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# ON SOME INEQUALITIES FOR ELEMENTARY SYMMETRIC FUNCTIONS

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**In this note, we prove certain inequalities for elementary symmetric funtions that are relevant to the study of partial differential equations associated with curvature problems.**

In this note, we prove certain inequalities for elementary symmetric funtions that are relevant to the study of partial differential equations associated with curvature problems, (see, for example, [**2**, **3**, **7**]). In particular our first theorem relates to the partial uniform ellipticity of the higher order mean curvature operators while our second one is an improvement of an inequality of Ivochkina that was crucial in her study of these operators in [**2**, **3**]. From these two inequalities, we deduce further inequalities that arose in our treatment of curvature quotients in [**5**].

We begin with some definitions and notation. First the $k$-th order elementary symmetric function of $n$ variables, $S_k$, is defined by

$$S_k(\lambda) = \sum_{1\leqslant i_1<i_2<\cdots<i_k\leqslant n} \lambda_{i_1}\lambda_{i_2}\ldots\lambda_{i_k} \tag{1}$$

where $1 \leqslant k \leqslant n$ and $\lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbf{R}^n$. For consistency, we extend $S_k$ by setting

$$S_0(\lambda) = 1,$$
$$S_k(\lambda) = 0 \qquad \text{for } k > n.$$

The function $S_k$ will be considered in the corresponding cone in $\mathbf{R}^n$, $\Gamma_k$, given by

$$\Gamma_k = \{\lambda \in \mathbf{R}^n \mid S_j(\lambda) > 0, \ \text{for all } j = 1, \ldots, k\}. \tag{2}$$

It is easily seen that $\Gamma_k$ is , in fact, a cone with vertex at the origin. Clearly $\Gamma_k \subseteq \Gamma_j$ for $k \geqslant j$ and $\Gamma_n$ is the positive cone $\{\lambda \in \mathbf{R}^n \mid \lambda_i > 0,\ i = 1, \ldots, n\}$. For any fixed $t$-tuple $\{i_1, i_2, \ldots, i_t\} \subseteq \{1, 2, \ldots, n\}$, we define

$$S_{k;i_1 i_2\ldots i_t}(\lambda) = S_k|_{\lambda_{i_1}=\lambda_{i_2}=\cdots=\lambda_{i_t}=0}, \tag{3}$$

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that is, $S_{k;i_1\dots i_t}$ is the $k$-th order elementary symmetric function of the $n-t$ variables $\{1,\dots,n\}\backslash\{i_1,\dots,i_t\}$.

The following properties of the functions $S_k$ will be used in this paper:

$$S_k(\lambda) = S_{k;i}(\lambda) + \lambda_i S_{k-1;i}(\lambda) \tag{4}$$

$$\sum_{i=1}^{n} S_{k;i}(\lambda) = (n-k)S_k(\lambda) \tag{5}$$

for all $\lambda \in \mathbf{R}^n$. Furthermore, if $\lambda \in \Gamma_k$, then at least $k$ of the numbers $\lambda_1,\dots,\lambda_n$ are positive and moreover

$$S_{l;i_1 i_2\dots i_s}(\lambda) > 0 \tag{6}$$

for all $\{i_1, i_2, \dots, i_s\} \subseteq \{1,2,\dots,n\}$, $l+s \leqslant k$. As well, we have the Newton inequalities

$$S_k S_{k-2}(\lambda) \leqslant \frac{(k-1)(n-k+1)}{k(n-k+2)}[S_{k-1}(\lambda)]^2 \tag{7}$$

for $\lambda \in \mathbf{R}^n$, $k \geqslant 2$ and the Maclaurin inequalities

$$\left[\frac{1}{\binom{n}{k}} S_k(\lambda)\right]^{1/k} \leqslant \left[\frac{1}{\binom{n}{l}} S_l(\lambda)\right]^{1/l} \tag{8}$$

for $\lambda \in \Gamma_k$, $k \geqslant l \geqslant 1$; (see [**6**]).

Throughout this paper we shall write $\lambda$ in decreasing order,

$$\lambda_1 \geqslant \dots \geqslant \lambda_k \geqslant \dots \geqslant \lambda_p > 0 \geqslant \lambda_{p+1} \dots \geqslant \lambda_n \tag{9}$$

where $p(\geqslant k)$ is the number of positive $\lambda_i$.

Our first theorem provides lower bounds for the ratios $S_{k-1;i}/S_{k-1}$.

**Theorem 1.** *There exists a positive constant $\theta$, depending on $n$ and $k$, such that*

$$\frac{S_{k-1;i}(\lambda)}{S_{k-1}(\lambda)} \geqslant \theta, \tag{10}$$

*for all $i \geqslant k$, $\lambda \in \Gamma_k$.*

REMARKS. (i) It suffices to prove (10) for the case $i = k$, because $S_{k-1;i} \leqslant S_{k-1,j}$ if $i \leqslant j$. This follows from the formula,

$$S_{k-1;i} = S_{k-1;ij} + \lambda_j S_{k-2;ij}$$

and the positivity of $S_{k-2;ij}$ on $\Gamma_k$.

(ii) Our proof will yield the following estimate for $\theta$, namely

$$\theta(n,k) \geqslant \left[1 + \sum_{j=2}^{k} \prod_{i=j}^{k} (1 + C_i)\right]^{-1}, \qquad C_i = \sqrt{\frac{(n-k)i}{n-1}}. \tag{11}$$

In the special case $n = 3, k = 2$, we obtain $\theta \geqslant 1/3$, which is sharp, as is evidenced by the example $\lambda_1 = \lambda_2 = 1$, $\lambda_3 = -1/2$. We provide an example later to show that the condition $i \geqslant k$ cannot be improved.

(iii) When $i > p$, that is $\lambda_i \leqslant 0$, the estimate (10) is already known, in conjunction with gradient bounds for curvature equations [1, 4]. In this case, (10) follows, with $\theta = 1$, immediately from the formula,

$$S_{k-1}(\lambda) = S_{k-1;i} + \lambda_i S_{k-2;i}.$$

PROOF OF THEOREM 1: We first prove the inequality,

$$|S_{k-1;1k}| \leqslant C_k S_{k-1;k}, \qquad C_k = \sqrt{\frac{k(n-k)}{n-1}}. \tag{12}$$

Using the formula (4), we have

$$\begin{aligned} S_{k;1k} + \lambda_1 S_{k-1;1k} = S_{k;k} &= S_k - \lambda_k S_{k-1;k} \\ &\geqslant -\lambda_k S_{k-1;k} \end{aligned} \tag{13}$$

$$S_{k-1;1k} + \lambda_1 S_{k-2;1k} = S_{k-1;k} \tag{14}$$

Eliminating $\lambda_1$ from (13) and (14), yields

$$\begin{aligned} (S_{k-1;1k})^2 - S_{k;1k} S_{k-2;1k} &\leqslant S_{k-1;k}(S_{k-1;1k} + \lambda_k S_{k-2;1k}) \\ &= S_{k-1;k} S_{k-1;1} \end{aligned}$$

so that by Newton's inequality (7) we obtain

$$\left[1 - \frac{(k-1)(n-k-1)}{k(n-k)}\right](S_{k-1;1k})^2 \leqslant (S_{k-1;k})^2,$$

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whence (12) follows.

Now from (12) and (14), we have

$$C_k S_{k-1;k} \geqslant -S_{k-1;k} + \lambda_1 S_{k-2;1k}$$

so that

$$S_{k-1;k} \geqslant \frac{\lambda_1}{1+C_k} S_{k-2;1k}.$$

Let us now suppose that (10) is valid wherever $k$ and $n$ are replaced by $k-1$ and $n-1$, that is for some positive constant $\theta = \theta(k-1, n-1)$, we have

$$S_{k-2;1k} \geqslant \theta S_{k-2;1}.$$

Here we are replacing $S_k$ by $S_{k-1;1}$. Then we obtain

$$\begin{aligned} S_{k-1;k} &\geqslant \frac{\lambda_1\theta}{1+C_k} S_{k-2;1} \\ &= \frac{\theta}{1+C_k}(S_{k-1} - S_{k-1;1}), \end{aligned}$$

whence we conclude

$$S_{k-1;k} \geqslant \frac{\theta}{1+C_k+\theta} S_{k-1}. \tag{15}$$

Since inequality (10) clearly holds for $k=1$, with $\theta = 1$, we are done. ∎

## Further examples

(i) Taking $k < n$ and $\lambda_1 = M$, $\lambda_i = 1$ for $1 < i < k$, $\lambda_k = M^{-1}$, $\lambda_i = 0$ for $i > k$, we clearly have

$$\begin{aligned} \frac{S_{k-1;k-1}}{S_{k-1}} &= \frac{1}{M + (k-2) + M^{-1}} \\ &\longrightarrow 0 \qquad \text{as } M \to \infty \end{aligned}$$

which shows Theorem 1 is impossible for $i < k$.

(ii) For $k = 2$ and $n \geqslant 3$, we get from (11),

$$\theta(n,2) \geqslant \frac{1}{2 + \sqrt{2(n-2)/(n-1)}}.$$

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As with the case $n = 3$, $\theta(3,2) = 1/3$, we also must have equality in this estimate. This follows from the example

$$\lambda_i = 1 \qquad \text{for } i = 1, \dots, k,$$
$$\lambda_i = s \qquad \text{for } i = k+1, \dots, n$$

where

$$s = \frac{1}{n-3}\left(\sqrt{\frac{2(n-1)}{n-2}} - 2\right), \qquad \text{for } n \geqslant 4.$$

Accordingly we have the sharp estimate

$$\frac{S_{1;2}}{S_1} \geqslant \frac{1}{2 + \sqrt{2(n-2)/(n-1)}} > \frac{1}{2+\sqrt{2}}. \tag{16}$$

The last inequality also follows directly from the formula

$$0 < 2S_2 = S_1^2 - \sum_{i=1}^{n} \lambda_i^2. \tag{17}$$

Note when $\theta$ is sharp, equality in (10) must be attained on $\partial\Gamma_k$, otherwise the $k^{th}$ partial derivative of the ratio must vanish, that is

$$\frac{S_{k-1;k}S_{k-2;k}}{S_{k-1}^2}(\lambda) = 0$$

which contradicts $\lambda \in \Gamma_k$ according to property (6).

Our second theorem is an improvement of the Ivochkina inequality [**2**].

**THEOREM 2.** *There exists a constant $C$ depending on $n$ and $k$ such that*

$$S_{k+1;r}(\lambda) \leqslant \sum_{i=k, i\neq r}^{n} S_{k-1;i}(\lambda)(\lambda_i)^2 \tag{18}$$

*for all $\lambda \in \Gamma_k$.*

PROOF: We separately estimate each of the terms $Q$ in the sum of the left hand side of (18). If $Q$ has an odd number of negative $\lambda_i$, we are clearly done. Let us assume first that $Q$ has an even number of negative factors and write $Q$ in the form

$$Q = A\lambda_i\lambda_j \qquad \lambda_i, \lambda_j < 0$$

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and

$$A = \lambda_{i_1} \cdots \lambda_{i_{k-1}}.$$

Without loss of generality, we can order $\lambda$,

$$\lambda_{i_1} \geqslant \cdots \geqslant \lambda_{i_s} > 0 > \lambda_{i_{s+1}} \cdots \geqslant \lambda_{i_{k-1}}.$$

Consequently

$$\begin{aligned} |A| &= (\lambda_{i_1} \cdots \lambda_{i_s}) \left|\lambda_{i_{s+1}} \cdots \lambda_{i_{k-1}}\right| \\ &\leqslant \lambda_{i_1} \cdots \lambda_{i_s} \left|\lambda_{i_{k-1}}\right|^{k-s-1}. \end{aligned}$$

Since $\lambda \in \Gamma_k$, the sum of any $n-k+1$ of the $\lambda_i$ is positive and hence

$$\left|\lambda_{i_{k-1}}\right| \leqslant (p-k+1)\lambda_k$$

and thus

$$\begin{aligned} A &\leqslant (n-k+1)^{k-s-1} \lambda_1 \lambda_2 \cdots \lambda_s (\lambda_k)^{k-s-1} \\ &\leqslant (n-k+1)^{k-1} \lambda_1 \cdots \lambda_s \lambda_{s+1} \cdots \lambda_{k-1}. \end{aligned}$$

Next by expanding

$$\begin{aligned} S_{k-1} &= S_{k-1;1} + \lambda_1 S_{k-2;1} \\ &= S_{k-1;1} + \lambda_1 S_{k-2;12} + \lambda_1 \lambda_2 S_{k-3;12} \\ &= S_{k-1;1} + \lambda_1 S_{k-2;12} + \cdots + \lambda_1 \cdots \lambda_{k-2} S_{1;12\cdots(k-1)} + \lambda_1 \cdots \lambda_{k-1} \end{aligned}$$

we have the inequality

$$\lambda_1 \lambda_2 \cdots \lambda_{k-1} \leqslant S_{k-1}. \tag{19}$$

Hence

$$\begin{aligned} Q &\leqslant \frac{1}{2}(n-k+1)^{k-1} S_{k-1}[(\lambda_i)^2 + (\lambda_j)^2] \\ &\leqslant \frac{1}{2}(n-k+1)^{k-1}[S_{k-1;i}(\lambda_i)^2 + S_{k-1;j}(\lambda_j)^2] \end{aligned}$$

by virtue of (10) with $\theta = 1$, Remark (iii). The case when $Q$ has all positive factors follows directly from (19) and Theorem 1, although it could be deduced independently of Theorem 1. ∎

REMARK. We have dispensed with an assumption made by Ivochkina [**2**, condition (2.7)] and our summation is taken from $k$ to $n$ rather than 2 to $n$. The latter improvement is essential for our applications to curvature quotients.

APPLICATION. Observing that

$$\frac{\partial S_k}{\partial \lambda_i} = S_{k-1,i}, \tag{20}$$

we may write the estimates (10) and (18) in the form

$$\frac{\partial S_k}{\partial \lambda_i}(\lambda) \geqslant \frac{\theta}{n-k+1}\sum_{j=1}^{n}\frac{\partial S_k}{\partial \lambda_j}(\lambda) \quad \text{for } \lambda \in \Gamma_k,\ i \geqslant k, \tag{21}$$

$$S_{k+1;r} \leqslant C \sum_{i=k, i\neq r}^{n} \frac{\partial S_k}{\partial \lambda_i}(\lambda_i)^2 \quad \text{for } \lambda \in \Gamma_k,\ \lambda_r > 0. \tag{22}$$

We need an extension of (22) for quotients of elementary symmetric functions

$$S_{k,l} = \frac{S_k}{S_l} \qquad \text{for } k > l \geqslant 1, \tag{23}$$

which follows by combination of Theorems 1 and 2.

**THEOREM 3.** *There exists a positive constant $C$ depending on $n$ and $k$ such that*

$$\frac{S_{k+1;r}(\lambda)}{S_l(\lambda)} \leqslant C \sum_{i=k, i\neq r}^{n} \frac{\partial S_{k,l}(\lambda)}{\partial \lambda_i}(\lambda_i)^2 \tag{24}$$

*for all $\lambda \in \Gamma_k$.*

PROOF: We calculate, (as in [7]),

$$\begin{aligned}\frac{\partial S_{k,l}}{\partial \lambda_i} &= \frac{S_l S_{k-1;i} - S_k S_{l-1;i}}{S_l^2}\\ &= \frac{S_{l;i} S_{k-1;i} - S_{k;i} S_{l-1;i}}{S_l^2}\\ &\geqslant \frac{n(k-l)}{k(n-l)}\frac{S_{l;i}S_{k-1;i}}{S_l^2} \qquad \text{(by Newton's inequality)}\\ &\geqslant \frac{n(k-l)}{k(n-l)}\theta(l+1,n)\frac{S_{k-1;i}}{S_l} \qquad (25)\end{aligned}$$

for $i \geqslant l+1$, by Theorem 1. The desired inequality (24) then follows directly from Theorem 2. □

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Next, following Ivochkina [**3**], we estimate

$$S_{k-1;r}\lambda_r^2 = \lambda_r S_k - \lambda_r S_{k;r}$$
$$= \lambda_r S_k + \frac{1}{k}\Big\{\sum_{i\neq r}^{n} S_{k-1;i}\lambda_i^2 + (k+1)S_{k+1;r} - S_k S_{1;r}\Big\}$$
$$\leqslant \Big(\lambda_r - \frac{1}{k}S_{1;r}\Big)S_k + C\sum_{i\neq r}^{n} S_{k-1;i}\lambda_i^2 \tag{26}$$

by Theorem 2, where $C$ is a further constant depending on $k$ and $n$. Consequently we have

$$\frac{\partial S_k}{\partial \lambda_r}\lambda_r^2 \leqslant S_1 S_k + C\sum_{i\neq r}^{n} \frac{\partial S_k}{\partial \lambda_i}\lambda_i^2 \tag{27}$$

with $S_1$ replaceable by $\lambda_r$, when $k \geqslant 2$. For quotients, we now prove a stronger version of (27).

**Theorem 4.** *There exists a positive constant $C$ depending on $n$ and $k$ such that for $l \geqslant 1$, $\lambda \in \Gamma_k$*

$$\frac{\partial S_{k,l}}{\partial \lambda_r}\lambda_r^2 \leqslant C\sum_{i\neq r}^{n} \frac{\partial S_{k,l}}{\partial \lambda_i}\lambda_i^2. \tag{28}$$

Proof: Let us write

$$F(\lambda) = S_{k,l}(\lambda), \quad F_i(\lambda) = \frac{\partial F}{\partial \lambda_i}, \quad i = 1, \dots, n. \tag{29}$$

From (25) we have

$$F_i\lambda_i^2 = F\lambda_i^2\Big(\frac{S_{k-1;i}}{S_k} - \frac{S_{l-1;i}}{S_l}\Big)$$
$$= F\lambda_i\Big(\frac{S_{l;i}}{S_l} - \frac{S_{k;i}}{S_k}\Big) \tag{30}$$

by (4). To estimate the second term on the right hand side of (29) for $i = r$, we sum (29) over $i \neq r$, to obtain

$$\sum_{i\neq r} F_i\lambda_i^2 = \frac{F}{S_l}[(l+1)S_{l+1} - S_{l;r}\lambda_r] - (k+1)\frac{S_{k+1;r}}{S_l} - k\frac{S_{k;r}\lambda_r}{S_l}, \tag{31}$$

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so that by Theorem 3,

$$-\frac{S_{k;r}}{S_l}\lambda_r \leqslant \frac{1}{k}[1+(k+1)C]\sum_{i\neq r} F_i\lambda_i^2 + \frac{F}{kS_l}[S_{l;r}\lambda_r - (l+1)S_{l+1}] \tag{32}$$

where $C$ is the constant in (24).

Since $S_{l+1}(\lambda) > 0$, it remains to estimate the term

$$\frac{F}{S_l}S_{l;r}\lambda_r$$

on the right hand side of (31). First we observe from the proof of Theorem 2, that any term $Q$ in $S_{l;r}\lambda_r$ can be estimated by

$$|Q| \leqslant (n-l)^{l+1}\lambda_1\lambda_2\cdots\lambda_{l+1}, \tag{33}$$

so that by (19),

$$\frac{S_{l;r}\lambda_r}{S_l} \leqslant \binom{n-1}{l}(n-l)^{l+1}\lambda_{l+1}. \tag{34}$$

Similarly we can prove the estimate

$$S_k \leqslant \binom{n}{k}(n-k+1)^k\lambda_j S_{k-1;j} \tag{35}$$

for any $j \leqslant k$. For, using

$$S_k = S_{k;j} + \lambda_j S_{k-1;j},$$

we see that (35) is automatically true if $S_{k;j} \leqslant 0$. Otherwise we estimate as above

$$\begin{aligned} S_k &\leqslant \binom{n}{k}(n-k+1)^k\lambda_1\cdots\lambda_k \\ &\leqslant \binom{n}{k}(n-k+1)^k\lambda_j S_{k-1;j} \end{aligned}$$

by (19). Similarly to the above estimation, we also have, for $j \leqslant l+1$,

$$\begin{aligned} \lambda_{l+1}S_l &\leqslant \binom{n}{l}(n-l+1)^l\lambda_1\cdots\lambda_{l+1} \\ &\leqslant \binom{n}{l}(n-l+1)^l\lambda_j S_{l;j}. \end{aligned} \tag{36}$$

Combining (34), (35) and (36), we thus obtain for $j \leqslant l+1$,

$$\begin{aligned} \frac{F}{S_l}S_{l;r}\lambda_r &\leqslant C\frac{S_{l;j}S_{k-1;j}}{S_l^2}\lambda_j^2 \\ &\leqslant CF_j\lambda_j^2 \end{aligned} \tag{37}$$

where $C$ depends on $n, k, l$. By choosing $j \neq r$, we thus complete the proof. □

REMARK. Theorem 4 is a crucial inequality in our derivation of second derivative estimates for solutions of prescribed curvature quotient equations [5].

## References


[1] S. Caffarelli, L. Nirenberg and J. Spruck, 'Nonlinear second-order elliptic equations V. The Dirichlet problem for Weingarten hypersurfaces', *Comm. Pure Appl. Math.* **41** (1988), 47–70.

[2] N.M. Ivochkina, 'Solution of the Dirichlet problem for curvature equations of order m', (English translation), *Math USSR Sb.* **67** (1990), 317–339.

[3] N.M. Ivochkina, 'The Dirichlet problem for the equations of curvature of order m', (English translation), *Leningrad Math J.* **2** (1991), 631–654.

[4] N.J. Korevaar, 'A priori interior gradient bounds for elliptic Weingarten equations', *Ann. Inst. H. Poincaré, Anal. Non Linéaire* **4** (1987), 405–421.

[5] M. Lin and N.S. Trudinger, 'The Dirichlet problem for the prescribed curvature quotient equations', *Topological Methods in Nonlinear Analysis* **3** (1994).

[6] D.S. Mitrinovic, *Analytic inequalities* (Springer-Verlag, Berlin, Heidelberg, New York, 1970).

[7] N.S. Trudinger, 'The Dirichlet problem for the prescribed curvature equations', *Arch. Rational Mech. Anal.* **111** (1990), 153–179.



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