$D(\tau\,;\,\mathscr{C})$-SPACES AND THE CLOSED-GRAPH THEOREM

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1. The problem considered in this paper is that of finding conditions on a range space such that the closed-graph theorem holds for linear mappings from a class of linear topological spaces. The concept of a $D(\tau;\,\mathscr{C})$-space, which is a result of this investigation, is meaningful for commutative topological groups but we limit our consideration in this paper to linear topological spaces. On restricting ourselves to locally convex linear topological spaces, we see that the notion of a $D(\tau;\,\mathscr{C})$-space is an extension of the powerful idea of a B-complete space.

The study in section 2 is of a preliminary nature. $D(\tau;\mathscr{C})$-spaces and $D_r(\tau;\mathscr{C})$-spaces are defined, and some general results which hold for them are presented. Section 3 contains closed-graph theorems. Theorem 3.1, for example, extends in different directions a result of V. Pták ((7), 4.9). The results in section 3 are used in section 4 to extend some results of A. P. Robertson and W. Robertson ((**8**), Theorems 2 and 3, corollary of Theorem 3). Throughout, we use the terminology and notation of (**3**) and (**4**). All our topological spaces shall be assumed separated (Hausdorff).

The method of presentation of the results in this paper is similar to that used in (**1**). The idea of writing this paper in this way was suggested to the author by Professor J. D. Weston to whom he hereby expresses his gratitude. The author wishes to thank Professor A. P. Robertson for his help and encouragement throughout the period of time when the work reported in this paper was being done. He is grateful to the Government of the Republic of Nigeria for financial support.

2. As in (**1**), we shall express our results in " category form ". Throughout, $\mathscr{C}$ shall denote a certain category; the objects in $\mathscr{C}$ shall be linear topological spaces (l.t.s.), and the maps shall be linear maps of such spaces. If u, v are different topologies on a linear space E such that (E, u), (E, v) are objects in $\mathscr{C}$, the two spaces (E, u), (E, v) are to be regarded as different objects in $\mathscr{C}$. However as $\mathscr{C}$ is a category, for objects E, F, G in $\mathscr{C}$ and maps in $\mathscr{C}$, t_1 of E into F and t_2 of F into G, the composition map $t_2 \circ t_1$ of E into G is in $\mathscr{C}$.

The concepts of B-completeness and B_r-completeness with respect to $\mathscr{C}$ are meaningful (see (**1**)). Precisely, an object E in $\mathscr{C}$ is said to be *B-complete* (in $\mathscr{C}$) if every continuous nearly open map in $\mathscr{C}$ of E onto an l.t.s. is open. The object E is said to be *B_r-complete* (in $\mathscr{C}$) if every continuous nearly open $(1-1)$ map in $\mathscr{C}$ of E onto an l.t.s. is open.

Let E, F be objects in $\mathscr{C}$ and t a map in $\mathscr{C}$ of F into E. Let $\mathscr{U}$, $\mathscr{V}$ be bases of balanced neighbourhoods of the origin for the topologies of F, E respectively. Let w be the linear topology on E with $(t(U)+V\colon\ U \in \mathscr{U},\ V \in \mathscr{V})$ as a base of neighbourhoods of the origin. Let us consider the following restriction on $\mathscr{C}$.

(C_1) With the notation above, (E, w) is an object in $\mathscr{C}$ whenever it is separated, and the identity map from E onto (E, w) is in $\mathscr{C}$.

If $\mathscr{C}$ is a category of linear maps of linear topological spaces satisfying (C_1), then it is not difficult to show that for a space E which is B_r-complete in $\mathscr{C}$, every closed nearly continuous map in $\mathscr{C}$ from any object in $\mathscr{C}$ into E is continuous.

If $\mathscr{C}$ is such that

(C_2) for two objects E_1, E_2 in $\mathscr{C}$ and a $(1-1)$ map t in $\mathscr{C}$ of E_1 onto E_2, the map t^{-1} of E_2 onto E_1 is also in $\mathscr{C}$, then the following are equivalent:

(i) An object E in $\mathscr{C}$ is B_r-complete.

(ii) Every closed nearly open $(1-1)$ map in $\mathscr{C}$ of E onto an l.t.s. is open.

(iii) Every closed nearly continuous map in $\mathscr{C}$ from an object in $\mathscr{C}$ into E is continuous.

If in addition,

(C_3) (a) for every object E in $\mathscr{C}$ and each closed linear subspace E_0 of E the quotient space E/E_0 is an object in $\mathscr{C}$ and the quotient map is in $\mathscr{C}$, (b) for every map t in $\mathscr{C}$ from an object E in $\mathscr{C}$, the induced map of t is in $\mathscr{C}$ whenever $E/t^{-1}(0)$ is separated, then the following are equivalent:

(i) An object E in $\mathscr{C}$ is B-complete.

(ii) Every closed nearly open map in $\mathscr{C}$ of E onto an l.t.s. is open.

(iii) Every closed nearly continuous map in $\mathscr{C}$ from any object in $\mathscr{C}$ into each quotient of E by a closed linear subspace is continuous.

We shall as from now assume that $\mathscr{C}$ satisfies conditions (C_2) and (C_3). Some examples are as follows.

(a) The category of all linear maps of all linear topological (semiconvex, locally convex) spaces. This shall be denoted by $\mathscr{C}_1$ $(\mathscr{C}_2, \mathscr{C}_3)$.

(b) For a fixed real $\lambda \geqq 2$, the category of all linear maps of all linear topological spaces having bases of neighbourhoods of the origin consisting of balanced λ-convex sets.

(c) The category of all linear maps of all ultrabarrelled (hyperbarrelled, barrelled) spaces.

(d) The category of all linear maps of all ultrabornological (hyperbornological, bornological) spaces.

(e) The category of all linear maps of all quasi-ultrabarrelled (quasi-hyperbarrelled, quasi-barrelled) spaces.

[For the definitions of ultrabarrelled, ultrabornological, quasi-ultrabarrelled {hyperbarrelled, hyperbornological, quasi-hyperbarrelled} spaces see for example, (3) {(4)}].

We observe that examples (*a*), (*b*) and (*c*) also satisfy condition (C_1). That of (*a*) and (*b*) is pretty obvious. That example (*c*) satisfies condition (C_1) can be seen from the following consideration. Let $\mathscr{C}$ be as in example (*c*). If with the notation of condition (C_1), t in $\mathscr{C}$ is a map of F into E, then t is nearly continuous. This implies that the continuous identity map from E onto (E, w) is nearly open. And since E is ultrabarrelled (hyperbarrelled, barrelled), so is (E, w).

If $\mathscr{C}$ is a category of linear maps of linear topological spaces, we shall denote by τ a class of spaces consisting of objects in $\mathscr{C}$ such that if G is an object in $\mathscr{C}$ and H is in τ, then (i) every map in $\mathscr{C}$ from G onto H is nearly open, and (ii) every map in $\mathscr{C}$ from H into G is nearly continuous. We say that an object E in $\mathscr{C}$ is a *$D(\tau;\ \mathscr{C})$-space* if for every F in τ, any closed map in $\mathscr{C}$ from E onto F is open. We call E a *$D_r(\tau;\ \mathscr{C})$-space* if the same condition is satisfied by $(1-1)$ maps in $\mathscr{C}$. The class of all $D(\tau;\ \mathscr{C})$-spaces ($D_r(\tau;\ \mathscr{C})$-spaces) shall be denoted by $D(\tau;\ \mathscr{C})$ ($D_r(\tau;\ \mathscr{C})$). It is assumed throughout that τ satisfies conditions (i) and (ii) above.

Clearly, for any $\mathscr{C}$ and τ, $D(\tau;\ \mathscr{C}) \subseteq D_r(\tau;\ \mathscr{C})$ and if $\tau_1 \subseteq \tau_2$ then $D(\tau_2;\ \mathscr{C}) \subseteq D(\tau_1;\ \mathscr{C})$ as well as $D_r(\tau_2;\ \mathscr{C}) \subseteq D_r(\tau_1;\ \mathscr{C})$. Also, if E is a $D(\tau;\ \mathscr{C})$-space, then for any closed linear subspace E_0 of E, E/E_0 is a $D(\tau;\ \mathscr{C})$-space. For let F be in τ and t in $\mathscr{C}$ be a closed map of E/E_0 onto F. If k is the canonical map of E onto E/E_0, then the map $t \circ k$ of E onto F is in $\mathscr{C}$ and is closed. Therefore $t \circ k$ is open. This implies that t is open, and thus E/E_0 is a $D(\tau;\ \mathscr{C})$-space. It is also not difficult to show that if E is a $D_r(\tau;\ \mathscr{C})$-space then it is a $D(\tau;\ \mathscr{C})$-space if and only if E/E_0 is a $D_r(\tau;\ \mathscr{C})$-space for every closed linear subspace E_0 of E.

Let τ be a class of barrelled spaces. Then according to T. Husain (2), a locally convex space E is a $B(\tau)$-space ($B_r(\tau)$-space) if, for every F in τ, any continuous (continuous $(1-1)$) nearly open linear map of E onto F is open. Clearly, every $D(\tau;\ \mathscr{C}_3)$-space ($\mathscr{C}_3$ is the category of all linear maps of all locally convex spaces) is a $B(\tau)$-space, and every $D_r(\tau;\ \mathscr{C}_3)$-space is a $B_r(\tau)$-space. However, I do not know of any $B(\tau)$-space ($B_r(\tau)$-space) which is not a $D(\tau;\ \mathscr{C}_3)$-space ($D_r(\tau;\ \mathscr{C}_3)$-space).

The letters α, α_u, α_h, β_n, β, β_1, β_{11}, $\mathfrak{F}$, $\mathfrak{F}_1$, $\mathfrak{F}_{11}$, Λ, η, η_1, η_{11}, $\mathfrak{F}_\delta$ shall respectively stand for the classes of all barrelled, ultrabarrelled, hyperbarrelled, Banach, second-category locally convex, second-category linear topological, second-category semiconvex, Frechet, complete metric linear, semiconvex complete metric linear, complete locally bounded, sequentially complete bornological, sequentially complete almost convex ultrabornological, sequentially complete almost convex hyperbornological, and finite-dimensional linear topological spaces. (For the definition of an almost convex l.t.s., see (3)).

Proposition 2.1. (*a*) *Let $\mathscr{C}$ be a category of linear maps of linear topological spaces, and let τ be a class of objects in $\mathscr{C}$. If E, E_1 are objects in $\mathscr{C}$ such that there is a continuous* (*continuous* $(1-1)$) *map in $\mathscr{C}$ of E onto E_1, then E_1 is a $D(\tau;\ \mathscr{C})$-space* (*$D_r(\tau;\ \mathscr{C})$-space*) *if E is.*

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(b) Let τ be a class of barrelled spaces. If u, v are locally convex topologies on a linear space E such that (E, u), (E, v) have the same dual, then (E, u) is a $D(\tau;\ \mathscr{C}_3)$-space ($D_r(\tau;\ \mathscr{C}_3)$-space) if and only if (E, v) is.

Proof. (*a*) Suppose that E is a $D(\tau;\ \mathscr{C})$-space and that h is a continuous map in $\mathscr{C}$ of E onto E_1. If f in $\mathscr{C}$ is a closed map of E_1 onto some H in τ then the map $f \circ h$ of E onto H is in $\mathscr{C}$ and is closed. Therefore $f \circ h$ is open. This implies that f is open, and thus E_1 is a $D(\tau;\ \mathscr{C})$-space. Similarly, E_1 is a $D_r(\tau;\ \mathscr{C})$ space if E is, provided that there is a continuous $(1-1)$ map in $\mathscr{C}$ of E onto E_1.

(*b*) Let m, w respectively denote the Mackey and weak topologies on E with dual $(E, u)'(=(E, v)')$. By (*a*), it is sufficient to prove that (E, m) is a $D(\tau;\ \mathscr{C}_3)$-space ($D_r(\tau;\ \mathscr{C}_3)$-space) if (E, w) is.

Suppose that (E, w) is a $D(\tau;\ \mathscr{C}_3)$-space, and let h be a closed linear map of (E, m) onto some H in τ. The graph of h is also closed in $(E, w)\times H$, since the graph is a linear subspace of $E\times H$, and the locally convex spaces $(E, w)\times H$, $(E, m)\times H$ have the same dual. Therefore h is an open map of (E, w) onto H. Now, $(E, w)/h^{-1}(0)$, $(E, m)/h^{-1}(0)$ have the weak and Mackey topologies respectively with the same dual. If f is the induced map of h then, since f^{-1} is a continuous linear map from H into $(E, w)/h^{-1}(0)$ and H has a Mackey topology, f^{-1} is a continuous linear map from H into $(E, m)/h^{-1}(0)$. Therefore h is an open map of (E, m) onto H, and thus (E, m) is a $D(\tau;\ \mathscr{C}_3)$-space. Similarly, (E, u) is a $D_r(\tau;\ \mathscr{C}_3)$-space if and only if (E, v) is.

Let (E, u) be a metrizable l.t.s. with dual E' separating the points of E. Let m_1 be the Mackey topology on E' with $(E, u^{\circ\circ})^\wedge$ (the completion of the linear space E considered under the locally convex topology derived from u) as dual. By an application of ((**8**), p. 9), we see that (E', m_1) is a B-complete locally convex space. If m_2 is the Mackey topology on E' with E as dual, it follows from Proposition 2.1 and the result stated after condition (C_3) that (E', m_2) is a $D(\tau;\ \mathscr{C})$-space, where $\mathscr{C} = \mathscr{C}_1$ ($\mathscr{C}_2$, $\mathscr{C}_3$) and $\tau = \alpha_u$ (α_h, α). In particular, this is so if E is a metrizable locally convex space and m_2 is the Mackey topology on E' with E as dual. We note that for $\mathscr{C}_3$, this result is at least as strong as Theorem 3 of (2).

Let us consider the following restriction on our category $\mathscr{C}$.

(C_4) If F is a linear space, $(E_\gamma\colon \gamma \in \Psi)$ are objects in $\mathscr{C}$, and each u_γ is a linear map of E_γ into F, then there is a finest topology, w say, on F such that

(*a*) The space (F, w) if separated, is an object in $\mathscr{C}$, and each map u_γ of E_γ into (F, w) is in $\mathscr{C}$.

(*b*) The topology w is the finest one on F satisfying (*a*) for which each u_γ is continuous.

(*c*) If G is an object in $\mathscr{C}$, then a map f in $\mathscr{C}$ from (F, w) into G is continuous if and only if each $f \circ u$ is continuous.

(*d*) If F_0 is a closed linear subspace of (F, w) and k is the canonical map of F onto F/F_0, then the quotient topology on $(F, w)/F_0$ is the finest topology on F/F_0 for which $(F, w)/F_0$ is an object in $\mathscr{C}$ and such that (i) each $k \circ u_\gamma$ is continuous and (ii) for an object H in $\mathscr{C}$, a map f in $\mathscr{C}$ of $(F, w)/F_0$ into H is continuous if and only if each $f \circ k \circ u_\gamma$ is continuous.

If (C_4) is satisfied and if the union of the linear subspaces $u_\gamma(E_\gamma)$ spans F, we say that (F, w) is the *$\mathscr{C}$-inductive limit* of $(E_\gamma;\ u_\gamma\colon \gamma \in \Psi)$.

Clearly the ordinary inductive limit of locally convex spaces satisfies the conditions for defining a $\mathscr{C}_3$-inductive limit. Also, the concept of a *-inductive limit of linear topological spaces (an *sc*-inductive limit of semiconvex spaces) defined in (3)((4)) satisfies the conditions for defining a $\mathscr{C}_1$-inductive limit ($\mathscr{C}_2$-inductive limit).

If F is the $\mathscr{C}$-inductive limit of $(E_\gamma;\ u_\gamma\colon \gamma \in \Psi)$ and F is the union over γ of $u_\gamma(E_\gamma)$, then we say that (F, w) is the *generalized strict $\mathscr{C}$-inductive limit* of $(E_\gamma;\ u_\gamma\colon \gamma \in \Psi)$.

If $\mathscr{C}$ is a category of linear maps of linear topological spaces satisfying (C_4) and τ is a class of objects in $\mathscr{C}$, we shall denote by τ^* the class of all objects in $\mathscr{C}$ each of which is the $\mathscr{C}$-inductive limit of some $(E_\gamma;\ u_\gamma\colon \gamma \in \Phi)$, where each E_γ is in τ.

Proposition 2.2. *Let $\mathscr{C}$ satisfy (C_4). Suppose further that if E, F are objects in $\mathscr{C}$ and t is a $(1-1)$ map of E into F, then $t(E)$ is an object in $\mathscr{C}$ and the map t^{-1} of $t(E)$ onto E is in $\mathscr{C}$. If τ is a class of objects in $\mathscr{C}$ such that every quotient by a closed linear subspace of each member of τ is also in τ, then an object in $\mathscr{C}$ is a $D(\tau;\ \mathscr{C})$-space if and only if it is a $D(\tau^*;\ \mathscr{C})$-space.*

Proof. It is sufficient to prove that $D(\tau;\ \mathscr{C}) \subseteq D(\tau^*;\ \mathscr{C})$. Let f be a closed map in $\mathscr{C}$ from a $D(\tau;\ \mathscr{C})$-space E onto some E_1 in τ^*. There is no loss of generality in assuming that f is a $(1-1)$ map, since otherwise f could be put in the form $f = f_1 \circ k$, where f_1 is the induced map of f and is in $\mathscr{C}$ as a map of $E/f^{-1}(0)$ onto E_1. Since by the hypothesis every quotient by a closed linear subspace of each member of τ is also in τ, we may also assume that E_1 is the $\mathscr{C}$-inductive limit of some $(F_\gamma;\ i_\gamma\colon \gamma \in \Gamma)$, where each F_γ is in τ and each i_γ is a $(1-1)$ map of F_γ into E_1. Because of the second restriction on $\mathscr{C}$ in the statement of this proposition, the map $i_\gamma^{-1} \circ f$ of E onto F_γ is in $\mathscr{C}$ for each γ. The graph of $i_\gamma^{-1} \circ f$ is closed, since it is the inverse image of the graph of f by the continuous map $(x, y) \to (x, i_\gamma(y))$ of $E \times F_\gamma$ into $E \times E_1$. Therefore $i_\gamma^{-1} \circ f$ is open and thus for every neighbourhood V of the origin in E, $i_\gamma^{-1} \circ f(V)$ $(= (f^{-1} \circ i_\gamma)^{-1}(V))$ is a neighbourhood of the origin in F_γ. This implies that $f^{-1} \circ i_\gamma$ is a continuous map of F_γ into E for each γ in Γ. Therefore f^{-1} is continuous. Thus f is open and E is a $D(\tau^*;\ \mathscr{C})$-space.

Corollary.

(i) $D(\beta_1;\ \mathscr{C}_1) = D(\beta_1^*;\ \mathscr{C}_1)$,

(ii) $D(\mathfrak{F}_1;\ \mathscr{C}_1) = D(\mathfrak{F}_1^*;\ \mathscr{C}_1)$,

(iii) $D(\Lambda;\ \mathscr{C}_1) = D(\eta_1;\ \mathscr{C}_1) = D(\Lambda^*;\ \mathscr{C}_1)$,

(iv) $D(\beta;\ \mathscr{C}_3) = D(\beta^*;\ \mathscr{C}_3)$,

(v) $D(\beta_n;\ \mathscr{C}_3) = D(\mathfrak{F};\ \mathscr{C}_3) = D(\eta;\ \mathscr{C}_3) = D(\beta_n^*;\ \mathscr{C}_3)$.

The proof of the following result is easy and is therefore omitted.

Proposition 2.3. *Let $\mathscr{C}$ be a category of linear maps of linear topological spaces, and let τ be a class of objects in $\mathscr{C}$. If every continuous (continuous $(1-1)$) nearly open linear image in $\mathscr{C}$ of each member of τ is also in τ, then every $D(\tau;\ \mathscr{C})$-space ($D_r(\tau;\ \mathscr{C})$-space which is in τ is B-complete (B_r-complete) in $\mathscr{C}$.*

In ((**5**), p. 195, problem $D(a)$), $G = E_1 \times E_2$, where E_1 is B-complete in $\mathscr{C}_3$, being a countable direct sum of reflexive Banach spaces, and E_2 is a Frechet space. Also G is barrelled, hyperbarrelled, and ultrabarrelled, but not B-complete in $\mathscr{C}_3$, $\mathscr{C}_2$, or $\mathscr{C}_1$. It follows that a product of two $D(\tau;\ \mathscr{C})$-spaces need not be a $D(\tau;\ \mathscr{C})$-space.

A separated quotient E of an *L.F.* space is in $D(\beta;\ \mathscr{C}_3)$ and $D(\beta_1;\ \mathscr{C}_1)$ (see, for example Theorem 4.2), but if E is not complete (there are such examples), then by Proposition 2.3, it is not in $D_r(\alpha;\ \mathscr{C}_3)$ or $D(\alpha_u;\ \mathscr{C}_1)$. Every separated locally convex space is in $D(\mathfrak{F}_\delta;\ \mathscr{C}_3)$ and $D(\mathfrak{F}_\delta;\ \mathscr{C}_1)$. But if E is an infinite-dimensional Banach space, then the space E considered under its finest locally convex topology is not in $D_r(\beta_n;\ \mathscr{C}_3)$ or $D_r(\mathfrak{F}_1;\ \mathscr{C}_1)$. In Proposition 2.4, (i), (ii), (v), and (vi) follow from these observations and the corollary of Proposition 2.2. Parts (iii) and (iv) of the same result are easily established.

Proposition 2.4.

(i) $D_r(\alpha_u;\ \mathscr{C}_1) \subset D_r(\beta_1;\ \mathscr{C}_1) \subseteq D_r(\mathfrak{F}_1;\ \mathscr{C}_1) \subset D_r(\mathfrak{F}_\delta;\ \mathscr{C}_1)$.

(ii) $D(\alpha_u;\ \mathscr{C}_1) \subset D(\beta_1;\ \mathscr{C}_1) = D(\beta_1^*;\ \mathscr{C}_1) \subseteq D(\mathfrak{F}_1;\ \mathscr{C}_1) = D(\mathfrak{F}_1^*;\ \mathscr{C}_1) \subset D(\mathfrak{F}_\delta;\ \mathscr{C}_1)$.

(iii) $D(\beta;\ \mathscr{C}_3) \subset D(\beta;\ \mathscr{C}_1)$.

(iv) $D(\beta_1;\ \mathscr{C}_1) \subseteq D(\beta;\ \mathscr{C}_1)$.

(v) $D_r(\alpha;\ \mathscr{C}_3) \subset D_r(\beta^*;\ \mathscr{C}_3) \subseteq D_r(\beta_n^*;\ \mathscr{C}_3) \subseteq D_r(\eta;\ \mathscr{C}_3) \subseteq D_r(\mathfrak{F};\ \mathscr{C}_3) \subseteq D_r(\beta_n;\ \mathscr{C}_3) \subset D_r(\mathfrak{F}_\delta;\ \mathscr{C}_3)$.

(vi) $D(\alpha;\ \mathscr{C}_3) \subset D(\beta^*;\ \mathscr{C}_3) = D(\beta;\ \mathscr{C}_3) \subseteq D(\beta_n^*;\ \mathscr{C}_3) = D(\beta_n;\ \mathscr{C}_3) = D(\eta;\ \mathscr{C}_3) = D(\mathfrak{F};\ \mathscr{C}_3) \subset D(\mathfrak{F}_\delta;\ \mathscr{C}_3)$.

3. We shall now suppose that $\mathscr{C}$ satisfies the following condition, one which is satisfied by examples (a), (b), (c), (d), (e) of $\mathscr{C}$ in section 2.

(C_5). (*a*) For any linear space E, there is a finest topology, denoted by ω, such that (E, ω) is an object in $\mathscr{C}$. (*b*) If for each of two topologies u, v on a linear space E, the space is an object in $\mathscr{C}$, then the identity map from (E, u) onto (E, v) is in $\mathscr{C}$.

Let $\mathscr{C}$ satisfy (C_4) and (*a*) of (C_5), and let τ be a class of objects in $\mathscr{C}$. We say that τ is a $\mathscr{C}\omega$-inductive class if (E, ω) is in τ for every linear space E, and for every $(E_i\colon i = 1, 2, \ldots, n)$ in τ any $\mathscr{C}$-inductive limit of $(E_i\colon i = 1, 2, \ldots, n)$ is in τ.

It is easy to see that

(*a*) $\alpha_u, \beta_n^*, \mathfrak{F}_1^*, \mathfrak{F}^*, \Lambda^*, \beta^*, \beta_1^*$, are $\mathscr{C}_1\omega$ inductive classes,

(*b*) $\alpha_h, \beta_n^*, \mathfrak{F}_{11}^*, \mathfrak{F}^*, \Lambda^*, \beta^*, \beta_{11}^*$, are $\mathscr{C}_2\omega$ inductive classes, and

(*c*) $\alpha, \beta_n^*, \beta^*, \mathfrak{F}^*$ are $\mathscr{C}_3\omega$ inductive classes.

Now, any inductive limit (in the usual sense) of a sequence of Banach spaces is a quotient of their topological direct sum. Since an inductive limit of a sequence of Banach spaces need not be sequentially complete ((6), p. 437), we see that $\eta(\eta_1, \eta_{11})$ is not a $\mathscr{C}_3\omega-$ $(\mathscr{C}_1\omega-, \mathscr{C}_2\omega-)$ class.

Theorem 3.1. *Let $\mathscr{C}$ satisfy (C_4) and (C_5), and let τ be a $\mathscr{C}\omega$-inductive class. Then, an object (E, u) in $\mathscr{C}$ is a $D_r(\tau;\ \mathscr{C})$-space if and only if for every F in τ, any closed map in $\mathscr{C}$ of F into (E, u) is continuous.*

Proof. Suppose that (E, u) is a $D_r(\tau;\ \mathscr{C})$-space. Let f be a closed map in $\mathscr{C}$ from some F in τ into (E, u). Since $F/f^{-1}(0)$ is also in τ and the induced map of f is in $\mathscr{C}$, we may also suppose that f is a $(1-1)$ map.

As f is closed and linear, f is continuous from F into (E, v_1), where v_1 is a separated linear topology on E which is coarser than u. Since f is $(1-1)$, we may identify F with the linear subspace $f(F) = E_1$ say of E. Let (E_1, p) be this space with the topology of F. The space (E_1, p) is in τ and p is finer than the v_1-induced topology on E_1. Let E_2 be an algebraic supplement of E_1 in E. As E is algebraically isomorphic to $E_1 \times E_2$, we may identify $E_1 \times E_2$ with E. With this identification, let (E, q) be the $\mathscr{C}$-inductive limit of (E_1, p) and (E_2, ω) by the injection maps. Clearly, q is finer than v_1, (E, q) is in τ, and the map f is continuous from F into (E, q). Now, the identity map, i say, from (E, u) onto (E, v_1) is closed, being continuous. Therefore, the graph of i is closed in $(E, u) \times (E, q)$. Moreover, by (*b*) of (C_5), the map i is in $\mathscr{C}$ as a map of (E, u) onto (E, q). Since (E, u) is a $D_r(\tau;\ \mathscr{C})$-space and (E, q) is in τ, it follows that u is coarser than q. Therefore f is continuous from F into (E, u).

The converse is easy, cf. ((7), 4.9).

By using Theorem 3.1, one can prove the following result.

Theorem 3.2. *Let $\mathscr{C}$ and τ be as in Theorem* 3.1. *Then, an object E in $\mathscr{C}$ is a $D(\tau;\ \mathscr{C})$-space if and only if, for every F in τ, any closed map in $\mathscr{C}$ of F into each quotient of E by a closed linear subspace is continuous.*

With $\mathscr{C}$ and τ as in Theorem 3.1, it is not difficult to show that if E is a $D_r(\tau;\ \mathscr{C})$-space, and E_0 in $\mathscr{C}$ is a closed linear subspace of E, then E_0 is also a $D_r(\tau;\ \mathscr{C})$-space, provided that a map into E_0 which is in $\mathscr{C}$ is also in $\mathscr{C}$ as a map into E. One can in a similar situation deduce from Theorem 3.2 that a closed linear subspace of a $D(\tau;\ \mathscr{C})$-space is also a $D\ (\tau;\mathscr{C})$-space. In this case, one uses the following result.

Lemma 3.3. *Let E_0 be a linear subspace of a separated l.t.s. E and suppose that for some l.t.s. F, t is linear map of E_0 into F, with induced map f. If the graph of t is closed in $E\times F$, then $E/t^{-1}(0)$ is separated. The graph of t is closed in $E\times F$ if and only if the graph of f is closed in $E/t^{-1}(0)\times F$.*

From Theorem 3.2 and Proposition 2.2, the following result is immediate.

Corollary to Theorem 3.2. *Let $\mathscr{C}$, τ be as in Theorem* 3.1. *Suppose that τ_1 is a subclass of τ such that every quotient by a closed linear subspace of each member of τ_1 is also in τ_1. If every member of τ is the $\mathscr{C}$-inductive limit of some $(E_\gamma;\ u_\gamma\colon\ \gamma\in\Phi)$, where each E_γ is in τ_1, then an object E in $\mathscr{C}$ is a $D(\tau_1;\ \mathscr{C})$-space if and only if every closed map in $\mathscr{C}$ from any member of τ_1 into each quotient of E by a closed linear subspace is continuous.*

The hypothesis of the above corollary is satisfied if

(a) for $\mathscr{C}_1$, τ is either β_n^*, $\mathfrak{F}^*$, $\mathfrak{F}_1^*$, $\mathfrak{F}_{11}^*$, β^*, β_1^*, β_{11}^* ,or Λ^*, and τ_1 is respectively chosen to be β_n, $\mathfrak{F}$, $\mathfrak{F}_1$, $\mathfrak{F}_{11}$, β, β_1, β_{11}, or Λ;

(b) for $\mathscr{C}_2$, τ is either β_n^*, $\mathfrak{F}^*$, $\mathfrak{F}_{11}^*$, β^*, β_{11}^*, or Λ^*, and τ_1 is β_n, $\mathfrak{F}$, $\mathfrak{F}_{11}$, β, β_{11}, or Λ;

(c) for $\mathscr{C}_3$, τ is β_n^*, $\mathfrak{F}^*$, or β, and τ_1 is β_n, $\mathfrak{F}$, or β.

Consider the case when $\mathscr{C}$ is $\mathscr{C}_1$ ($\mathscr{C}_2$, $\mathscr{C}_3$) and τ is η_1 (η_{11}, η). Then every member of τ is the $\mathscr{C}$-inductive limit of some $(E_\gamma;\ u_\gamma\colon\ \gamma\in\Phi)$, where each E_γ is in Λ (Λ, β_n). It is then easy to see that for an object F in $\mathscr{C}$, every closed linear map from each member of τ into F is continuous if and only if every closed linear map from each member of Λ (Λ, β_n) into F is continuous. By using the method of proof of Theorem 3.1, one can then show that in this case, an object E in $\mathscr{C}$ is a $D_r(\tau;\ \mathscr{C})$-space if and only if every closed linear map from any member of τ into E is continuous. This then implies that

$$\text{(i)}\ D_r(\eta_1;\ \mathscr{C}_1)=D_r(\Lambda^*;\ \mathscr{C}_1)=D_r(\eta_1^*;\ \mathscr{C}_1),$$

$$\text{(ii)}\ D_r(\eta_{11};\ \mathscr{C}_2)=D_r(\Lambda^*;\ \mathscr{C}_2)=D_r(\eta_{11}^*;\ \mathscr{C}_2),$$

$$\text{(iii)}\ D_r(\eta;\ \mathscr{C}_3)=D_r(\beta_n^*;\ \mathscr{C}_3)=D_r(\eta^*;\ \mathscr{C}_3).$$

4. For any $\mathscr{C}$, we shall throughout this section assume that τ is the class of all objects in $\mathscr{C}$ each of which is of the second category in itself. This does not contradict the restriction imposed on τ when the concept of a $D(\tau;\ \mathscr{C})$-space was defined.

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An object E in $\mathscr{C}$ is said to be *extracomplete* if every quotient of E by a closed linear subspace is complete.

For $\mathscr{C}_3$, any B-complete space is extracomplete, since each quotient by a closed linear subspace of a B-complete locally convex space is B-complete and by (7), a B-complete locally convex space is complete. However, an extracomplete space space need not be B-complete. For, any linear space is extracomplete under its finest locally convex topology, but such a space need not be B-complete.

We call an object E in $\mathscr{C}$ a $D_1(\tau;\ \mathscr{C})$*-space* if there exists a continuous linear map from some F onto E, where F is either an extracomplete $D(\tau;\ \mathscr{C}_1)$-space or is the generalized strict *-inductive limit of a sequence of extracomplete $D(\tau;\ \mathscr{C}_1)$-spaces.

If (E, u) is a B-complete l.t.s. (i.e. if (E, u) is B-complete in $\mathscr{C}_1$) or an L.F. space then, for any separated linear (semiconvex, locally convex) topology v on E coarser than u, (E, v) is a $D_1(\tau;\ \mathscr{C}_1)$-($D_1(\tau;\ \mathscr{C}_2)$-, $D_1(\tau;\ \mathscr{C}_3)$-) space. Also if (E, u) is the *-direct sum of a sequence (E_i) of linear topological spaces, where for each i, $E_i = l^p$ or H^p for some p in the open interval (0, 1) then, for any linear (semiconvex, locally convex) topology v on E coarser than $u^{\circ\circ}$, (E, v) is a $D_1(\tau;\ \mathscr{C}_1)$- ($D_1(\tau;\ \mathscr{C}_2)$-, $D_1(\tau;\ \mathscr{C}_3)$-) space.

There may not exist a continuous linear map from a B-complete locally convex space onto a $D_1(\tau;\ \mathscr{C}_3)$-space. For, let (E, u) be the sequence space $l^{\frac{1}{2}}$. Then the incomplete barrelled normed space $(E, u^{\circ\circ})$ is a $D_1(\tau;\ \mathscr{C}_3)$-space. If there were to exist a continuous linear map f say, from a B-complete locally convex space onto $(E, u^{\circ\circ})$ then f would be open and this would imply that $(E, u^{\circ\circ})$ is complete.

From the corollary of Theorem 3.2, we derive the following result.

Lemma 4.1. *Let $\mathscr{C}$ be a category of linear maps of linear topological spaces satisfying (C_4) and (C_5). Then, an object E in $\mathscr{C}$ is a $D(\tau;\ \mathscr{C})$-space if and only if every closed map in $\mathscr{C}$ from any F in τ into each quotient of E by a closed linear subspace is continuous.*

Let F be an object in $\mathscr{C}$ and (F_n) be a sequence of $D_1(\tau;\ \mathscr{C})$-spaces. Suppose that, for each n, u_n is a continuous map in $\mathscr{C}$ from F_n into F and that F is the union over n of $u_n(F_n)$. For each n, there is a continuous map g_n say, in $\mathscr{C}$ from G_n onto F_n, where G_n is an extracomplete $D(\tau:\ \mathscr{C}_1)$-space or G_n is the generalized strict *-inductive limit of some $(G_{n_i};\ w_{n_i}\colon i = 1, 2, \ldots)$ where each G_{n_i} is an extracomplete $D(\tau;\ \mathscr{C}_1)$-space.

For each n such that G_n is an extracomplete $D(\tau;\ \mathscr{C}_1)$-space, the induced map v_n of the continuous map $u_n \circ g_n$ from G_n into F is continuous and $G_n/(u_n \circ g_n)^{-1}(0)$ is an extracomplete $D(\tau;\ \mathscr{C}_1)$-space. Let J_1 be the union over n of $v_n(G_n/(u_n \circ g_n)^{-1}(0))$.

For each n such that G_n is the generalized strict *-inductive limit of some $(G_{n_i};\ w_{n_i}\colon i = 1, 2, \ldots)$, the induced map v'_{n_i} of the map $u_n \circ g_n \circ w_{n_i}$ from G_{n_i} into F is continuous. Also, $G_{n_i}/(u_n \circ g_n \circ w_{n_i})^{-1}(0)$ is an extracomplete $D(\tau;\ \mathscr{C}_1)$-space. Let J_2 be the union over n and i of $v'_{n_i}(G_{n_i}/(u_n \circ g_n \circ w_{n_i})^{-1}(0))$.

Clearly, F is the union of J_1 and J_2. We may thus suppose that each u_n is a continuous linear $(1-1)$ map and that each F_n is an extracomplete $D(\tau;\ \mathscr{C}_1)$-space. With this observation, on using the method of proof of Theorems 2 and 3 (ii) of (8), but this time applying Lemma 4.1 instead of Theorem 1 of (8), one can prove the following result.

Theorem 4.2. *Let $\mathscr{C}$ be a category of linear maps of linear topological spaces satisfying (C_4) and (C_5). Let E be the $\mathscr{C}$-inductive limit of $(E_\gamma;\ u_\gamma;\ \gamma \in \Phi)$, where each E_γ is in τ. Suppose that F is an object in $\mathscr{C}$ and that for each positive integer n, v_n is a continuous linear map from a $D_1(\tau;\ \mathscr{C})$-space F_n into F. If F is the union over n of $v_n(F_n)$, then any closed linear map from E into F is continuous, and any closed linear map from F onto E is open. This is so in particular if F is the generalized strict $\mathscr{C}$-inductive limit of $(F_n;\ v_n\colon n = 1, 2, \ldots)$.*

Theorem 4.3. *Let $\mathscr{C}$ satisfy (C_4) and (C_5). For each positive integer n, let u_n be a continuous linear map from a $D_1(\tau;\ \mathscr{C})$-space E_n into an l.t.s. E, and suppose that E is the union of the subspaces $u_n(E_n)$. If t is a closed linear map from E into an object F in $\mathscr{C}$ such that $t(E)$ is in τ, then $t(E)$ is closed in F.*

Proof. By an argument similar to that preceding Theorem 4.2, one can show that we may assume that each E_n is an extracomplete $D(\tau;\ \mathscr{C}_1)$-space and that E is the union of the subspaces E_n such that the topology of E_n is finer than that induced from E.

Since $t(E) = \cup_{n \geq 1} t(E_n)$ is of the second category in itself, there is a positive integer N such that $t(E_N)$ is of the second category in $t(E)$ and $t(E_N)$ is dense in $t(E)$. ($t(E)$, $t(E_N)$ are considered under the respective topologies induced from F). The space $t(E_N)$ is of the second category in itself and the graph of the map t from E_N onto $t(E_N)$ is closed in $E_N \times t(E_N)$. As E_N is a $D(\tau;\ \mathscr{C}_1)$-space, it follows from Lemma 4.1 that t is an open map of E_N onto $t(E_N)$.

Since $E_N/t^{-1}(0)$ is an extracomplete $D(\tau;\ \mathscr{C}_1)$-space, we may assume that t is $(1-1)$ and thus consider E_N as the same space $t(E_N)$ under a coarser topology v. Moreover, $(t(E_N), v)$ is complete and the identity map, i say, from $(t(E_N), v)$ into F is closed.

Let $(y_\alpha\colon \alpha \in \Psi)$ be a net in $t(E_N)$ converging to y_0 in F. Because i is an open map from $(t(E_N), v)$ onto $t(E_N)$, $(y_\alpha\colon \alpha \in \Psi)$ is v-Cauchy and must therefore converge to some point y_0' in $(t(E_N);\ v)$, since this space is complete. As the graph of i is closed in $(t(E_N), v) \times F$, $y_0 = y_0'$ and thus $t(E_N)$ is closed in F. The result now follows from this, since $t(E_N)$ is dense in $t(E)$.

Corollary. *Let $\mathscr{C}$ satisfy (C_4) and (C_5). Let F be an object in $\mathscr{C}$ and E the generalized strict $\mathscr{C}$-inductive limit of $(E_n;\ u_n\colon n = 1, 2, \ldots)$, where each E_n is a $D_1(\tau;\ \mathscr{C})$-space. If t is a closed linear map of E into F, then either $t(E)$ is of the first category in F or $t(E) = F$.*

Proof. If $t(E)$ is of the second category in F then $t(E)$ is of the second category in itself and $t(E)$ is dense in F. By the theorem, $t(E)$ is closed in F, and this gives the result.

cf. (**8**), Theorem 3, Corollary.

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