psi(\sqrt{\gamma y}) + \sqrt{2\gamma y} \bigg\},$$

$$b_2(x, y) = \mathrm{e}^{-(l_1 - l_2)\sqrt{2\gamma} - \gamma x} \sqrt{y},$$

$$G = \mathrm{e}^{-2(l_1 - l_2)\sqrt{2\gamma}} \{ \sqrt{d_4} \Psi(\sqrt{\gamma d_1}) + \sqrt{d_1} \Psi(\sqrt{\gamma d_4}) \} + \frac{(1 - \mathrm{e}^{-2(l_1 - l_2)\sqrt{2\gamma}})}{2\sqrt{2\gamma}} \{ \Psi(\sqrt{\gamma d_1}) + \sqrt{\gamma \pi d_1} \} \bigg\{ \sqrt{\frac{2}{\pi}} \Psi(\sqrt{\gamma d_4}) + \sqrt{2\gamma d_4} \bigg\}.$$

Proof. See Appendix A.

5. Pricing double-barrier Parisian options

We want to price a double-barrier Parisian call option with the current price of its underlying asset being x, $L_1 < x < L_2$, the owner of which will obtain the right to exercise it when either

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the length of the excursion above the barrier L_1 reaches d_1, or the length of the excursion below the barrier L_2 reaches d_2 before T. Its price formula is given by

$$\mathrm{DP}_{\text{min-in call}} = \mathrm{e}^{-rT}\,\mathrm{E}_Q((S_T - K)^+ \mathbf{1}_{\{\tau^S < T\}}),$$

where S is the underlying stock price, Q denotes the risk neutral measure, and τ^S is defined with respect to barriers L_1 and L_2. The subscript *min-in call* means it is a call option which will be triggered when the minimum of two stopping times, τ_1^S and τ_4^S, is less than T, i.e. $\tau^S < T$. We assume that S is a geometric Brownian motion, i.e.

$$\mathrm{d}S_t = rS_t\,\mathrm{d}t + \sigma S_t\,\mathrm{d}W_t, \qquad S_0 = x,$$

where $L_1 < x < L_2$, r is the risk free rate, and W_t with $W_0 = 0$ is a standard Brownian motion under Q. Set

$$m = \frac{1}{\sigma}\biggl(r - \frac{1}{2}\sigma^2\biggr), \qquad b = \frac{1}{\sigma}\ln\biggl(\frac{K}{x}\biggr), \qquad B_t = mt + W_t,$$

$$l_1 = \frac{1}{\sigma}\ln\biggl(\frac{L_1}{x}\biggr), \qquad l_2 = \frac{1}{\sigma}\ln\biggl(\frac{L_2}{x}\biggr).$$

We have

$$S_t = x\exp\bigl\{\bigl(r - \tfrac{1}{2}\sigma^2\bigr)t + \sigma W_t\bigr\} = x\exp\{\sigma(mt + W_t)\} = x\mathrm{e}^{\sigma B_t}.$$

By applying Girsanov's theorem we have

$$\mathrm{DP}_{\text{min-in call}} = \mathrm{e}^{-(r+m^2/2)T}\,\mathrm{E}_P[(x\mathrm{e}^{\sigma B_T} - K)^+ \mathrm{e}^{mB_T}\mathbf{1}_{\{\tau^B < T\}}],$$

where P is a new measure, under which B_t is a standard Brownian motion with $B_0 = 0$, and τ^B is the stopping time defined with respect to barriers l_1 and l_2. We also define

$$\mathrm{DP}^*_{\text{min-in call}} = \mathrm{e}^{(r+m^2/2)T}\,\mathrm{DP}_{\text{min-in call}}.$$

We are going to show that we can obtain the Laplace transform of $\mathrm{DP}^*_{\text{min-in call}}$ with respect to T, denoted by $\mathcal{L}_T$.

Firstly, assuming that $\tilde{T}$ is a random variable with an exponential distribution of parameter γ which is independent of W_t, we have

$$\begin{aligned}
&\mathrm{E}_P[(x\mathrm{e}^{\sigma B_{\tilde{T}}} - K)^+ \mathrm{e}^{mB_{\tilde{T}}}\mathbf{1}_{\{\tau^B < \tilde{T}\}}] \\
&\quad = \int_b^\infty (x\mathrm{e}^{\sigma y} - K)\mathrm{e}^{my}\,\mathrm{P}(B_{\tilde{T}} \in \mathrm{d}y,\ \tau^B < \tilde{T}) \\
&\quad = \int_0^\infty \gamma\mathrm{e}^{-\gamma T}\int_b^\infty (x\mathrm{e}^{\sigma y} - K)\mathrm{e}^{my}\,\mathrm{P}(B_T \in \mathrm{d}y,\ \tau^B < T)\,\mathrm{d}T \\
&\quad = \gamma\int_0^\infty \mathrm{e}^{-\gamma T}\,\mathrm{E}_P[(x\mathrm{e}^{\sigma B_T} - K)^+ \mathrm{e}^{mB_T}\mathbf{1}_{\{\tau^B < T\}}]\,\mathrm{d}T \\
&\quad = \gamma\mathcal{L}_T.
\end{aligned}$$

Hence, we have

$$\mathcal{L}_T = \frac{1}{\gamma}\int_b^\infty (x\mathrm{e}^{\sigma y} - K)\mathrm{e}^{my}\,\mathrm{P}(B_{\tilde{T}} \in \mathrm{d}y,\ \tau^B < \tilde{T}).$$

Using the results of Theorem 3, this Laplace transform can be calculated explicitly.

When $b \geq l_1$, i.e. $K \geq L_1$, we have

$$\mathcal{L}_T = \frac{x}{\gamma} F_1(\sigma + m) - \frac{K}{\gamma} F_1(m),$$

where

$$\begin{aligned}
F_1(x) = a_1(0)&\bigg\{\frac{1}{\sqrt{2\gamma}} - \mathrm{e}^{\gamma d_1}\sqrt{2\pi d_1}\mathcal{N}(-\sqrt{2\gamma d_1})\bigg\}\frac{\exp\{\sqrt{2\gamma}l_1 + (x - \sqrt{2\gamma})b\}}{\sqrt{2\gamma} - x} \\
&+ a_2(0)\bigg\{\frac{1}{\sqrt{2\gamma}} - \mathrm{e}^{\gamma d_4}\sqrt{2\pi d_4}\mathcal{N}(-\sqrt{2\gamma d_4})\bigg\}\frac{\exp\{\sqrt{2\gamma}l_2 + (x - \sqrt{2\gamma})b\}}{\sqrt{2\gamma} - x} \\
&+ a_1(0)\sqrt{2\pi d_1}\mathrm{e}^{\gamma d_1} \\
&\quad\times \bigg\{\frac{2x\exp\{xl_1 - rd_1 + d_1x^2/2\}\mathcal{N}(x\sqrt{d_1} - (b - l_1)/\sqrt{d_1})}{2\gamma - x^2} \\
&\qquad + \frac{\exp\{\sqrt{2\gamma}l_1 + (x - \sqrt{2\gamma})b\}\mathcal{N}((b - l_1)/\sqrt{d_1} - \sqrt{2\gamma d_1})}{\sqrt{2\gamma} - x} \\
&\qquad + \frac{\exp\{-\sqrt{2\gamma}l_1 + (x + \sqrt{2\gamma})b\}\mathcal{N}(-(b - l_1)/\sqrt{d_1} - \sqrt{2\gamma d_1})}{\sqrt{2\gamma} + x}\bigg\}.
\end{aligned}$$

When $l_2 < b < l_1$, i.e. $L_2 < K < L_1$, we have

$$\mathcal{L}_T = \frac{x}{\gamma} F_2(\sigma + m) - \frac{K}{\gamma} F_2(m),$$

where

$$\begin{aligned}
F_2(x) = &\frac{2a_1(0)\mathrm{e}^{l_1 x}}{2\gamma - x^2}\{1 + x\sqrt{2\pi d_1}\mathrm{e}^{d_1x^2/2}\mathcal{N}(x\sqrt{d_1})\} \\
&- a_1(0)\bigg\{\frac{1}{\sqrt{2\gamma}} - \mathrm{e}^{\gamma d_1}\sqrt{2\pi d_1}\mathcal{N}(-\sqrt{2\gamma d_1})\bigg\}\frac{\exp\{-\sqrt{2\gamma}l_1 + (x + \sqrt{2\gamma})b\}}{\sqrt{2\gamma} - x} \\
&+ a_2(0)\bigg\{\frac{1}{\sqrt{2\gamma}} - \mathrm{e}^{\gamma d_4}\sqrt{2\pi d_4}\mathcal{N}(-\sqrt{2\gamma d_4})\bigg\}\frac{\exp\{\sqrt{2\gamma}l_2 + (x - \sqrt{2\gamma})b\}}{\sqrt{2\gamma} - x}.
\end{aligned}$$

When $b \leq l_2$, i.e. $K \leq L_2$, we have

$$\mathcal{L}_T = \frac{x}{\gamma} F_3(\sigma + m) - \frac{K}{\gamma} F_3(m),$$

where

$$\begin{aligned}
F_2(x) = &\frac{2a_1(0)\mathrm{e}^{l_1 x}}{2\gamma - x^2}\{1 + x\sqrt{2\pi d_1}\mathrm{e}^{d_1x^2/2}\mathcal{N}(x\sqrt{d_1})\} \\
&- a_1(0)\bigg\{\frac{1}{\sqrt{2\gamma}} - \mathrm{e}^{\gamma d_1}\sqrt{2\pi d_1}\mathcal{N}(-\sqrt{2\gamma d_1})\bigg\}\frac{\exp\{-\sqrt{2\gamma}l_1 + (x + \sqrt{2\gamma})b\}}{\sqrt{2\gamma} - x} \\
&+ \frac{2a_2(0)\mathrm{e}^{l_2 x}}{2\gamma - x^2}\{1 - 2\sqrt{\pi d_4\gamma}\mathrm{e}^{d_4x^2/2}\mathcal{N}(x\sqrt{d_4})\} \\
&- a_2(0)\bigg\{\frac{1}{\sqrt{2\gamma}} - \mathrm{e}^{\gamma d_4}\sqrt{2\pi d_4}\mathcal{N}(-\sqrt{2\gamma d_4})\bigg\}\frac{\exp\{-\sqrt{2\gamma}l_2 + (x + \sqrt{2\gamma})b\}}{\sqrt{2\gamma} - x}
\end{aligned}$$

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$$
\begin{aligned}
&+ a_2(0)\sqrt{2\pi d_4}\mathrm{e}^{\gamma d_4} \\
&\quad\times \biggl\{ \frac{2\sqrt{2\gamma}\exp\{xl_2 - rd_4 + d_4x^2/2\}\mathcal{N}(x\sqrt{d_4} - (b-l_2)/\sqrt{d_4})}{2\gamma - x^2} \\
&\qquad - \frac{\exp\{\sqrt{2\gamma}l_2 + (x-\sqrt{2\gamma})b\}\mathcal{N}((b-l_2)/\sqrt{d_4} - \sqrt{2\gamma d_4})}{\sqrt{2\gamma} - x} \\
&\qquad - \frac{\exp\{-\sqrt{2\gamma}l_2 + (x+\sqrt{2\gamma})b\}\mathcal{N}(-(b-l_2)/\sqrt{d_4} - \sqrt{2\gamma d_4})}{\sqrt{2\gamma} + x} \biggr\}.
\end{aligned}
$$

Remark. The price can be calculated by numerical inversion of the Laplace transform.

So far, we have shown how to obtain the Laplace transform of

$$\mathrm{DP}^*_{\text{min-call in}} = \mathrm{e}^{(r+m^2/2)T}\mathrm{DP}_{\text{min-call in}}.$$

For

$$\mathrm{DP}_{\text{min-call out}} = \mathrm{e}^{-rT}\,\mathrm{E}_Q((S_T - K)^+\mathbf{1}_{\{\tau^S > T\}}),$$

we can get the result from the relationship

$$\mathrm{DP}_{\text{min-call out}} = \mathrm{e}^{-rT}\,\mathrm{E}_Q((S_T - K)^+) - \mathrm{DP}_{\text{min-call in}}.$$

Furthermore, if we set

$$\tilde{\tau}^S = \tau_1^S \vee \tau_2^S,$$

we can define another type of Parisian option by $\tilde{\tau}^Y$:

$$\mathrm{DP}_{\text{max-call in}} = \mathrm{e}^{-rT}\,\mathrm{E}_Q((S_T - K)^+\mathbf{1}_{\{\tilde{\tau}^S < T\}}).$$

In order to get its pricing formula, we should use the following relationship:

$$\mathbf{1}_{\{\tilde{\tau}^S < T\}} = \mathbf{1}_{\{\tau_1^S < T\}} + \mathbf{1}_{\{\tau_2^S < T\}} - \mathbf{1}_{\{\tau^S < T\}}.$$

We therefore have

$$\mathrm{DP}_{\text{max-call in}} = \mathrm{DP}_{\text{up-in call}} + \mathrm{P}_{\text{down-in call}} - \mathrm{DP}_{\text{min-call in}}.$$

Similarly, from

$$\mathrm{DP}_{\text{max-call out}} = \mathrm{e}^{-rT}\,\mathrm{E}_Q((S_T - K)^+) - \mathrm{DP}_{\text{max-call in}},$$

we can work out $\mathrm{DP}_{\text{max-call out}}$.

Appendix A. Proof of Theorem 3

Let T be the final time. According to the definition of $\Psi(x)$, we have

$$\Psi(x) = 2\sqrt{\pi}x\mathcal{N}(\sqrt{2}x) - \sqrt{\pi}x + \mathrm{e}^{-x^2} = \sqrt{\pi}x - \sqrt{\pi}x\,\mathrm{Erfc}(x) + \mathrm{e}^{-x^2}.$$

It is not difficult to show that

$$\mathrm{E}(\exp\{-\beta\tau^{W^\mu}\}) = \mathrm{E}\biggl(\int_0^\infty \beta\mathrm{e}^{-\beta T}\mathbf{1}_{\{\tau^{W^\mu} < T\}}\,\mathrm{d}T\biggr).$$

By Girsanov's theorem, this is equal to

$$\int_0^\infty \beta \exp\left\{-(\beta + \frac{\mu^2}{2})T - \mu x_0\right\} \mathrm{E}(\mathrm{e}^{\mu W_T} \mathbf{1}_{\{\tau^W < T\}})\,\mathrm{d}T.$$

Setting $\gamma = \beta + \frac{1}{2}\mu^2$ gives

$$\begin{aligned}\mathrm{E}(\exp\{-\beta\tau^{W^\mu}\}) &= \int_0^\infty \left(\gamma - \frac{1}{2}\mu^2\right)\mathrm{e}^{-\gamma T - \mu x_0}\,\mathrm{E}(\mathrm{e}^{\mu W_T}\mathbf{1}_{\{\tau^W < T\}})\,\mathrm{d}T \\ &= \frac{\gamma - \mu^2/2}{\gamma}\mathrm{e}^{-\mu x_0}\,\mathrm{E}(\mathrm{e}^{\mu W_{\tilde{T}}}\mathbf{1}_{\{\tau^W < \tilde{T}\}}),\end{aligned}$$

where $\tilde{T}$ is a random variable with an exponential distribution of parameter γ which is independent of W_t. Therefore, we have

$$\mathrm{E}(\mathrm{e}^{\mu W_{\tilde{T}}}\mathbf{1}_{\{\tau^W < \tilde{T}\}}) = \frac{\gamma \mathrm{e}^{\mu x_0}}{\gamma - \mu^2/2}\mathrm{E}(\exp\{-\beta\tau^{W^\mu}\}).$$

In order to invert the above moment generating function, we first need to invert the following expressions:

$$\begin{aligned}\frac{\mu}{\gamma - \mu^2/2} &= \int_0^\infty \mathrm{e}^{\mu x}\mathrm{e}^{-\sqrt{2\gamma}x}\,\mathrm{d}x - \int_{-\infty}^0 \mathrm{e}^{\mu x}\mathrm{e}^{\sqrt{2\gamma}x}\,\mathrm{d}x, \\ \frac{1}{\gamma - \mu^2/2} &= \int_0^\infty \mathrm{e}^{\mu x}\frac{1}{\sqrt{2\gamma}}\mathrm{e}^{-\sqrt{2\gamma}x}\,\mathrm{d}x + \int_{-\infty}^0 \mathrm{e}^{\mu x}\frac{1}{\sqrt{2\gamma}}\mathrm{e}^{\sqrt{2\gamma}x}\,\mathrm{d}x, \\ \mathrm{e}^{d_1\mu^2/2} &= \int_{-\infty}^\infty \mathrm{e}^{\mu x}\frac{1}{\sqrt{2\pi d_1}}\mathrm{e}^{-x^2/2d_1}\,\mathrm{d}x, \\ 1 - \sqrt{\frac{d_i}{2}\pi}\,\mu\mathrm{e}^{d_i\mu^2/2}\,\mathrm{Erfc}\left(\sqrt{\frac{d_i}{2}}\mu\right) &= \int_{-\infty}^0 \mathrm{e}^{\mu x}\frac{-x}{d_i}\mathrm{e}^{-x^2/2d_i}\,\mathrm{d}x.\end{aligned}$$

Therefore, the inversion of $\mu\mathrm{e}^{d_1\mu^2/2}/(\gamma - \mu^2/2)$ is

$$\begin{aligned}&\int_0^\infty \mathrm{e}^{-\sqrt{2\gamma}y}\frac{1}{\sqrt{2\pi d_1}}\exp\left\{-\frac{(x-y)^2}{2d_1}\right\}\mathrm{d}y - \int_{-\infty}^0 \mathrm{e}^{\sqrt{2\gamma}y}\frac{1}{\sqrt{2\pi d_1}}\exp\left\{-\frac{(x-y)^2}{2d_1}\right\}\mathrm{d}y \\ &\quad = \mathrm{e}^{\gamma d_1}\left\{\mathrm{e}^{-\sqrt{2\gamma}x}\mathcal{N}\left(\frac{x}{\sqrt{d_1}} - \sqrt{2\gamma d_1}\right) - \mathrm{e}^{\sqrt{2\gamma}x}\mathcal{N}\left(-\frac{x}{\sqrt{d_1}} - \sqrt{2\gamma d_1}\right)\right\}.\end{aligned}$$

The inversion of $1 - \sqrt{d_i\pi/2}\mu\mathrm{e}^{d_i\mu^2/2}\,\mathrm{Erfc}(\sqrt{d_i\mu/2})/(\gamma - \mu^2/2)$ is given below.

For $x \geq 0$,

$$\int_{-\infty}^0 \frac{-y}{d_i}\mathrm{e}^{-y^2/2d_i}\frac{1}{\sqrt{2\gamma}}\mathrm{e}^{-\sqrt{2\gamma}(x-y)}\,\mathrm{d}y = \frac{\mathrm{e}^{-\sqrt{2\gamma}x}}{\sqrt{2\gamma}} - \mathrm{e}^{\gamma d_i - \sqrt{2\gamma}x}\sqrt{2\pi d_i}\,\mathcal{N}(-\sqrt{2\gamma d_i}).$$

For $x < 0$,

$$\begin{aligned}&\int_{-\infty}^x \frac{-y}{d_i}\mathrm{e}^{-y^2/2d_i}\frac{1}{\sqrt{2\gamma}}\mathrm{e}^{-\sqrt{2\gamma}(x-y)}\,\mathrm{d}y + \int_x^0 \frac{-y}{d_i}\mathrm{e}^{-y^2/2d_i}\frac{1}{\sqrt{2\gamma}}\mathrm{e}^{\sqrt{2\gamma}(x-y)}\,\mathrm{d}y \\ &\quad = \frac{\mathrm{e}^{\sqrt{2\gamma}x}}{\sqrt{2\gamma}} - \mathrm{e}^{\gamma d_i - \sqrt{2\gamma}x}\sqrt{2\pi d_i}\,\mathcal{N}\left(\frac{x}{\sqrt{d_i}} - \sqrt{2\gamma d_i}\right) \\ &\qquad + \mathrm{e}^{\gamma d_i + \sqrt{2\gamma}x}\sqrt{2\pi d_i}\left\{\mathcal{N}(\sqrt{2\gamma d_i}) - \mathcal{N}\left(\frac{x}{\sqrt{d_i}} + \sqrt{2\gamma d_i}\right)\right\}.\end{aligned}$$

Consequently, we can get Theorem 3.

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References

[1] Bertoin, J. (1997). *Lévy Processes* (Camb. Tracts Math. **121**). Cambridge University Press.

[2] Borodin, A. N. and Salminen, P. (1996). *Handbook of Brownian Motion—Facts and Formulae.* Birkhäuser, Basel.

[3] Breiman, L. (1968). *Probability.* Addison-Wesley, Reading, MA.

[4] Chesney, M., Jeanblanc-Picque, M. and Yor, M. (1997). Brownian excursions and Parisian barrier options. *Adv. Appl. Prob.* **29,** 165–184.

[5] Chung, K. L. (1976). Excursions in Brownian motion. *Ark. Math.* **14,** 155–177.

[6] Dassios, A. and Wu, S. (2010). Two-sided Parisian option with single barrier. *Finance Stoch.* **14,** 473–494.

[7] Davis, M. H. A. (1994). *Markov Models and Optimization.* Chapman and Hall, London.

[8] Labart, C. and Lelong, J. (2005). Pricing Parisian options. Tech. Rep., ENPC. Available at http://cermics.enpc.fr/reports/CERMICS-2005/CERMICS-2005-294.pdf.

[9] Revuz, D. and Yor, M. (1999). *Continuous Martingales and Brownian Motion*, 3rd edn. Springer, Berlin.

[10] Rogers, L. C. G. (1989). A guided tour through excursions. *Bull. London Math. Soc.* **21,** 305–341.

[11] Rogers, L. C. G. and Williams, D. (1987). *Diffusions, Markov Processes and Martingales*, Vol. 2. John Wiley, New York.

[12] Rogers, L. C. G. and Williams, D. (1994). *Diffusions, Markov Processes and Martingales*, Vol. 1, 2nd edn. John Wiley, Chichester.

[13] Schröder, M. (2003). Brownian excursions and Parisian barrier options: a note. *J. Appl. Prob.* **40,** 855–864.

[14] Yor, M. (1997). *Some Aspects of Brownian Motion. Part II.* Birkhäuser, Basel.