CORRESPONDENCE.

The Hamiltonian Revival.

To the Editor of the *Mathematical Gazette.*

Sir,—Professor E. T. Whittaker's courtesy in answering the questions I raised recently relating to a sentence in his article " The Hamiltonian Revival " is deeply appreciated. At the same time many who use the calculus of vectors and dyadics will regret to find in Professor Whittaker's letter little realisation of its beauties and powers. Professor Whittaker approaches the question as a pure mathematician interested in " any set of objects of thought which may be called *generalised numbers* provided they satisfy certain definitions " concerning equality, addition and multiplication and form a " group ". But he shows no recognition of the point of view of the natural philosopher who wishes to deal with physical phenomena in the universe in which we live. This ignoring of the needs which presumably inspired Gibbs and Heaviside pervades much of what Professor Whittaker has to say, in both his article and his letter, on the subject of vectors.

In his article he condemned dyadics and vector-analysis as reproducing " the limitations imposed by (their) close association with the geometry of ordinary space ". But it is precisely because vector analysis embodies the quality of " three-dimensionality " that it is so appropriate to the handling of physical problems. In *tensor* analysis the " vector product " of two vectors A_μ and B_ν requires for its expression the use of the " alternate tensor ", of rank n equal to the number of dimensions of the space being used; the resulting tensor is of rank $(n-2)$. In three-dimensional space $(n=3)$, the resulting tensor is itself of rank 1, *i.e.* is a vector; hence in *vector* analysis the " vector product " of two vectors is so-called because amongst the various products of two vectors which can be constructed this one is also a vector. Hence arises the power of vector analysis proper, a power which tensor analysis in general, whether for " flat " space or for a general Riemannian space, exercises only by an excessive use of suffixes. Professor Whittaker objects to the calling of $\mathbf{B}$ a true product in the equation $\mathbf{A} \wedge \mathbf{X} = \mathbf{B}$, because the vector $\mathbf{X}$ is not uniquely determined by $\mathbf{A}$ and $\mathbf{B}$. But multiplying each side vectorially by $\mathbf{A}$ and using the " continued vector product " theorem (which may be regarded as the " signature-tune " of ordinary space), we have

$$-\mathbf{A}(\mathbf{X} \, . \, \mathbf{A}) + \mathbf{X}\mathbf{A}^2 = \mathbf{B} \wedge \mathbf{A}$$

or

$$\mathbf{X} = \frac{\mathbf{B} \wedge \mathbf{A}}{\mathbf{A}^2} + \lambda \mathbf{A}.$$

where λ is an arbitrary constant. Does Professor Whittaker condemn differential equations as not " true " equations because their solution involves arbitrary constants? The component $\lambda\mathbf{A}$ of $\mathbf{X}$ is arbitrary in modulus because this component is irrelevant in many physical applications. Mechanics, in vector presentation, turns largely on the use of the two relations $\mathbf{L} = \mathbf{r} \wedge \mathbf{P}$ and $\mathbf{v} = \mathbf{\Omega} \wedge \mathbf{r}$, where $\mathbf{L}$ is the moment about the origin of a force $\mathbf{P}$ acting at $\mathbf{r}$, and $\mathbf{v}$ is the velocity at $\mathbf{r}$ associated with an angular velocity $\mathbf{\Omega}$ of the rigid body of which the point located at the origin is at rest; in these relations, the component of $\mathbf{P}$ or $\mathbf{\Omega}$ along $\mathbf{r}$ is physically irrelevant, and the vector-product is useful precisely because it seizes on the relevant and rejects the irrelevant.

To require a combination of two entities to be what Professor Whittaker calls a " true product " ignores that two given entities may have many different multiplicative combinations, whether we call them " products " or

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not; *e.g.* two vectors **A**, **B** have the combinations **A** . **B**, **A** ∧ **B** and the dyad-product **AB**, which form the hierarchy scalar, vector and tensor of rank two. The flexibility of the calculus invented by Gibbs arises from a wise employment of these different " products ", and in the ordinary space most used by physicists these products may readily be handled without the use of suffixes.

In this connection I may refer to Professor Whittaker's statement that Gibbs and Heaviside " did not rise to the idea of treating vectors by means of a calculus of generalised numbers of a new type : all they wanted was some way of writing ordinary Cartesian analysis without putting the axes of coordinates in evidence ". In the first place, their calculus *was* new, on Professor Whittaker's own showing, in that it departed from the group property ; in the second place, it is necessary to distinguish between the frame of reference, the axes of coordinates and the components of vectors with respect to these axes. The frame of reference—the observer's observational apparatus—must always be in evidence, though the pure mathematician may cloak it in a mantle of invisibility ; particular axes of reference have certain conveniences when specific directions are of interest ; the components of vectors with respect to arbitrary axes have little physical interest. *Tensor* calculus is, indeed, a condensed method of writing out relations between numerical components or coordinates ; it always mentions coordinates, but avoids mentioning each one separately by the magic of its summation convention. In *vector* analysis we suppress suffixes altogether : $P_\mu Q_\mu$ becomes **P** . **Q**, $P_\mu Q_\nu$ becomes **PQ**, $E_{\alpha\mu\nu} P_\mu Q_\nu$ becomes the α component of **P** ∧ **Q** ; and we deal directly with the entities represented by the symbols, not their components. It is *not* a " syncopated form of ordinary Cartesian analysis ". The power of vector analysis, again, is seen in the circumstance that we can pass readily to the suffix notation if for the moment it proves more convenient. But vector analysis has the further advantage that it can pick out axes of physical interest which are not mutually perpendicular, and yet retain the advantages of an orthogonal set of axes.

For example, in handling the motion of a top, vector analysis concentrates attention on two directions of interest, the axis of the top and the vertical. It needs to mention no others. The immense logical superiority of vector analysis over Cartesian is evident when we reflect on that perversion by which in some text-books, the laws of nature are formulated in vector form, embodied in Cartesian equations, applied to a sleeping top, recombined to give equations in a complex variable $x+iy$, the resulting differential equation in z solved, and z re-dissected into its component scalars, which then need to be mentally recombined to give the physical kinematic picture. In the solution of a problem by the methods of vector analysis, there is no need for an eventual interpretation of the final result : the problem is formulated vectorially and solved vectorially, and the vectorial solution proclaims the resulting motion.

In conclusion, if quaternions are so useful and natural a mathematical tool, why are they not habitually employed in research and teaching?

Yours truly, E. A. Milne.

To the Editor of the *Mathematical Gazette*.

Sir,—What I tried to show in my previous letter was that vector-analysis has certain inherent defects which must always prevent it from developing into a satisfactory calculus. It is not difficult to illustrate this, for example, on the ground of Professor Milne's own choosing, namely the different multiplicative combinations. If the vector-analyst forms the different kinds of products of three vectors, such as (**A** ∧ **B**) . **C**, he has a choice of 3 alternatives for the first multiplicative sign, 3 for the second multiplicative sign, 2 for the position of the brackets denoting association, and 6 for the permutations of A, B, C, that is, he has $3 . 3 . 2 . 6$ or 108 formally different products of 3 vectors, of which, however, many are meaningless, many equal to each other, and many equal to each other with sign reversed. Yet with all this multitude of forms, there is no vector-analytic expression, formally symmetric in the 3 vectors, which represents the simplest symmetric function of the vectors, namely the volume of the parallelepiped of which they are three edges. How much simpler is the quaternion theory, where there is only one product of the three vectors in a given order. It is a quaternion, and the volume of the parallelepiped is its scalar.

Between the many forms there are many identical relations. But how much better to do away with both!

Professor Milne suggests that quaternions are less physical than vector-analysis. But any vector-analysis solution of a problem can be at once transliterated into quaternions, the two solutions differing merely in notation. The converse however is not true, since the powerful concept of division by a vector, which is used in quaternions, does not exist in vector-analysis.

At the end of his letter, Professor Milne puts the interesting question, why quaternions have not been used habitually by physicists instead of vector-analysis. The reason is, I think, to be found in the combination of two circumstances. Firstly, that the problems which physicists solve by vector-analysis are practically always of a linear or quadratic character, and therefore so simple from the mathematical point of view that the defects of vector-analysis don't cause any trouble : and secondly, that vector-analysis can be picked up in an hour or so, involving as it does no methods radically different from those of Cartesian analysis, whereas the quaternion calculus is to the physicist a new kind of pure mathematics, which he is reluctant to learn. I am inclined to conjecture that if the present range of problems were not extended, physicists would continue to prefer vector-analysis ; but that the development of relativity and quantum mechanics will sooner or later require quaternion methods for the more difficult problems (as in Professor Conway's paper, referred to in my original article) ; and that when the physicists have thus been forced to learn quaternions, they will use them for all purposes, and vector-analysis will be forgotten. Yours truly, E. T. Whittaker.

[This correspondence is now closed.—Ed.]

The Mathematician and the Community.

To the Editor of the *Mathematical Gazette.*

Sir,—In *A Mathematician's Apology*, Professor Hardy writes : "There is no instance, so far as I know, of a first-rate mathematician abandoning mathematics and attaining first-rate distinction in any other field. There may have been young men who would have been first-rate mathematicians if they had stuck to mathematics, but I have never heard of a really plausible example. And all this is fully borne out by my own very limited experience. Every young mathematician of real talent whom I have known has been faithful to mathematics, and not from lack of ambition but from abundance of it ; they have all recognized that there, if anywhere, lay the road to a life of any distinction."

It is easy to maintain this argument by simply denying that any other form of success is comparable with a mathematician's in his own line. That mathe-