

A NOTE ON NET REPLACEMENT IN TRANSPOSED SPREADS

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ABSTRACT. Let P be a translation plane containing a net N which is replaceable by \bar{N} . Let P' denote the transposed plane. We note that N' is replaceable by $(\bar{N})'$. This result shows how to relate the various constructions of the two translation planes of order 16 that admit $\text{PSL}(2, 7)$.

Let P denote a finite translation plane and S a spread set of matrices for P . Let S' denote the spread set obtained by transposing the matrices of S . We denote the associated translation plane by P' (the transposed plane). If P is a semifield plane of order q^2 , let N be a net coordinatized by a middle nucleus M of a coordinatizing semifield. Then N' may be coordinatized by a right nucleus R . Furthermore, if M is isomorphic to $\text{GF}(q)$ then R is also isomorphic to $\text{GF}(q)$ so that P and P' are both derivable semifield planes. More generally, we may ask the following question: Let P be any finite translation plane containing a net N which is replaceable by \bar{N} . Is N' replaceable by $(\bar{N})'$?

In [1], Bruen discusses the connections between “indicator sets” and the spreads obtained by Ostrom’s net extension techniques (via transversal functions) and shows that given a transversal function the two spreads obtained by indicator sets and net extension are related by a polarity of the associated projective space and it is implicit in Bruen’s work that the two translation planes are transposes of each other (see [1], section 5, part B).

Once the polarity-transpose connection has been made, the result on net replacement is almost immediate. That is,

THEOREM. *Let P be a finite translation plane and V the underlying vector space of dimension $2r$ over $\text{GF}(p)$ for p a prime.*

(1) *Let N denote any net and \bar{N} a net which replaces N . Let α be any polarity of V . Then N^α is replaceable by $(\bar{N})^\alpha$.*

(2) *If N is a derivable net, let \bar{N} denote the unique replaceable net of Baer subplanes of N . Then $(\bar{N})^\alpha = \bar{N}^\alpha$. That is, to derive N^α , one may either first derive N and then apply the polarity or apply the polarity and then derive.*

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(3) Let $x = 0$, and $y = xM$ for M in S be a spread set of matrices for P where x, y denote the associated vectors. Let P' denote the transposed translation plane obtained from the spread S' of transposed matrices of S . If N is a derivable net, denote the derived net by \bar{N} . If $P = N \cup M$, denote P' by $N' \cup M'$. Then $(\bar{N})' = \bar{N}'$ (derive-transpose = transpose-derive).

PROOF. (2) and (3) are immediate from Bruen's work once (1) is established.

Note, finiteness is essential by Bruen and Fisher [2]. Let $\{V_i$ for $i = 1$ to $k\}$ denote the components of N and let $\{W_i$ for $i = 1$ to $k\}$ denote the components of N . Then $\cup_{i=1}^k V_i = \cup_{i=1}^k W_i$ and $\dim V_i = r$ for $i = 1$ to k . Let the $\dim V_i \cap W_j = r_{ij}$. Then since V_i is contained in $\cup_{i=1}^k W_i$, we have $V_i = \cup_{j=1}^k (V_i \cap W_j)$. So, $\sum_{j=1}^k (p^{r_{ij}} - 1) = p^r - 1$ for each $i = 1$ to k . Now, for any subspaces S, R of V we have $(S \cap R)^a = S^a + R^a$. So,

$$\dim (V_i^a \cap W_j^a) = \dim (V_i \cap W_j).$$

That is,

$$\dim V_i^a \cap W_j^a = \dim (V_i + W_j)^a = 2r - \dim (V_i + W_j) = \dim V_i \cap W_j$$

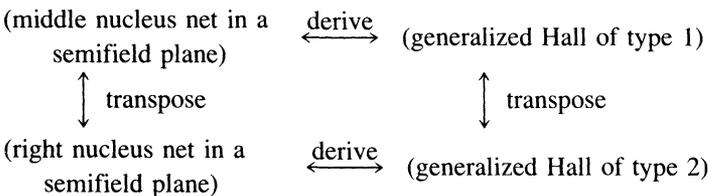
Thus, $\cup_{j=1}^k V_i^a \cap W_j^a$ has cardinality $\sum_{j=1}^k (p^{r_{ij}} - 1) = p^r - 1$ and since $V_i^a \cap W_j^a \subseteq V_i^a$ we have $\cup_{j=1}^k V_i^a \cap W_j^a = V_i^a$ for $i = 1$ to k . Thus, by symmetry, we have $\cup_{i=1}^k V_i^a = \cup_{i=1}^k W_i^a$.

Note that net replacements are not always unique and for every replacement N^* for N , $(N^*)'$ is a replacement for N' .

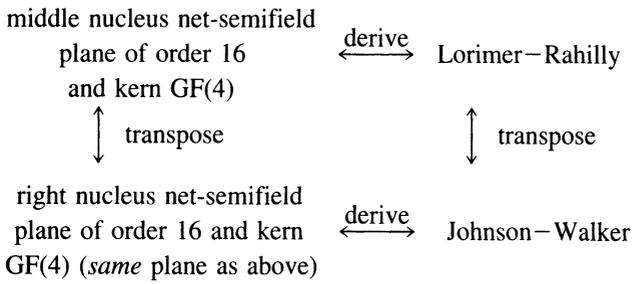
Several authors have studied translation planes of order 16 that admit $\text{PSL}(2, 7)$. It turns out that there are exactly two such planes which have lately been called the Johnson-Walker and the Lorimer-Rahilly planes.

Johnson [3], has given a construction of both these planes by deriving the unique semifield plane of order 16 and kern $\text{GF}(4)$. The J-W plane may be derived by replacing a net coordinatized by a right nucleus and the L-R plane may be derived by replacing a net coordinatized by a middle nucleus.

Walker [4] constructs the Lorimer-Rahilly plane from the group $\text{PSL}(2, 7)$ and then observes the same construction by the dual representation on the dual space produces another plane admitting $\text{PSL}(2, 7)$. By Bruen [1], these two planes are transposes of each other. Thus, we have the connection between the two constructions. Generally,



In particular,



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