BV, CW will all be perpendicular to GH; and the triangle UVW will circumscribe the triangle ABC.

Let N, P, Q be the feet of the interior bisectors of the angles A, B, C, and N′, P′, Q′ the feet of the exterior bisectors; then the six straight lines UN, VP, WQ, UN′, VP′, WQ′ pass three and three through four points which are the points of contact of the nine-point circle with the inscribed and escribed circles.*

Geometrical Note.

By R. Tucker, M.A.

If in a triangle ABC, points are taken on the sides such that

$$BP : CP = CQ : AQ = AR : BR = m : n = CP' : BP'$$
$$= AQ' : CQ' = BR' : AR'$$

then the radical axis of the circles PQR, P′Q′R′ passes through the centroid and "S." points of ABC; and if QR, Q′R′ cut in 1, RP, R′P′ in 2, PQ, P′Q′ in 3, then the equation to the circle 123 is

$$abc\Sigma\alpha\beta\gamma = mn\Sigma a\alpha . \Sigma a\alpha\{-mna^2 + (m^2+mn+n^2)(b^2+c^2)\}.$$

FIGURE 20.

The points P, Q, R are given by

$$(0,\ nc,\ mb),\ (mc,\ 0,\ na),\ (nb,\ ma,\ 0),$$

i.e., P, in trilinear co-ordinates, is (0, nc sinA, mb sinA), etc.;
and P′, Q′, R′ by

$$(0,\ mc,\ nb),\ (nc,\ 0,\ ma),\ (mb,\ na,\ 0).$$

It is hence evident that the pairs of triangles are concentroidal with each other and with ABC.

It is also evident that PQ′, P′Q are parallel to AB, and so on; also that P′Q, PR′ intersect on the median through A; and so on.

The triangle PQR $= (m^2 - mn + n^2)\Delta =$ the triangle P′Q′R′.

The equation to the circle PQR is

$$(m^2 - mn + n^2)abc.\Sigma(a\beta\gamma) = mn\Sigma(a\alpha).\Sigma(a\alpha . -mna^2 + m^2b^2 + n^2c^2),$$

and to P′Q′R′ is

$$(m^2 - mn + n^2)abc.\Sigma(a\beta\gamma) = mn\Sigma(a\alpha).\Sigma(a\alpha . -mna^2 + n^2b^2 + m^2c^2).$$

---

* (20) Rev. W. A. Whitworth in *Mathematical Questions from the Educational Times*, X. 51 (1868).

https://doi.org/10.1017/S0013091500031175 Published online by Cambridge University Press

The radical axis of these circles is, therefore,

$$\Sigma(a\alpha.b^2-c^2)=0, \text{ hence } \quad \dots \quad \dots \quad \dots \quad (1)$$

The radical axis of either of the circles and of the circumcircle is of the form $l^2\mathrm{P}-l\mathrm{Q}+\mathrm{R}=0$, where P, Q, R are linear functions of α, β, γ; and the envelope of each of these axes is the conic

$$(a^3\alpha+b^3\beta+c^3\gamma)^2=4(ab^2\alpha+bc^2\beta+ca^2\gamma)(ac^2\alpha+ba^2\beta+cb^2\gamma) \quad \dots \quad (a).$$

The tangents in (a) intersect in the point $a\alpha/(a^4-b^2c^2)=\dots=\dots$.
The radical centre of the three circles is

$$a\alpha/[a^4-b^2c^2+mnk(k-3a^2)]=\dots=\dots;$$

where $$k\equiv a^2+b^2+c^2.$$

The equations to QR, Q′R′ are

$$\left.\begin{array}{l}-mna\alpha+n^2\,b\beta+m^2c\gamma=0\\ -mna\alpha+m^2b\beta+n^2\,c\gamma=0\end{array}\right\} \quad \dots \quad \dots \quad (b);$$

and 1, their point of intersection, is on the median through A, and is given by

$$a\alpha/(m^2+n^2)=b\beta/(mn)=c\gamma/mn.$$

Similarly 2, 3 are

$$a\alpha/mn=b\beta/(m^2+n^2)=c\gamma/mn,$$
$$a\alpha/mn=b\beta/mn=c\gamma/(m^2+n^2).$$

The above lines (b) envelope the parabola $a^2\alpha^2=4bc\beta\gamma$, and so on.
The triangle 123 is readily found to be

$$=(m^2-mn+n^2)^2\Delta.$$

The circle 123 has its equation

$$abc\,\Sigma(\alpha\beta\gamma)=mn\Sigma(a\alpha).\Sigma\{a\alpha.-mna^2+(m^2+mn+n^2)(b^2+c^2)\} \quad \dots \quad (2)$$

The radical axis of this circle and the circumcircle can be written

$$(1-mn)k\Sigma(a\alpha)=\Sigma(a^3\alpha),$$

hence it is a straight line parallel to the chord of contact of the conic (a).

The lines PR, P′Q′, ... intersect in 4, 5, 6, given by

$$a\alpha/(mn-n^2)=b\beta/m^2=c\gamma/m^2, \dots,$$

showing that these points are also on the medians, as is evident from the symmetry of the figure.

https://doi.org/10.1017/S0013091500031175 Published online by Cambridge University Press

The lines PR′, P′Q, ... intersect in p, q, r, where p is given by

$$a\alpha/(m-n) = b\beta/n = c\gamma/n.$$

The conic through PP′QQ′RR′ has for its equation

$$mn(a\alpha + b\beta + c\gamma)^2 = bc\beta\gamma + ca\gamma\alpha + ab\alpha\beta \quad \dots \quad (4),$$

which, in the figure, is an ellipse, concentric, similar and similarly situated with the minimum circum-ellipse of ABC.

The polar of A, with regard to (4), is

$$2amn\alpha - (m^2+n^2)(b\beta + c\gamma) = 0,$$

therefore it is parallel to BC, and cuts AC in J (say); so that $\mathrm{AJ} = 2mn.\mathrm{AC}$. The triangle formed by the three polars (for A, B, C) is

$$= 4(m^2 - mn + n^2)^2\Delta.$$

The tangents to the conic at P, P′ are given by

$$a\alpha(m^2+n^2) + b\beta m(m-n) - c\gamma n\,(m-n) = 0,$$
$$a\alpha(m^2+n^2) - b\beta n\,(m-n) + c\gamma m(m-n) = 0,$$

and intersect, on the median through A, in the point

$$\frac{a\alpha}{-(m-n)^2} = \frac{b\beta}{m^2+n^2} = \frac{c\gamma}{m^2+n^2};$$

and the triangle formed by this and the corresponding points equals the above triangle.

To find the "S." points of PQR, P′Q′R′, assume the sides of these triangles to be p, q, r; p', q', r'; then

$$\left.\begin{aligned} p^2 &= m^2c^2 + n^2b^2 - 2mnbc\cos\mathrm{A} \\ p'^2 &= m^2b^2 + n^2c^2 - 2mnbc\cos\mathrm{A} \end{aligned}\right\} \text{etc.} = \text{etc.};$$

$$\therefore \quad \Sigma(p^2) = (m^2 - mn + n^2)(a^2+b^2+c^2) = \mathrm{K}\ (\text{suppose}) = \Sigma(p'^2).$$

The "S." lines through Q, R, respectively, are

$$\begin{vmatrix} \alpha & \beta & \gamma \\ nbcr^2 & ca(mr^2+np^2) & mabp^2 \\ mc & o & na \end{vmatrix}, \quad \begin{vmatrix} \alpha & \beta & \gamma \\ mbcq^2 & ncap^2 & (mp^2+nq^2)ab \\ nb & ma & o \end{vmatrix},$$

i.e.,

$$-na\alpha(mr^2+np^2) + (n^2r^2 - m^2p^2)b\beta + mc\gamma(mr^2+np^2) = 0,$$
$$ma\alpha(mp^2+nq^2) - (mp^2+nq^2)nb\beta - c\gamma(m^2q^2 - n^2p^2) = 0;$$

whence we get, for the "S." point of $\mathrm{PQR}(\mathrm{K}_1)$,

$$\frac{a\alpha}{mq^2+nr^2} = \frac{b\beta}{mr^2+np^2} = \frac{c\gamma}{mp^2+nq^2} = \frac{2\Delta}{\mathrm{K}}.$$

Similarly, for the "S." point of $P'Q'R'(K_2)$, we have

$$\frac{a\alpha}{nq'^2+mr'^2}=\frac{b\beta}{nr'^2+mp'^2}=\frac{c\gamma}{np'^2+mq'^2}=\frac{2\Delta}{K}.$$

The triangle 123 is directly in perspective with ABC, and has the centroid of the triangles for centre of perspective; hence we can readily obtain the co-ordinates of the principal points.

For (1) the "S." point

$$a\alpha/[a^2+mn(b^2+c^2+2a^2)]=\ldots=\ldots;$$

(2*a*) the positive "B." point

$$a\alpha/[(m^2+n^2)c^2a^2+mnb\,(c^2+a^2)]=\ldots=\ldots;$$

(2*b*) the negative "B." point

$$a\alpha/[(m^2+n^2)a^2b^2+mnc^2(a^2+b^2)]=\ldots=\ldots;$$

(3) the in-centre

$$a\alpha/[a(m^2+n^2)+(b+c)mn]=\ldots=\ldots;$$

(4) the orthocentre

$$\alpha/[(m^2+n^2)\cos B\cos C+mn\cos A]=\ldots=\ldots;$$

(5) the circumcentre

$$\alpha/[(m^2+n^2)\cos A+mn\cos(B-C)]=\ldots=\ldots.$$

It is readily seen that the lines (AP, BQ), (AP′, BQ′) intersect on the conic $c^2\gamma^2=ab\alpha\beta$, which touches CA, CB at A and B, and passes through the centroid.

The co-ordinates of the centre are

$$\{\tfrac{1}{3}(2c\sin B),\ \tfrac{1}{3}(2c\sin A),\ \tfrac{1}{3}(-a\sin B)\};$$

like results hold for the other points of intersection.

[The preceding Note consists of a solution of Questions 11599 and 11670 of the *Educational Times*, and is published in vol. lviii. (pp. 119–123) of the "Reprint" from that journal. It is given here with the editor's kind consent. Part also of Question 11599 was proposed by Prof. Neuberg as Question 787 of *Mathesis*. In the number for January 1893, Prof. Neuberg points out that (α) *supra* is a conic touching the Brocardians of the Lemoine-line, where they meet the reciprocal of that line.]

https://doi.org/10.1017/S0013091500031175 Published online by Cambridge University Press