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A NOTE ON NET REPLACEMENT IN TRANSPOSED SPREADS

BY

N. L. JOHNSON

ABSTRACT. Let P be a translation plane containing a net N which is replaceable by $\bar{N}$. Let P' denote the transposed plane. We note that N' is replaceable by $(\bar{N})'$. This result shows how to relate the various constructions of the two translation planes of order 16 that admit PSL(2, 7).

Let P denote a finite translation plane and S a spread set of matrices for P. Let S' denote the spread set obtained by transposing the matrices of S. We denote the associated translation plane by P' (the transposed plane). If P is a semifield plane of order q^2, let N be a net coordinatized by a middle nucleus M of a coordinatizing semifield. Then N' may be coordinatized by a right nucleus R. Furthermore, if M is isomorphic to $\mathrm{GF}(q)$ then R is also isomorphic to $\mathrm{GF}(q)$ so that P and P' are *both* derivable semifield planes. More generally, we may ask the following question: Let P be any finite translation plane containing a net N which is replaceable by $\bar{N}$. Is N' replaceable by $(\bar{N})'$?

In [1], Bruen discusses the connections between "indicator sets" and the spreads obtained by Ostrom's net extension techniques (via transversal functions) and shows that given a transversal function the two spreads obtained by indicator sets and net extension are related by a polarity of the associated projective space and it is implicit in Bruen's work that the two translation planes are transposes of each other (see [1], section 5, part B).

Once the polarity-transpose connection has been made, the result on net replacement is almost immediate. That is,

THEOREM. *Let P be a finite translation plane and V the underlying vector space of dimension $2r$ over $\mathrm{GF}(p)$ for p a prime.*

(1) *Let N denote any net and $\bar{N}$ a net which replaces N. Let a be any polarity of V. Then N^a is replaceable by $(\bar{N})^a$.*

(2) *If N is a derivable net, let $\bar{N}$ denote the unique replaceable net of Baer subplanes of N. Then $(\bar{N})^a = \overline{N^a}$. That is, to derive N^a, one may either first derive N and then apply the polarity or apply the polarity and then derive.*

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(3) *Let $x = 0$, and $y = xM$ for M in S be a spread set of matrices for P where x, y denote the associated vectors. Let P^t denote the transposed translation plane obtained from the spread S^t of transposed matrices of S. If N is a derivable net, denote the derived net by $\bar{N}$. If $P = N \cup M$, denote P^t by $N^t \cup M^t$. Then $(\bar{N})^t = \overline{N^t}$ (derive-transpose = transpose-derive).*

PROOF. (2) and (3) are immediate from Bruen's work once (1) is established.

Note, finiteness is essential by Bruen and Fisher [2]. Let $\{V_i$ for $i = 1$ to $k\}$ denote the components of N and let $\{W_i$ for $i = 1$ to $k\}$ denote the components of N. Then $\dot{\bigcup}_{i=1}^{k} V_i = \dot{\bigcup}_{i=1}^{k} W_i$ and $\dim V_i = r$ for $i = 1$ to k. Let the $\dim V_i \cap W_j = r_{ij}$. Then since V_i is contained in $\dot{\bigcup}_{i=1}^{k} W_i$, we have $V_i = \dot{\bigcup}_{j=1}^{k} (V_i \cap W_j)$. So, $\sum_{j=1}^{k} (p^{r_{ij}} - 1) = p^r - 1$ for each $i = 1$ to k. Now, for any subspaces S, R of V we have $(S \cap R)^a = S^a + R^a$. So,

$$\dim (V_i^a \cap W_j^a) = \dim (V_i \cap W_j).$$

That is,

$$\dim V_i^a \cap W_j^a = \dim (V_i + W_j)^a = 2r - \dim (V_i + W_j) = \dim V_i \cap W_j$$

Thus, $\dot{\bigcup}_{j=1}^{k} V_i^a \cap W_j^a$ has cardinality $\sum_{j=1}^{k} (p^{r_{ij}} - 1) = p^r - 1$ and since $V_i^a \cap W_j^a \subseteq V_i^a$ we have $\dot{\bigcup}_{j=1}^{k} V_i^a \cap W_j^a = V_i^a$ for $i = 1$ to k. Thus, by symmetry, we have $\dot{\bigcup}_{i=1}^{k} V_i^a = \dot{\bigcup}_{i=1}^{k} W_i^a$.

Note that net replacements are not always unique and for every replacement N^* for N, $(N^*)^t$ is a replacement for N^t.

Several authors have studied translation planes of order 16 that admit PSL(2, 7). It turns out that there are exactly two such planes which have lately been called the Johnson–Walker and the Lorimer–Rahilly planes.

Johnson [3], has given a construction of both these planes by deriving the unique semifield plane of order 16 and kern GF(4). The J–W plane may be derived by replacing a net coordinatized by a right nucleus and the L–R plane may be derived by replacing a net coordinatized by a middle nucleus.

Walker [4] constructs the Lorimer-Rahilly plane from the group PSL(2, 7) and then observes the same construction by the dual representation on the dual space produces another plane admitting PSL(2, 7). By Bruen [1], these two planes are transposes of each other. Thus, we have the connection between the two constructions. Generally,

$$\begin{array}{ccc}
\text{(middle nucleus net in a semifield plane)} & \xleftrightarrow{\text{derive}} & \text{(generalized Hall of type 1)} \\
\updownarrow \text{ transpose} & & \updownarrow \text{ transpose} \\
\text{(right nucleus net in a semifield plane)} & \xleftrightarrow{\text{derive}} & \text{(generalized Hall of type 2)}
\end{array}$$

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In particular,

middle nucleus net-semifield plane of order 16 and kern GF(4)	$\overset{\text{derive}}{\longleftrightarrow}$	Lorimer–Rahilly
$\updownarrow$ transpose		$\updownarrow$ transpose
right nucleus net-semifield plane of order 16 and kern GF(4) (*same* plane as above)	$\overset{\text{derive}}{\longleftrightarrow}$	Johnson–Walker

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Department of Mathematics
The University of Iowa
Iowa City, IA 52242