ANNIHILATOR AND COMPLEMENTED BANACH*-ALGEBRAS

B. J. TOMIUK and PAK-KEN WONG

(Received 18 August 1969)

Communicated by E. Strzelecki

1. Introduction

The study of complemented Banach*-algebras taken up in [1] was confined mainly to B^*-algebras. In the present paper we extend this study to (right) complemented Banach*-algebras in which $x^*x = 0$ implies $x = 0$. We show that if A is such an algebra then every closed two-sided ideal of A is a *-ideal. Using this fact we obtain a structure theorem for A which states that if A is semi-simple then A can be expressed as a topological direct sum of minimal closed two sided ideals each of which is a complemented Banach*-algebra. It follows that A is an A^*-algebra and is a dense subalgebra of a dual B^*-algebra $\mathfrak{A}$, which is determined uniquely up to *-isomorphism.

A Banach*-algebra A is said to have the weak (β_k) property if for every minimal left ideal I of A there exists a constant $k > 0$ such that $||x||^2 \leqq k||x^*x||$ for all $x \in I$. This concept is introduced in **5**, where we also show its relation to annihilator properties in Banach*-algebras. An A^*-algebra which is a dense two-sided ideal of a dual B^*-algebra has the weak (β_k) property. A semi-simple complemented Banach*-algebra with the weak (β_k) property is a dual A^*-algebra. In **6** we look at the weakly completely continuous A^*-algebras. Lemma 5.5 plays a prominent role in the development of **5** and **6**, as well as that of **7**. (In this context see [6] Lemmas 8 and 9.

In **7** we study dual A^*-algebras. We give several characterizations of duality for A^*-algebras, one of which is expressed in terms of (right) complementors. We show, in particular, that if A is a dense two-sided ideal of a B^*-algebra then A is dual if and only if it is complemented. In **8** we look at complementors induced by given complementors. More precisely, let A be an A^*-algebra which is a dense subalgebra of a B^*-algebra $\mathfrak{A}$ and let p be a complementor on $\mathfrak{A}$ and q a complementor on A. We find conditions on $\mathfrak{A}$, A and the complementors p and q such that: (a) The mapping $I \to \mathrm{cl}(I)^p \cap A$ on the closed right ideals I of A is a complementor on A. (b) The mapping $R \to \mathrm{cl}((R \cap A)^q)$ on the closed right ideals R of $\mathfrak{A}$ is a complementor on $\mathfrak{A}$.

In **9** we discuss an example of a complemented A^*-algebra.

https://doi.org/10.1017/S1446788700010594 Published online by Cambridge University Press

2. Preliminaries

Let A be a complex Banach algebra and let L_r be the set of all closed right ideals of A. Following [10], we shall say that A is a *right complemented Banach algebra* if there exists a mapping $p : R \to R^p$ of L_r into itself having the following properties:

(C_1) $R \cap R^p = (0)$ $(R \in L_r)$;

(C_2) $R + R^p = A$ $(R \in L_r)$;

(C_3) $(R^p)^p = R$ $(R \in L_r)$;

(C_4) if $R_1 \subseteq R_2$, then $R_2^p \subseteq R_1^p$ $(R_1, R_2 \in L_r)$.

The mapping p is called a *right complementor* on A. Analogously we define a *left complemented Banach algebra* and a *left complementor*. Thus a complex Banach algebra is left (right) complemented if and only if it has a left (right) complementor defined on it. A left and right complemented Banach algebra is called bicomplemented. We shall restrict our attention to right complemented Banach algebras. Therefore, unless mentioned otherwise, a complementor on a Banach algebra will always mean a right complementor and a complemented Banach algebra will always mean a right complemented Banach algebra. All Banach algebras and Banach spaces under consideration are over the complex field C.

For any set S in a Banach algebra A, let $l(S)$ and $r(S)$ denote the left and right annihilators of S respectively. A Banach algebra A is called an annihilator algebra if $l(A) = r(A) = (0)$, and if for every proper closed right ideal I and every proper closed left ideal J, $l(I) \neq (0)$ and $r(J) \neq (0)$. If, in addition, $r(l(I)) = I$ and $l(r(J)) = J$, then A is called a dual algebra.

A Banach algebra A is called simple if it is semi-simple and if (0) and A are the only closed two-sided ideals of A. An idempotent e in a Banach algebra A is said to be minimal if eAe is a division algebra. In case A is semi-simple, this is equivalent to saying that Ae (eA) is a minimal left (right) ideal of A.

A Banach algebra with an involution $x \to x^*$ is called a Banach*-algebra. A Banach*-algebra A is called a B^*-algebra if the norm and the involution satisfy the condition $\|x^*x\| = \|x\|^2$, $x \in A$. If A is a Banach*-algebra on which there is defined a second norm $|\cdot|$ which satisfies, in addition to the multiplicative condition $|xy| \leqq |x|\,|y|$, the B^*-algebra condition $|x^*x| = |x|^2$, then A is called an A^*-algebra. The norm $|\cdot|$ is called an auxiliary norm. Let A be an A^*-algebra. Then A is semi-simple, the involution in A is continuous with respect to the given norm $\|\cdot\|$ and the auxiliary norm $|\cdot|$ and $|\cdot| \leqq \beta\|\cdot\|$ for a real constant β (see [8] p. 187).

Let H be a Hilbert space with inner product (,). If x and y are elements of H, then $x \otimes y$ will denote the operator on H defined by the relation $(x \otimes y)(h) =$

$(h, y)x$ for all $h \in H$. Let $L(H)$ be the algebra of all continuous linear operators on H into itself with the usual operator bound norm. $LC(H)$ will denote the subalgebra of $L(H)$ consisting of all compact operators on H.

Let $\{A_\lambda : \lambda \in \Lambda\}$ be a family of Banach algebras A_λ, and let $(\sum A_\lambda)_0$ be the set of all functions f defined on Λ such that $f(\lambda) \in A_\lambda$ for each $\lambda \in \Lambda$ and such that, for arbitrary $\varepsilon > 0$, the set $\{\lambda : \|f(\lambda)\| \geqq \varepsilon\}$ is finite. It is easy to see that $(\sum A_\lambda)_0$ is closed under the usual operations of addition, multiplication and scalar multiplication for functions. $(\sum A_\lambda)_0$ is a Banach algebra under the norm $\|f\| = \sup\{\|f(\lambda)\| : \lambda \in \Lambda\}$. If each A_λ is a B^*-algebra, then $(\sum A_\lambda)_0$ is also a B^*-algebra under the norm $\|f\|$ and the involution $f \to f^*$ given by $(f^*)(\lambda) = f(\lambda)^{*\lambda}$, where $*\lambda$ is the involution on A_λ. $(\sum A_\lambda)_0$ is called the $B^*(\infty)$-sum of A_λ. If, in addition, A_λ are dual, then $(\sum A_\lambda)_0$ is dual ([8], Theorem (4.10.25)).

Let A be a dual B^*-algebra and $\{I_\lambda : \lambda \in \Lambda\}$ the family of all minimal closed two-sided ideals of A. Then A is isometrically *-isomorphic to $(\sum I_\lambda)_0$. Since each I_λ is isometrically *-isomorphic to $LC(H_\lambda)$, for some Hilbert space H_λ, we see that A is isometrically *-isomorphic to $(\sum LC(H_\lambda))_0$ (see [8] Chap. IV, § 10). A B^*-algebra is dual if and only if it is complemented ([1] Theorem 3.6). We shall often use, without explicitly mentioning, the following fact about dual B^*-algebras: If A is a dual B^*-algebra then the mapping $R \to l(R)^*$ on the set of all closed right ideals R of A is a complementor on A (see [10] p. 652).

Let X be a topological space and S a subset of X. Then $\mathrm{cl}(S)$ will denote the closure of S in X. The norm in a B^*-algebra will always be denoted by $|\cdot|$.

We shall need the following lemma:

Lemma 2.1. *Let A be a semi-simple Banach algebra with a dense socle. Then for every proper closed two-sided ideal I of A, $l(I) = r(I) \neq (0)$. Moreover, every closed left (right) ideal of the algebra I is also a closed left (right) ideal of A.*

Proof. If A is simple, the lemma is trivially true. So suppose A is not simple. Since the socle is dense in A, there exists a minimal idempotent e of A such that $e \notin I$. Let J be the closed two-sided ideal generated by e. By the proof of [2] Theorem 5, J is a minimal closed two-sided ideal of A. Since $e \notin I$, $I \cap J = (0)$ and so $J \subset l(I)$, which shows that $l(I) \neq (0)$. By the proof of [8] Lemma (2.8.10), we have that $l(I) = r(I)$ and that, if $R = \mathrm{cl}(I + l(I))$, then $l(R) = (0)$. Since every proper closed two-sided ideal of A has a non-zero annihilator, we must have $R = A$. The second part of the lemma now follows from the proof of [8] Lemma (2.8.11).

3. Annihilator complemented Banach algebras

In this section, as well as in the rest of the paper, a complemented Banach algebra will always mean a right complemented Banach algebra.

Let A be a complemented Banach algebra with a complementor p. We shall

https://doi.org/10.1017/S1446788700010594 Published online by Cambridge University Press

call an idempotent e in A a *p-projection* if $(eA)^p = \{x - ex : x \in A\}$. If ,moreover, e is a minimal idempotent, we shall say that e is a *minimal p-projection.* (In [10], a p-projection is called a left projection).

LEMMA 3.1. *Let A be a semi-simple annihilator complemented Banach algebra with a complementor p. Then every non-zero right ideal I contains a minimal p-projection. Moreover, if I is a closed non-zero right ideal and $\{e_\alpha\}$ is the family of minimal p-projections in I, then $I = \mathrm{cl}(\sum_\alpha e_\alpha A)$.*

PROOF. Let R be a minimal right ideal contained in I. Since R^p is a maximal closed right ideal, by [2] Theorem 1, R^p is modular. The existence of a minimal p-projection in I now follows from [10] Lemma 2. To prove the second part of the lemma, suppose that $I \neq \mathrm{cl}(\sum_\alpha e_\alpha A)$; let $J = \mathrm{cl}(\sum_\alpha e_\alpha A)$. Then there exists $x \in I$ such that $x \notin J$. Write $x = x_1 + x_2$ with $x_1 \in J$ and $x_2 \in J^p$. Then $0 \neq x_2 = x - x_1 \in I$ and so $I \cap J^p \neq (0)$. Hence there exists a minimal p-projection e in $I \cap J^p \subset I$ which does not belong to J; a contradiction. Therefore $I = J$.

Combining Lemma 3.1 and [1] Lemma 2.1, we obtain the following result:

COROLLARY 3.2. *Let A be an annihilator semi-simple complemented Banach algebra. Then every closed right ideal of A is the intersection of maximal modular right ideals containing it.*

THEOREM 3.3: *Let A be a semi-simple complemented Banach algebra with a complementor p. Then the following statements are equivalent*:

(i) *A is an annihilator algebra.*

(ii) *Every non-zero right ideal contains a minimal p-projection.*

(iii) *Every maximal closed right ideal is modular.*

(iv) *Every maximal closed right ideal has a non-zero left annihilator.*

PROOF. (i) ⇒ (ii). This follows from Lemma 3.1.

(ii) ⇒ (iii). Suppose (ii) holds and let M be a maximal closed right ideal of A. Then M^p is a minimal right ideal and hence $M^p = eA$, where e is a minimal p-projection.

(iii) ⇒ (iv). Let M be a maximal closed right ideal. If M is modular, [10] Lemma 2 shows that $M = \{x - ex : x \in A\}$, for some idempotent e, and hence $l(M) \neq (0)$.

(iv) ⇒ (i). Let I be a proper closed right ideal and R a minimal right ideal contained in I ([10] Corollary Theorem 1). Then R^p is a maximal closed right ideal and $I \subset R^p$. Hence if $l(R^p) \neq (0)$, then $l(I) \neq (0)$ and so, by [10] Theorem 8, A is an annihilator algebra.

THEOREM 3.4. *Let A be an annihilator semi-simple complemented Banach algebra. Then every closed two-sided ideal of A is an annihilator semi-simple complemented Banach algebra.*

https://doi.org/10.1017/S1446788700010594 Published online by Cambridge University Press

Proof. Let M be a closed two-sided ideal of A. Since, by [10] Lemma 1, $M^p = l(M) = r(M)$, every closed left (right) ideal of M is a closed left (right) ideal of A; so that M is semi-simple. Now $p_M : I \to I^{p_M} = I^p \cap M$ is a complementor on the closed right ideals of M. Hence if I is a maximal closed right ideal of of M, then I^{p_M} is a minimal right ideal of M and also of A. Thus $(I^{p_M})^p$ is a maximal closed right ideal of A and therefore modular. But, by [10] Lemma 2,

$$(I^{p_M})^p = \{x - ex : x \in A\},$$

where e is an idempotent in I^{p_M}. Hence, since $I = (I^{p_M})^p \cap M$,

$$I = \{x - ex : x \in M\},$$

i.e., I is modular. Therefore, by Theorem 3.3, M is an annihilator algebra.

4. Complemented Banach*-algebras

Throughout this section, p will denote the given complementor on the complemented Banach*-algebra A.

Lemma 4.1. *Let A be a semi-simple complemented Banach*-algebra. Then the involution in A is continuous and hence A is bicomplemented.*

Proof. By [10] Lemma 5, the socle of A is dense in A and therefore, by [8] Corollary (2.5.8), A has a unique norm topology. Hence the involution is continuous and consequently the mapping

$$q : J \to J^q = ((J^*)^p)^*$$

on the closed left ideals J of A is a left complementor on A.

Lemma 4.2. *Let A be a complemented Banach*-algebra in which $x^*x = 0$ implies $x = 0$. Then every closed two-sided ideal I of A is a complemented Banach*-algebra.*

Proof. Since $x^*x = 0$ implies $x = 0$, we have $r(A) = (0)$ and therefore, by [10] Lemma 1, $l(I) = r(I) = I^p$ which also implies that I is a complemented algebra. Now let $x \in I$ and $y \in I^p$. Then

$$(x^*y)^*(x^*y) = y^*xx^*y \in I \cap I^p = (0),$$

so that $(x^*y)^*(x^*y) = 0$. Thus $x^*y = 0$ and hence $x^* \in l(I^p) = I^{pp} = I$, for all $x \in I$. Therefore $I^* = I$.

Theorem 4.3 (Structure Theorem). *Let A be a semi-simple right complemented Banach*-algebra in which $x^*x = 0$ implies $x = 0$. Then A is the topological direct sum of its minimal closed two-sided ideals, each of which is a simple right complemented Banach*-algebra.*

https://doi.org/10.1017/S1446788700010594 Published online by Cambridge University Press

PROOF. Follows from Lemma 4.2 and [10] Theorem 4.

LEMMA 4.4. *Let A be a simple complemented Banach*-algebra in which $x^*x = 0$ implies $x = 0$. Then there exists a faithful *-representation of A on a Hilbert space H such that the image of A' of A in $L(H)$ is a dense subalgebra of $LC(H)$; A is an A^*-algebra.*

PROOF. Let I be a minimal left ideal of A. Since $I = Ae$, where e is a self-adjoint minimal idempotent, the scalar-valued function (x, y) on I given by $(x, y)e = y^*x$, $x, y \in I$, is an inner product on I. Let H be the completion of I in the norm $|x|_0 = (x, x)^{\frac{1}{2}}$. The left regular representation $x \to T_x$ of A on I is faithful and is a *-representation with respect to this inner product and, for each $x \in A$, T_x is a bounded operator relative to the norm $|\cdot|_0$. Therefore A has a faithful *-representation on H whose image A' contains all operators of the form $g \otimes h$, $g, h \in I$ ([8] theorem (4.10.5)). Since I is dense in H, $\mathrm{cl}(A') \supset LC(H)$. Now the socle $\mathfrak{S}$ of A is dense in A and every element of $\mathfrak{S}$ gives rise to an operator of finite rank on I ([2] Lemma 5) and hence on H. Therefore $A' \subset LC(H)$ and so $\mathrm{cl}(A') = LC(H)$.

THEOREM 4.5. *Let A be a semi-simple complemented Banach*-algebra in which $x^*x = 0$ implies $x = 0$. Then A is an A^*-algebra which is a dense subalgebra of a dual B^*-algebra $\mathfrak{A}$; A is uniquely determined up to *-isomorphism.*

PROOF. Let $\{I_\lambda : \lambda \in \Lambda\}$ be the family of all minimal closed two-sided ideals of A. By Lemma 4.4, each I_λ may be identified with a dense subalgebra of $LC(H_\lambda)$, for some Hilbert space H_λ. Let $\mathfrak{A} = (\sum LC(H_\lambda))_0$. By Theorem 4.3, A can be identified as a subalgebra of $\mathfrak{A}$ so that A is an A^*-algebra. Considering A as a subalgebra of $\mathfrak{A}$, we have $LC(H_\lambda) \subset \mathrm{cl}(A)$ for all λ, and so $\mathfrak{A} \subset \mathrm{cl}(A)$, i.e., A is dense in $\mathfrak{A}$. Since the socle is dense in A, by [6] Theorem 3, $\mathfrak{A}$ is uniquely determined up to *-isomorphism.

THEOREM 4.6. *Let A be a complemented Banach*-algebra in which $x^*x = 0$ implies $x = 0$. Then the radical $\mathscr{R}$ and the *-radical $\mathscr{R}^{(*)}$ ([**8**, p. 210]) of A coincide.*

PROOF. By [8] Theorem (4.4.10), $\mathscr{R}^{(*)} \supset \mathscr{R}$. We may assume $\mathscr{R} \neq A$; for if $\mathscr{R} = A$, then $\mathscr{R} = \mathscr{R}^{(*)} = A$. By [10, Theorem 2] and Lemma 4.2, $\mathscr{R}^p$ is a semi-simple right complemented Banach*-algebra; clearly, $x^*x = 0$ implies $x = 0$ for all $x \in \mathscr{R}^p$. Hence, by Theorem 4.5, $\mathscr{R}^p$ is an A^*-algebra. It is easy to show that the natural homomorphism $x \to x'$ (where $x' = x + \mathscr{R}$) is a *-isomorphism of $\mathscr{R}^p$ onto $A/\mathscr{R}$. Therefore $A/\mathscr{R}$ is an A^*-algebra and, by [8] Corollary (4.8.12), $A/\mathscr{R}$ is *-semi-simple. Hence $\mathscr{R}^{(*)}/\mathscr{R} = (0)$ and so $\mathscr{R}^{(*)} = \mathscr{R}$.

5. Annihilator and weak (β_k) properties in Banach*-algebras

If A is a Banach*-algebra in which $x^*x = 0$ implies $x = 0$, then, by [8] Lemma (4.10.1), every minimal left ideal I of A is of the form $I = Ae$, where e

is a minimal self-adjoint idempotent. A similar result holds for minimal right ideals. It follows from the proof of [8] Theorem (4.10.3) that the scalar-valued function (x, y) defined by $(x, y)e = y^*x$ $(x, y \in I)$ is an inner product on I. Hence $|x|_0 = (x, x)^{\frac{1}{2}}$ is a norm on I. Since this inner product will be used on several occasions in the rest of the paper, to avoid repeating ourselves in the future we will adopt the following notation: the bracket $(\cdot)$ will always denote the inner product on the minimal left ideal I defined by $(x, y)e = y^*x$ $(x, y \in I)$ and $|\cdot|_0$ the inner product norm on I given by $|x|_0 = (x, x)^{\frac{1}{2}}$, for all $x \in I$.

It is easy to see that if A is a B^*-algebra, then the norm $|\cdot|_0$ coincides with the given norm on every minimal left ideal of A.

DEFINITION. A Banach*-algebra A is said to have the *weak* (β_k) *property* if, for every minimal left ideal I of A, there exists a constant k (depending on I) such that $||x||^2 \leqq k||x^*x||$ for all $x \in I$.

REMARK. A has the weak (β_k) property if and only if every minimal left ideal I is complete under the inner product norm $|\cdot|_0$, or equivalently, the norms $|\cdot|_0$ and $||\cdot||$ are equivalent on every minimal left ideal I (see [8] Theorem (4.10.6) and its proof).

THEOREM 5.1. *Let A be an A^*-algebra which is a dense subalgebra of a B^*-algebra $\mathfrak{A}$. Then A has the weak (β_k) property if and only if every minimal left (right) ideal of A is also a minimal left (right) ideal of $\mathfrak{A}$.*

PROOF. Suppose that every minimal left ideal of A is also a minimal left ideal of $\mathfrak{A}$, and let I be a minimal left ideal of A. Then I is complete in the inner product norm $|\cdot|_0$. Hence by the above Remark, A has the weak (β_k) property. Conversely suppose A has the weak (β_k) property and let I be a minimal left ideal of A. Then $I = Ae$ with e a self-adjoint idempotent in A. Since eAe is one-dimensional and dense in $e\mathfrak{A}e$, e is a minimal idempotent of $\mathfrak{A}$. But $|\cdot|_0$ and $|\cdot|$ are equal on $\mathfrak{A}e$ and I is complete under $|\cdot|_0$. Since I is dense in $\mathfrak{A}e$, we have $Ae = \mathfrak{A}e$. The same argument holds for minimal right ideals.

COROLLARY 5.2. *Let A be an A^*-algebra which is a dense two-sided ideal of a dual B^*-algebra $\mathfrak{A}$. Then A has the weak (β_k) property.*

PROOF. This follows from Theorem 5.1, since in this case A and $\mathfrak{A}$ have the same minimal left (right) ideals.

LEMMA 5.3. *Let A be a Banach*-algebra with socle $\mathfrak{S}$ such that $a\mathfrak{S} = (0)$ implies $a = 0$. If A has the weak (β_k) property, then $x^*x = 0$ implies $x = 0$.*

PROOF. By [8] Corollary (2.5.8), A has a unique norm topology and hence the involution is continuous. Let $x \in A$ be such that $x^*x = 0$, and let I be any minimal left ideal of A. Then, for each $a \in I$, $(xa)^*(xa) = a^*x^*xa = 0$. Hence by the weak (β_k) property of A, $||xa||^2 = 0$ which gives $xa = 0$ and therefore $xI = (0)$. As I

https://doi.org/10.1017/S1446788700010594 Published online by Cambridge University Press

is an arbitrary minimal left ideal of A, it follows that $x\mathfrak{S} = (0)$ and consequently $x = 0$.

THEOREM 5.4. *Let A be a semi-simple Banach*-algebra. Then the following statements are equivalent*:

(i) *A is an annihilator algebra in which $x^*x = 0$ implies $x = 0$.*

(ii) *A has the waak (β_k) property and the socle $\mathfrak{S}$ of A is dense in A.*

PROOF. (i) $\Rightarrow$ (ii). Suppose (i) holds. By [2] Theorem 4, the socle $\mathfrak{S}$ of A is dense in A and therefore the involution is continuous. Let I be a minimal left ideal of A; $I = Ae$, where e is a self-adjoint idempotent. Let J be the closed two-sided ideal generated by I. Then J is a minimal closed two-sided ideal of A ([2] Theorem 5) with $J^* = J$ and therefore a simple annihilator Banach*-algebra; moreover, I is a minimal left ideal of J. Applying the proof of [8] Theorem (4.10.16) to J and I, we see that I is complete under the inner product norm $|\cdot|_0$ and so A has the weak (β_k) property.

(ii) $\Rightarrow$ (i). Suppose (ii) holds. By Lemma 5.3, $x^*x = 0$ implies $x = 0$. Assume first that A is simple, and let I be a minimal left ideal of A. Since A has the weak (β_k) property, I is a Hilbert space under the inner product $(\cdot)$. Therefore the image A' of A by the left regular representation $x \to T_x$ of A on I contains the set F of all operators of finite rank on I (see the proof of Lemma 4.4). But the elements of the socle give rise to operators of finite rank on I and, since $A = \mathrm{cl}(\mathfrak{S})$, F is dense in A' relative to the norm $\|\cdot\|$. Hence, by [8] Theorem (2.8.23), A' is an annihilator algebra and therefore A is an annihilator algebra, since the representation is faithful.

Now suppose that A is not simple. Let I be a minimal left ideal of A and J the closed two-sided ideal generated by I. Then J is a minimal closed two-sided ideal of A (see the proof of Lemma 2.1) with $J^* = J$. Since $A = \mathrm{cl}(\mathfrak{S})$, Lemma 2.1 shows that I is a minimal left ideal of J and since J is simple, J has a dense socle and so is an annihilator algebra by the argument above. Thus, by [8] Theorem (2.8.29), A is an annihilator algebra.

LEMMA 5.5. *Let A be an annihilator A^*-algebra, I a closed right ideal of A and $\mathfrak{A}$ the completion of A in an auxiliary norm $|\cdot|$. Then the following statements are true*:

(i) *$\mathfrak{A}$ is a dual B^*-algebra which is uniquely determined up to *-isomorphism.*

(ii) *A and $\mathfrak{A}$ have the same socle.*

(iii) *If $\mathfrak{S}$ is the socle of A, then $\mathrm{cl}(I)\mathfrak{S} \subset I$.*

(iv) $l(\mathrm{cl}(I)) = \mathrm{cl}(l_A(I))$.

(v) $\mathrm{cl}(I) \cap A = r_A(l_A(I))$.

(*Where* $\mathrm{cl}(S)$ (*resp.* $\mathrm{cl}_A(S)$) *denotes the closure of the set S in $\mathfrak{A}$* (*resp.* A) *and* $l(S)$ (*resp.* $l_A(S)$) *the left annihilator of S in $\mathfrak{A}$* (*resp.* A).)

PROOF. (i). Since $\mathfrak{S}$ is dense in A, A has a unique auxiliary norm and therefore $\mathfrak{A}$ is uniquely determined up to *-isomorphism (see the proof of Theorem 4.5). Since A has the weak (β_k) property, $\mathfrak{S}$ is contained in the socle of $\mathfrak{A}$ by Theorem 5.1. Thus the socle of $\mathfrak{A}$ is dense in $\mathfrak{A}$ and so $\mathfrak{A}$ is dual by [5] Theorem 2.1.

(ii) By the weak (β_k) property, $\mathfrak{S}$ is a two-sided ideal of $\mathfrak{A}$. Let f be a minimal idempotent in $\mathfrak{A}$. Then clearly $I = f\mathfrak{A} \cap S$ is a non-zero right ideal of $\mathfrak{A}$ contained in A. As $f\mathfrak{A}$ is a minimal right ideal of $\mathfrak{A}$, $f\mathfrak{A} = I \subset A$ and so $f \in A$. This proves (ii).

(iii). It clearly suffices to show that $xye \in I$ for $x \in \mathrm{cl}(I)$, $y \in A$ and e a minimal idempotent. Now $Ae = \mathfrak{A}e$ and the two norms $|\cdot|$ and $||\cdot||$ are equivalent on Ae (by the weak (β_k) property in A). Hence

$$||xye|| \leqq c|x|\,||ye||,$$

for some constant c. Let $\{x_n\}$ be a sequence in I such that $|x_n - x| \to 0$ as $n \to \infty$. Since

$$||x_n ye - xye|| \leqq c|x_n - x|\,||ye||,$$

$||x_n ye - xye|| \to 0$ as $n \to \infty$, which shows that $xye \in I$. Hence $\mathrm{cl}(I)\mathfrak{S} \subset I$.

(iv). Let $\{e_\beta\}$ be the set of all minimal idempotents in $l(\mathrm{cl}(I))$;

$$e_\beta \in l(\mathrm{cl}(I)) \cap A = l(I) \cap A = l_A(I),$$

for all β. Now $\mathrm{cl}(\sum_\beta \mathfrak{A}e_\beta) = l(\mathrm{cl}(I))$ (Lemma 3.1) and so

$$\mathrm{cl}(l_A)I)) \supset \mathrm{cl}(\textstyle\sum_\beta Ae_\beta) = \mathrm{cl}(\sum_\beta \mathfrak{A}e_\beta) = l(\mathrm{cl}(I)).$$

But $l_A(I) \subset l(\mathrm{cl}(I))$. Hence $\mathrm{cl}(l_A(I)) = l(\mathrm{cl}(I))$.

(v) By the duality of $\mathfrak{A}$ and (iv), we have

$$r_A(l_A(I)) = r(l_A(I)) \cap A = r(l(\mathrm{cl}(I))) \cap A = \mathrm{cl}(I) \cap A.$$

This completes the proof.

From Theorem 4.5 and Lemma 5.5 we see that if A is either a complemented or an annihilator A^*-algebra, then A can be imbedded as a dense subalgebra in a unique (up to *-isomorphism) B^*-algebra $\mathfrak{A}$. From now on we shall refer to $\mathfrak{A}$ as the *completion* of A.

THEOREM 5.6. *Let A be a semi-simple complemented Banach*-algebra with the weak (β_k) property. Then A is a dual A^*-algebra.*

PROOF. We use the notation of Lemma 5.5. Since the socle is dense, Theorems 4.5 and 5.4 show that A is an annihilator A^*-algebra. Let $\mathfrak{A}$ be the completion of A and let I be a closed right ideal of A. We claim that $\mathrm{cl}(I) \cap A = I$. Let $J = \mathrm{cl}(I) \cap A$. Then J is a closed right ideal of A, and clearly $I \subset J$. Let p be the given complementor on A and let $\{e_\alpha\}$ be the family of all minimal p-projections in I. If $I \neq J$, then $I^p \cap J \neq (0)$ and so, by Lemma 3.1, contains a minimal p-projection f. Since $e_\alpha \in I$ and $f \in I^p$, we have $e_\alpha f = fe_\alpha = 0$ for all α and hence, since $\mathrm{cl}(I) = \mathrm{cl}(\sum_\alpha e_\alpha A)$, it follows that $f\,\mathrm{cl}(I) = (0)$. But this is a contradiction since $f \in \mathrm{cl}(I)$ and $f^2 = f \neq 0$. Hence $J = I$ and consequently, by Lemma 5.5, $I = l_A(r_A(I))$. Applying now the continuity of the involution, we obtain that A is dual.

COROLLARY 5.7. *An annihilator complemented A^*-algebra is dual.*

PROOF. Follows from Theorems 5.4 and 5.6.

From Theorem 3.3 and Corollary 5.7, we have the following result:

THEOREM 5.8. *Let A be a complemented A^*-algebra with a complementor p. Then the following statements are equivalent:*

(i) *A is dual.*

(ii) *Every non-zero right ideal contains a minimal p-projection.*

(iii) *Every maximal closed right ideal is modular.*

(iv) *Every maximal closed right ideal has non-zero left annihilator.*

DEFINITION. A Banach algebra A is said to be *completely continuous* (c.c.) if the left- and right-multiplication operators of every element in A are completely continuous on A.

THEOREM 5.9. *A complemented c.c. A^*-algebra is dual.*

PROOF. By Theorem 4.3, A is the direct topological sum of all its minimal closed two-sided ideals I_λ, each of which is a simple c.c. complemented A^*-algebra. Since each I_λ is finite dimensional, it is a full matrix algebra and hence dual. Therefore, by [8] Theorem (2.8.9), A is an annihilator algebra and so, by Corollary 5.7, A is dual.

6. Weakly completely continuous A^*-algebras

DEFINITION. A Banach algebra is said to be weakly completely continuous (w.c.c.) if the left- and right-multiplication operators of every element in A are weakly completely continuous on A.

THEOREM 6.1. *An annihilator A^*-algebra A is w.c.c.*

PROOF. Let $\mathfrak{A}$ be the completion of A. $\mathfrak{A}$ is dual and hence w.c.c. by [6] Theorem 6. Let e be a minimal idempotent of A. From Lemma 5.5 we have $eA = \mathfrak{A}e$ and from its proof that $\|ex\| \leqq c\|e\|\,|x|$ for all $x \in \mathfrak{A}$ (see the proof of (iii)). Let $y \in A$ and let $\{y_n\}$ be any bounded sequence in A. As $\mathfrak{A}$ is w.c.c. and $\{y_n\}$ is bounded in $|\cdot|$, there exists a subsequence $\{y_{n_k}\}$ such that $\{yy_{n_k}\}$ converges weakly

https://doi.org/10.1017/S1446788700010594 Published online by Cambridge University Press

to an element $z \in \mathfrak{A}$. For each continuous linear functional f on A let g be the linear functional on $\mathfrak{A}$ given by $g(x) = f(ex)$ $(x \in \mathfrak{A})$. Since

$$|g(x)| = |f(ex)| \leqq ||f|| \, ||ex|| \leqq c||f|| \; ||e|| \, |x| \; (x \in \mathfrak{A}),$$

where $||f||$ denotes the norm of f with respect to $|| \cdot ||$, it follows that g is continuous on $\mathfrak{A}$. Now $ez \in A$ and

$$f(eyy_{n_k} - ez) = g(yy_{n_k} - z) \to 0 \text{ as } n \to \infty,$$

and so ey is a w.c.c. element of A. This shows that every element of the socle $\mathfrak{S}$ of A is w.c.c. Since $\mathfrak{S}$ is dense in A and the set of all w.c.c. elements is closed in A, A is w.c.c.

THEOREM 6.2. *Let A be an A^*-algebra which is a dense two-sided ideal of a B^*-algebra $\mathfrak{A}$. Then A is an annihilator algebra if and only A is w.c.c. and A^2 is dense in A.*

PROOF. If A is an annihilator algebra, Theorem 6.1 shows that A is w.c.c., and since A^2 contains the socle of A, A^2 is dense in A. Conversely, suppose that A is w.c.c. and A^2 is dense in A. Then, by [6] Lemma 9, $\mathfrak{A}$ is w.c.c. (therefore dual) and hence, by Corollary 5.2, A has the weak (β_k) property. Let $\mathfrak{S}$ be the socle of A and let $\{e_\alpha\}$ be a maximal orthogonal family of minimal self-adjoint idempotents in A. Then, for all $x, y \in A$, we have $xy = \sum e_\alpha xy$, the summation being taken relative to the norm $|| \cdot ||$ (see the proof of [6] Theorem 16). Thus (in the notation of Lemma 5.5) we have that $xy \in \text{cl}_A(\mathfrak{S})$, which shows that $\text{cl}_A(\mathfrak{S}) = \text{cl}_A(A^2) = A$. Theorem 5.4 now completes the proof.

7. Dual A^*-algebras

In this section we shall give several characterizations of duality in A^*-algebras.

THEOREM 7.1. *Let A be an annihilator A^*-algebra. Then the following statements are equivalent*:

(i) *A is dual.*

(ii) *x belongs to the closure of xA for every x in A.*

(iii) *For every closed right ideal I of A and $x \in A$, $xx^* \in I$ implies $x \in I$.*

(iv) *Every closed right ideal I of A is the intersection of maximal closed right ideals containing it.*

PROOF. We use the notation of Lemma 5.5. Let $\mathfrak{A}$ be the completion of A and $\mathfrak{S}$ the socle of A; $\mathfrak{A}$ is dual and $\text{cl}_A(x\mathfrak{S}) = \text{cl}_A(xA)$ for all $x \in A$. In the ensuing arguments let I be a closed right ideal of A and $R = \text{cl}(I)$.

(i) $\Rightarrow$ (ii). This is [8] Corollary (2.8.3).

https://doi.org/10.1017/S1446788700010594 Published online by Cambridge University Press

(ii) ⇒ (iii). Suppose $xx^* \in I$. Then $xx^* \in R$ and therefore, since R is a closed right ideal of $\mathfrak{A}$, [8] Corollary (4.9.3) implies that $x \in R \cap A$. Hence, if $x \in \mathrm{cl}_A(xA)$, then $x \in \mathrm{cl}_A(R\mathfrak{S}) \subset I$ by Lemma 5.5 (ii), whence (iii).

(iii) ⇒ (iv). Suppose (iii) holds and let $x \in R \cap A$. Then clearly $x \in \mathrm{cl}_A(xA)$ and so $x \in I$ by the argument above. Hence $I = R \cap A$. Now, by [3] Theorem (2.9.5) (iii), $R = \bigcap_\alpha \mathfrak{M}_\alpha$, where $\{\mathfrak{M}_\alpha\}$ is the family of all maximal closed right ideals of A containing R. Therefore $I = \bigcap_\alpha(\mathfrak{M}_\alpha \cap A)$. [2], Theorem 1 and Lemma 5.5 (ii) show that each $M_\alpha = \mathfrak{M}_\alpha \cap A$ is a maximal closed right ideal of A, whence (iv).

(iv) ⇒ (i). Suppose (iv) holds. Since every maximal closed right ideal M of A is of the form $M = \{x - ex : x \in A\}$, where e is a minimal idempotent, $\mathrm{cl}(M)$ is a maximal closed right ideal of $\mathfrak{A}$ and clearly $\mathrm{cl}(M) \cap A = M$. Hence if $\{M_\alpha\}$ is the family of all closed right ideals of A containing I and $\mathfrak{M}_\alpha = \mathrm{cl}(M_\alpha)$ for each α, then $R = \bigcap_\alpha \mathfrak{M}_\alpha$ and $R \cap A = \bigcap_\alpha(\mathfrak{M}_\alpha \cap A) = \bigcap_\alpha M_\alpha = I$. Therefore, by Lemma 5.5 (v) and the continuity of the involution, A is dual.

THEOREM 7.2. *Let A be an A^*-algebra which is a dense two-sided ideal of a B^*-algebra $\mathfrak{A}$. Then A is dual if and only if every maximal commutative *-subalgebra of A is dual.*

PROOF. If A is dual then, by [6] Theorem 19, every maximal commutative *-subalgebra of A is also dual. Conversely suppose that every maximal commutative *-subalgebra of A is dual. Let $\mathfrak{S}$ be the socle of A (and hence of $\mathfrak{A}$). Let $x \in A$ and write $x = x_1 + ix_2$, where x_1 and x_2 are hermitian elements of A and let B_1 and B_2 be maximal commutative *-subalgebras containing x_1, x_2 respectively. Since B_1, B_2 have dense socles, it follows that x_1 and x_2 belong to $\mathrm{cl}_A(\mathfrak{S})$. Hence $x \in \mathrm{cl}_A(\mathfrak{S})$ and so $\mathrm{cl}_A(\mathfrak{S}) = A$. It follows now that $\mathfrak{S}$ is dense in $\mathfrak{A}$ and consequently $\mathfrak{A}$ is dual by [5] Theorem 2.1. Therefore by Corollary 5.2 and Theorem 5.4, A is an annihilator algebra. Since B_i is dual, $x_i \in \mathrm{cl}_A(x_iB_i) \subset \mathrm{cl}_A(x_iA)$ $(i = 1, 2)$. Let $\{e_\alpha\}$ be a maximal orthogonal family of minimal self-adjoint idempotents in A. By the proof of [6] Theorem 16, $x_i = \sum_\alpha e_\alpha x_i$ $(i = 1, 2)$ in the norm $\|\cdot\|$ and hence $x = \sum_\alpha e_\alpha x$ in the norm $\|\cdot\|$. Therefore $x \in \mathrm{cl}_A(xA)$ and so, by Theorem 7.1, A is dual. This completes the proof.

THEOREM 7.3. *Let A be an A^*-algebra which is a dense two-sided ideal of a B^*-algebra $\mathfrak{A}$. Then A is dual if and only if it is complemented.*

PROOF. We use the notation of Lemma 5.5. Suppose A is complemented. By Theorem 4.5, $\mathfrak{A}$ is dual and therefore, by Corollary 5.2, A has the weak (β_k) property. Theorem 5.6 now shows that A is dual. Conversely, suppose A is dual. Let I be a closed right ideal of A and let $R = \mathrm{cl}(I)$; R is a closed right ideal of $\mathfrak{A}$. Let $\{e_\alpha\}$ be a maximal orthogonal family of minimal self-adjoint idempotents contained in R. By Lemma 5.5, $\{e_\alpha\} \subset R \cap A = I$. Now $\mathfrak{A} = R + l(R)^*$, so that

$x = y+z$ with $y \in R$ and $z \in l(R)^*$, for every $x \in A$. Hence $e_\alpha x = e_\alpha y$ for all e_α and so, by [6] Lemma 6, $y = \sum e_\alpha y = \sum e_\alpha x$, where the summations are taken in the norm $|\cdot|$. Since, by [6] Theorem 16, $\sum e_\alpha x$ is also summable in $\|\cdot\|$, $y \in A \cap R = I$. Hence $z \in A \cap l(R)^* = l_A(I)^*$. Thus $A = I + l_A(I)^*$. It is easy to see that the mapping $I \to l_A(I)^*$ also has properties (C_1), (C_3) and (C_4). Hence A is complemented.

We shall need the following result in **8**.

THEOREM 7.4. *Every complemented A^*-algebra A which is a dense two-sided ideal of $LC(H)$ is a two-sided ideal of $L(H)$.*

PROOF. By Theorem 7.3, A is dual. Let $x \in A$, $y \in L(H)$ and let $\{e_\alpha\}$ be a maximal orthogonal family of minimal selfadjoint idempotents in A. By [6] Theorem 16, $\sum_\alpha e_\alpha x$ is summable to x in the norm $\|\cdot\|$ and hence there is only a countable number of e_α for which $e_\alpha x \neq 0$, say $e_{\alpha_1}, e_{\alpha_2}, \cdots$. Clearly $ye_{\alpha_i} \in A$ $(i = 1, 2, \cdots)$. For any two positive integers m, n $(m \leqq n)$, [6] Lemma 4 shows that

$$\Big\|\sum_{i=1}^{n} ye_{\alpha_i} x - \sum_{i=1}^{m} ye_{\alpha_i} x\Big\| = \Big\|\Big(y \sum_{i=m+1}^{n} e_{\alpha_i}\Big)\Big(\sum_{i=m+1}^{n} e_{\alpha_i} x\Big)\Big\|$$
$$\leqq k\Big|y \sum_{i=m+1}^{n} e_{\alpha_i}\Big| \, \Big\|\sum_{i=m+1}^{n} e_{\alpha_i} x\Big\| \leqq k|y| \, \Big\|\sum_{i=m+1}^{n} e_{\alpha_i} x\Big\|,$$

where k is a constant. Therefore $\{\sum_{i=1}^{n} ye_{\alpha_i} x\}$ is a Cauchy sequence in A and so there exists an element $z \in A$ such that $z = \sum_{i=1}^{\infty} ye_{\alpha_i} x$. Since $\sum_{i=1}^{\infty} e_{\alpha_i} x$ also converges to x in the norm $|\cdot|$, we have $yx = \sum_{i=1}^{\infty} ye_{\alpha_i} x$. Hence $yx = z \in A$. Similarly we can show that $xy \in A$, and this completes the proof.

8. Induced complementors

Throughout this section we shall use the notation introduced in Lemma 5.5.

Let A be an A^*-algebra which is a dense subalgebra of a B^*-algebra $\mathfrak{A}$. Let p be a complementor on $\mathfrak{A}$ and q a complementor on A. In this section we are going to give conditions on A, $\mathfrak{A}$ and the complementors p and q such that: (a) The mapping $\boldsymbol{q} : I \to \mathrm{cl}(I)^p \cap A$ on the closed right ideals I of A is a complementor on A. (b) The mapping $\boldsymbol{p} : R \to \mathrm{cl}((R \cap A)^q)$ on the closed right ideals R of $\mathfrak{A}$ is a complementor on $\mathfrak{A}$.

We shall say that the complementor $\boldsymbol{q}$ is induced on A by p and the complementor $\boldsymbol{p}$ is induced on $\mathfrak{A}$ by q.

LEMMA 8.1. *Let A be a dual A^*-algebra which is a dense two-sided ideal of the B^*-algebra $LC(H)$. Then, for every complementor p on $LC(H)$, the mapping $\boldsymbol{q} : I \to \mathrm{cl}(I)^p \cap A$ on the closed right ideals I of A is a complementor on A.*

PROOF. Let p be a complementor on $LC(H)$. If the dimension of H is finite,

https://doi.org/10.1017/S1446788700010594 Published online by Cambridge University Press

then $A = LC(H)$ and therefore $\boldsymbol{q} = p$, so that $\boldsymbol{q}$ is a complementor on A. Now suppose the dimension of H is infinite. Then, by [1] Theorem 6.8, p is continuous and hence, by [1] Theorem 6.11, there exists an involution $*'$ on $LC(H)$ such that $R^p = l(R)^{*'}$, for every closed right ideal R of $LC(H)$. This means, by [1] Corollary 6.14, that there exists a positive operator $Q \in L(H)$ with continuous inverse Q^{-1} such that $a^{*'} = Q^{-1}a^*Q$ for all $a \in LC(H)$. Now, from Theorem 7.4 we know that A is a two-sided ideal of $L(H)$. Hence $a^{*'} \in A$ for all $a \in A$ and therefore A is an A^*-algebra under the involution $*'$ (and an auxiliary norm $|\cdot|'$ equivalent to $|\cdot|$). Since A is dual, $I \to l_A(I)^{*'}$ is a complementor on A (see the proof of Theorem 7.3) and we have

$$\begin{aligned} I^q = \mathrm{cl}(I)^p \cap A = l(\mathrm{cl}(I))^{*'} \cap A &= (l(\mathrm{cl}(I)) \cap A)^{*'} \\ &= l_A(I)^{*'}. \end{aligned}$$

Thus $\boldsymbol{q}$ is a complementor on A and the proof is complete.

Definition. Let p be a complementor on a B^*-algebra A and P the p-derived mapping (see [1] Definition 3.7). We shall say that p is uniformly continuous if P is uniformly continuous.

Theorem 8.2. *Let A be a dual A^*-algebra which is a dense two-sided ideal of a B^*-algebra $\mathfrak{A}$. Suppose that $\mathfrak{A}$ has no minimal left ideals of dimension less than three. Then, for every uniformly continuous complementor p on $\mathfrak{A}$, the mapping $\boldsymbol{q}$: $I \to \mathrm{cl}(I)^p \cap A$ on the closed right ideals I of A is a complementor on A.*

Proof. Let p be a uniformly continuous complementor on $\mathfrak{A}$. Let $\{I_\lambda : \lambda \in \Lambda\}$ be the family of all minimal closed two-sided ideals of A. It is easy to see that, for each λ, $\mathrm{cl}(I_\lambda)$ is a minimal closed two-sided ideal of $\mathfrak{A}$ and hence $*$-isomorphic to $LC(H_\lambda)$, for some Hilbert space H_λ. Since A is the direct topological sum of I_λ, $\mathfrak{A}$ is $*$-isomorphic to $(\sum LC(H_\lambda))_0$. In the rest of the proof we identify $\mathfrak{A}$ with $(\sum LC(H_\lambda))_0$. For each λ, let p_λ be the complementor on $LC(H_\lambda)$ induced by p. Then, by [1] Theorem 3.9, each p_λ is continuous on $LC(H_\lambda)$. Therefore each p_λ gives rise to an involution $*'_\lambda$ on $LC(H_\lambda)$ and a positive operator $Q_\lambda \in L(H_\lambda)$ with continuous inverse Q_λ^{-1} such that

$$a_\lambda^{*'\lambda} = Q_\lambda^{-1} a_\lambda^* Q_\lambda$$

for all $a_\lambda \in LC(H_\lambda)$ (see the proof of Lemma 8.1); we may clearly take $|Q_\lambda| = 1$, for all λ. By the proof of [1] Theorem 7.4, $a \to a^{*'} = (a_\lambda^{*'\lambda})$ is an involution on $\mathfrak{A}$ under which $\mathfrak{A}$ is a B^*-algebra and $R^p = l(R)^{*'}$, for all closed right ideals R of $\mathfrak{A}$. We show that A is closed under the involution $*'$. Let $H = \oplus_\lambda H_\lambda$, the Hilbert direct sum of H_λ and $Q = (Q_\lambda)$. Then Q is a positive operator in $L(H)$ with bounded inverse and $|Q| = 1$. Let $\{e_\alpha\}$ be a maximal orthogonal family of minimal self-adjoint idempotents in A. Since $\sum_\alpha e_\alpha x$ converges to x in the norm $\|\cdot\|$, $e_\alpha x \neq 0$ for only a countable number of e_α, say $e_{\alpha_1}, e_{\alpha_2}, \cdots$. Now each e_{α_i} belongs

https://doi.org/10.1017/S1446788700010594 Published online by Cambridge University Press

to some I_λ and $QI_\lambda = Q_\lambda I_\lambda \subset I_\lambda$ (by Theorem 7.4); hence each $Qe_{\alpha_i} \in A$ and so $\sum_{i=1}^n Qe_{\alpha_i} x \in A$ for $n = 1, 2, \cdots$ (we identify A as a subalgebra of $L(H)$.) Since $\{\sum_{i=1}^n Qe_{\alpha_i} x\}$ converges to Qx in the norm $\|\cdot\|$ (see the proof of Theorem 7.4), $Qx \in A$ and so $x^*Q = (Qx)^* \in A$; similarly $Q^{-1}x \in A$. Therefore $x^{*\prime} = Q^{-1}x^*Q \in A$, for all $x \in A$. Thus $*'$ is an involution on A and therefore, since A is dual, $I \to l(I_A)^{*\prime}$ is a complementor on A. Now, applying the argument in the proof of Lemma 8.1, we obtain that $I^{\mathbf{q}} = \mathrm{cl}(I)^{\mathbf{p}} \cap A = l_A(I)^{*\prime}$, which shows that $\mathbf{q}$ is a complementor on A.

COROLLARY 8.3. *Let A, $\mathfrak{A}$, p and $\mathbf{q}$ be as in Theorem 8.2. Then there exists an involution $*'$ in A such that $I^{\mathbf{q}} = l_A(I)^{*\prime}$ for every closed right ideal I of A.*

NOTATION. Let A be an algebra of operators on a normed space X. For every closed subspace S of X, let

$$\mathscr{J}_A(S) = \{a \in A = a(X) \subseteq S\}.$$

For every right ideal I of A, let $\mathscr{S}_A(I)$ be the smallest closed subspace of X that contains the range $a(X)$ of each operator a in I. We shall write $\mathscr{J}(S)$ for $\mathscr{J}_A(S)$ and $\mathscr{S}(I)$ for $\mathscr{S}_A(I)$ if $A = LC(H)$ and $X = H$.

LEMMA 8.4. *Let A be a dual A^*-algebra which is a dense subalgebra of $LC(H)$. Then, for every closed right ideal I of A, $I = \mathscr{J}_A(\mathscr{S}_A(I))$ and, for every closed subspace S of H, $\mathscr{J}_A(S)$ is a closed right ideal of A and $S = \mathscr{S}_A(\mathscr{J}_A(S))$.*

PROOF. It is easy to see that A is simple and that the set of all operators of finite rank on H is dense in A. The proof can now be completed by using the argument (with obvious modifications) given in the proof of [1] Lemma 4.1.

REMARK. Lemma 8.4 shows that $I \to \mathscr{S}_A(I)$ defines a one-to-one correspondence between the closed right ideals of A and the closed subspaces of H. Moreover if q is a complementor on A, then the mapping

$$S \to S' = \mathscr{S}_A(\mathscr{J}_A(S)^q)$$

defines a complementor on the closed subspaces S of H in the sense of [4] Theorem 1.

LEMMA 8.5. *Let A be a dual A^*-algebra which is a dense subalgebra of $LC(H)$. Then, for every complementor q on A, the mapping $\mathbf{p} : R \to \mathrm{cl}((R \cap A)^q)$ on the closed right ideals R of $LC(H)$ is a complementor on $LC(H)$.*

PROOF. It is clear that A is simple. Let q be a complementor on A. Then, by the Remark above, the mapping $S \to S' = \mathscr{S}_A(\mathscr{J}_A(S)^q)$ defines a complementor on the closed subspaces S of H. By the Remark following [1] Lemma 4.1, the mapping $S \to S'$ induces a complementor p' on $LC(H)$ given by the relation $R^{p'} = J(S(R)')$, for every closed right ideal R of $LC(H)$. It is easy to see that

$\mathrm{cl}(R \cap A) = R$. In fact, let $\mathfrak{A} = LC(H)$ and let $\{e_\alpha\}$ be the family of all minimal self-adjoint idempotents in R. Then clearly $R = \mathrm{cl}(\sum_\alpha e_\alpha \mathfrak{A})$. But from Lemma 5.5 we have $e_\alpha \mathfrak{A} \subset R \cap A$ for all α; hence $R = \mathrm{cl}(R \cap A)$. Similarly $R^{p'} = \mathrm{cl}(R^{p'} \cap A)$. Now $\mathscr{J}_A(S) = \mathscr{J}(S) \cap A$ and, by Lemma 8.4,

$$\mathscr{J}_A(\mathscr{S}(R)) = \mathscr{J}(\mathscr{S}(R)) \cap A = R \cap A = I.$$

Therefore

$$\begin{aligned} R^{p'} \cap A &= \mathscr{J}(\mathscr{S}(R)') \cap A = \mathscr{J}(\mathscr{S}_A(\mathscr{J}_A(\mathscr{S}(R))^q)) \cap A = \mathscr{J}(\mathscr{S}_A(I^q)) \cap A \\ &= \mathscr{J}_A(\mathscr{S}_A(I^q)) = I^q = (R \cap A)^q. \end{aligned}$$

Hence $R^{p'} = \mathrm{cl}(R^{p'} \cap A) = \mathrm{cl}((R \cap A)^q$, so that p is a complementor on $\mathfrak{A}$.

LEMMA 8.6. *Let $\mathfrak{A}$ be a B^*-algebra which has no minimal left ideals of dimension less than three. Let p be a continuous complementor on $\mathfrak{A}$ and let $\mathscr{E}_p$ be the set of all minimal p-projections in $\mathfrak{A}$. Then p is uniformly continuous if and only if the set $\{|e| : e \in \mathscr{E}_p\}$ is bounded.*

PROOF. Suppose p is uniformly continuous. By [1] Theorem 7.4, there exists an involution $*'$ on $\mathfrak{A}$ for which $R^p = l(R)^{*'}$, for every closed right ideal R of $\mathfrak{A}$, and an equivalent norm $|\cdot|'$ on $\mathfrak{A}$ satisfying the B^*-condition for $*'$. Since, by [1] Corollary 4.4, $e^{*'} = e$ and hence $|e|' = 1$, it follows that $\{|e| : e \in \mathscr{E}_p\}$ is bounded.

Conversely, suppose that $\sup \{|e| : e \in \mathscr{E}_p\} \leqq k$, for some constant k. We use the notation of the proof of [1] Theorem 7.4. Let $\{T_\lambda\}$ be the family of all p_λ-representing operators such that $\|T_\lambda^{-1}\| = 1$ for all λ. Then the set $\{\|T_\lambda\|\}$ is bounded; for if not, by the proof of [1] Theorem 7.4, there would exist a sequence $\{H_{\lambda_n}\} \subset \{H_\lambda\}$ and elements $x_n, y_n \in H_{\lambda_n}$ $(n = 1, 2, \cdots)$ such that $|f_{y_n} - f_{x_n}| \to 0$ and $|e_{y_n} - e_{x_n}| \to \infty$, as $n \to \infty$, which would contradict the fact that $|e_{y_n} - e_{x_n}| \leqq 2k$. It follows now from the proof of [1] Theorem 7.4 that p is uniformly continuous. This completes the proof.

Now let A be a dual A^*-algebra which is a dense subalgebra of a B^*-algebra $\mathfrak{A}$, and let $\{I_\lambda : \lambda \in \Lambda\}$ be the family of all minimal closed two-sided ideals of A. Clearly each $\mathrm{cl}(I_\lambda)$ is a minimal closed two-sided ideal of $\mathfrak{A}$ and hence $*$-isomorphic to $LC(H_\lambda)$, for some Hilbert space H_λ. Suppose q is a complementor on A and, for each $\lambda \in \Lambda$, let q_λ be the complementor on I_λ induced by q. Identifying I_λ as a subalgebra of $LC(H_\lambda)$, q_λ induces the complementor p_λ on $LC(H_\lambda)$ (Lemma 8.5). For each closed right ideal R_λ of $LC(H_\lambda)$, let P_{R_λ} be the projection on R_λ along $R_\lambda^{p_\lambda}$. Then P_{R_λ} is a bounded linear operator on $LC(H_\lambda)$ whose operator bound we denote by $|P_{R_\lambda}|$. Let

$$\begin{aligned} m_\lambda &= \sup \{|P_{R_\lambda}| : R_\lambda \subset LC(H_\lambda)\}, \\ m &= \sup \{m_\lambda : \lambda \in \Lambda\}; \end{aligned}$$

m may be finite or infinite.

LEMMA 8.7. *If I is a closed right ideal of A, then $I^q \cap I_\lambda = (I \cap I_\lambda)^{q\lambda}$, for every $\lambda \in \Lambda$.*

PROOF. Since $I \cap I_\lambda \subset I$, we have $I^q \subset (I \cap I_\lambda)^q$ and hence

$$I^q \cap I_\lambda \subset (I \cap I_\lambda)^{q\lambda}.$$

Now, by [1] Lemma 2.1, $\mathrm{cl}(I + I_\lambda^q) = I^q \cap I_\lambda$; hence

$$\mathrm{cl}(I + I_\lambda^q) \cap I_\lambda = (I^q \cap I_\lambda)^q \cap I_\lambda = (I^q \cap I_\lambda)^{q\lambda}.$$

Let $x \in (I^q \cap I_\lambda)^{q\lambda}$. Then $x \in I_\lambda$ and $x = \lim_n x_n$, where $x_n = y_n + z_n$ with $y_n \in I$ and $z_n \in I_\lambda^q$ $(n = 1, 2, \cdots)$. Since, by [10] Lemma 1, $I_\lambda^q = l(I_\lambda)$ and since $x^* \in I_\lambda$, we obtain that $xx^* = \lim_n x_n x^* = \lim_n y_n x^* \in I$. But, by Theorem 7.1, this means that $x \in I$ and therefore $x \in I \cap I_\lambda$. Hence

$$(I^q \cap I_\lambda)^{q\lambda} \subset I \cap I_\lambda$$

and consequently

$$I^q \cap I_\lambda = (I \cap I_\lambda)^{q\lambda}.$$

THEOREM 8.8. *Let A be a dual A^*-algebra which is a dense sub-algebra of a B^*-algebra $\mathfrak{A}$. Then, for every complementor q on A for which m is finite, the mapping $\mathbf{p} : R \to \mathrm{cl}((R \cap A)^q)$ on the closed right ideals R of $\mathfrak{A}$ is a complementor on $\mathfrak{A}$. If, moreover, $\mathfrak{A}$ has no minimal left ideals of dimension less than three and $\mathbf{p}$ is continuous, then there exists an involution $*'$ on $\mathfrak{A}$ such that $R^p = l(R)^{*'}$.*

PROOF. We use the notation of the paragraph preceding Lemma 8.7. It is clear that $\mathfrak{A}$ is *-isomorphic to $(\sum LC(H_\lambda))_0$. In what follows we identify $\mathfrak{A}$ with $(\sum LC(H_\lambda))_0$. Let q be a complementor on A for which m is finite. Let R be a closed right ideal of $\mathfrak{A}$ and, for each $\lambda \in \Lambda$, let $R_\lambda = R \cap LC(H_\lambda)$. Then, by [1] Lemma 7.1, $R = (\sum R_\lambda)_0$. Define

$$R' = (\textstyle\sum [R \cap LC(H_\lambda)]^{p_\lambda})_0,$$

where p_λ is the complementor on $LC(H_\lambda)$ induced by q_λ. Clearly R' is a closed right ideal of $\mathfrak{A}$ and

$$R' \cap LC(H_\lambda) = R_\lambda^{p_\lambda}.$$

Hence

$$(R')' = (\textstyle\sum [R' \cap LC(H_\lambda)]^{p_\lambda})_0 = (\sum R_\lambda)_0 = R.$$

It is easy to see that the mapping $R \to R'$ has properties (C_1), (C_3) and (C_4). For $x = (x_\lambda) \in \mathfrak{A}$, write $x_\lambda = y_\lambda + z_\lambda$, $y_\lambda \in R_\lambda$ and $z_\lambda \in R_\lambda^{p_\lambda}$. We have

$$|y_\lambda| = |P_{R_\lambda} x_\lambda| \leqq m|x_\lambda| \qquad (\lambda \in \Lambda);$$

similarly $|z_\lambda| \leqq m|x_\lambda|$ $(\lambda \in \Lambda)$. Hence, since m is finite,

$$(y_\lambda) \in (\textstyle\sum R_\lambda)_0 = R \text{ and } (z_\lambda) \in (\sum R_\lambda^{p_\lambda})_0 = R'.$$

Thus $R + R' = \mathfrak{A}$ and consequently $R \to R'$ is a complementor on $\mathfrak{A}$.

We show next that $R' = \mathrm{cl}((R \cap A)^q) = R^p$. Let $I = R \cap A$. Since, by [1] Theorem 7.1, we have $\mathrm{cl}(I^q) = (\sum[\mathrm{cl}(I^q) \cap LC(H_\lambda)])_0$, it suffices to show that

$$R_\lambda^{p_\lambda} = \mathrm{cl}(I^q) \cap LC(H_\lambda) \quad (\lambda \in \Lambda).$$

Now, by the duality of A, we have

$$\mathrm{cl}(I^q) \cap I_\lambda = \mathrm{cl}(I^q) \cap A \cap I_\lambda = I^q \cap I_\lambda.$$

Therefore, the duality of I_λ and Lemma 8.7 give

$$\begin{aligned}\mathrm{cl}(I^q) \cap LC(H_\lambda) &= \mathrm{cl}([\mathrm{cl}(I^q) \cap LC(H_\lambda)] \cap I_\lambda) \\ &= \mathrm{cl}(\mathrm{cl}(I^q) \cap I_\lambda) = \mathrm{cl}(I^q \cap I_\lambda) = \mathrm{cl}((I \cap I_\lambda)^{q_\lambda}) \\ &= \mathrm{cl}((R_\lambda \cap I_\lambda)^{q_\lambda}) = R_\lambda^{p_\lambda}.\end{aligned}$$

To prove the second part of the theorem, we see that, by [1, Theorem 7.4] and Lemma 8.6, it suffices to show that $\{|f| : f \in \mathscr{E}_p\}$ is bounded. Let $f \in \mathscr{E}_p$. Since $f\mathfrak{A} \subset LC(H_\lambda)$, for some λ, $|P|_{f\mathfrak{A}} \leqq m$. But $fa = P_{f\mathfrak{A}}a$ for all $a \in \mathfrak{A}$, and so $|f| \leqq m$. This completes the proof of the theorem.

9. Examples

As an immediate example of a complemented A^*-algebra we have an H^*-algebra (see [10]). We shall now give another example, which we believe has not yet been discussed from this point of view.

Let H be a Hilbert space and $\tau c(H)$ the trace class operators on H with the trace norm $\|\cdot\|$. $\tau c(H)$ is an A^*-algebra which is a dense two-sided ideal of $LC(H)$ and, as a Banach space, it is isometrically isomorphic to the conjugate space of $LC(H)$ (see [9] p. 47). Clearly $\tau c(H)$ contains all operators of finite rank as a dense set and hence is an annihilator algebra, in fact it is dual as we shall see.

Now let $\{H_\lambda : \lambda \in \Lambda\}$ be a family of Hilbert spaces H_λ and let $(\sum_\lambda \tau c(H_\lambda))_1$ denote the family of all functions f defined on Λ such that $f(\lambda) \in \tau c(H_\lambda)$ for each λ and such that $\sum_\lambda \|f(\lambda)\| < \infty$. It follows that $(\sum \tau c(H_\lambda))_1$ is a Banach algebra under the norm $\|f\| = \sum_\lambda \|f(\lambda)\|$ and the usual operations for functions. It is easily verified that, as a Banach space, $(\sum \tau c(H_\lambda))_1$ is isometrically isomorphic to the conjugate space of $(\sum LC(H_\lambda))_0$. It is clearly a sub-algebra of $(\sum LC(H_\lambda))_0$ and an A^*-algebra under the involution $f \to f^*$, where $f^*(\lambda) = f(\lambda)^{*\lambda}$ ($*\lambda$ being the adjoint operation in $\tau c(H_\lambda)$).

LEMMA 9.1. *$\tau c(H)$ is a dual A^*-algebra and the mapping $I \to l(I)^*$ on the closed right ideals I is a complementor on $\tau c(H)$.*

PROOF. Let $A = \tau c(H)$ and let I be a closed right ideal of A. We show that $\mathscr{I}_A(\mathscr{S}_A(I)) = I$. Clearly $I \subset \mathscr{I}_A(\mathscr{S}_A(I))$. Let $T \in \mathscr{I}_A(\mathscr{S}_A(I))$ and $\{T_n\}$ a sequence of operators of finite rank on H such that $\|T_n - T\| \to 0$ as $n \to \infty$. Let

https://doi.org/10.1017/S1446788700010594 Published online by Cambridge University Press

P be the orthogonal projection on $\mathscr{S}_A(I)$. Since PT_n is finite dimensional with range in $\mathscr{S}_A(I)$, [8] Theorem (2.4.18) shows that $PT_n \in I$ for all $n = 1, 2, \cdots$. Clearly $PT = T$. By [9] Lemma 8, we have

$$||PT_n - T|| = ||PT_n - TP|| \leqq |P|\, ||T_n - T||,$$

so that $||PT_n - T|| \to 0$ as $n \to \infty$. Hence $T \in I$ and consequently $\mathscr{J}_A \mathscr{S}_A(I)) = I$. Thus, by [8] Lemma (2.8.24), and the continuity of the involution, A is dual. Let $T \in A$. Then $T = PT + P'T$ where $P' = 1 - P$, and, since $PT \in I$ and $P'T \in l(I)^*$, we have $I + l(I)^* = A$. It is now easy to see that the mapping $I \to l(I)^*$ is a complementor on A.

THEOREM 9.2. *$(\sum \tau c(H_\lambda))_1$ is a dual A^*-algebra which is a dense two-sided ideal of $(\sum LC(H_\lambda))_0$.*

PROOF. Let $A = (\sum \tau c(H_\lambda))_1$ and $\mathfrak{A} = (\sum LC(H_\lambda))_0$. Identifying $\tau c(H_\lambda)$ as a subalgebra of A, we see that $\tau c(H_\lambda)$ is a closed two-sided ideal of A and that A is the direct topological sum of the $\tau c(H_\lambda)$. Therefore, by [8] Theorem (2.8.29), A is an annihilator algebra. Since each $\tau c(H_\lambda)$ is dense in $LC(H_\lambda)$, it is easy to show that A is dense in $\mathfrak{A}$. Moreover, since for all $x_\lambda \in LC(H_\lambda)$ and $y_\lambda \in \tau c(H_\lambda)$, we have $||x_\lambda y_\lambda|| \leqq |x_\lambda|\, ||y_\lambda||$ ([9] Lemma 8, p. 39), it readily follows that A is a two-sided ideal of $\mathfrak{A}$. Let $x = (x_\lambda) \in A$. Identifying $\tau c(H_\lambda)$ as a subalgebra of A, we have $x\tau c(H_\lambda) = x_\lambda \tau c(H_\lambda)$ for all λ. Therefore, by the duality of $\tau c(H_\lambda)$, $x_\lambda \in \mathrm{cl}(xA)$ for all λ. It is now easy to show that $x \in \mathrm{cl}(xA)$, and so, by Theorem 7.1, A is dual.

COROLLARY 9.3. *The mapping $I \to l(I)^*$ on the closed right ideals I of $(\sum \tau c(H_\lambda))_1$ is a complementor on $(\sum \tau c(H_\lambda))_1$.*

PROOF. This follows from Theorem 9.2 and the proof of Theorem 7.3.

We do not know of an example of a complemented A^*-algebra which is not a dense two-sided ideal of a B^*-algebra. Also we do not know if every dual A^*-algebra is complemented, and conversely if every complimented A^*-algebra is dual.

References

[1] F. E. Alexander and B. J. Tomiuk, 'Complemented B^*-algebras', *Trans. Amer. Math. Soc.* 137 (1969), 459–480.

[2] F. F. Bonsall and A. W. Goldie, 'Annihilator algebras', *Proc. London Math. Soc.* (3), 4 (1954), 154–167.

[3] J. Dixmer, *C*-algèbres et leurs représentations* (Gauthier-Villars, Paris, 1964).

[4] I. Kakutani and G. W. Mackey, 'Ring and lattice characterizations of complex Hilbert space', *Bull. Amer. Math. Soc.* 52 (1964), 727–733.

[5] I. Kaplansky, 'The structure of certain operator algebras', *Trans. Amer. Math. Soc.* 70 (1951), 219–255.

[6] T. Ogasawara and K. Yoshinaga, 'Weakly completely continuous Banach*-algebras', *J. Sci. Hiroshima Uni. Ser. A* 18 (1954), 15–36.

[7] T. Ogasawara and K. Yoshinaga, 'A characterization of dual B^*-algebras', *J. Sci. Hiroshima Uni. Ser.* A 18 (1954), 179–182.
[8] C. E. Rickart, *General theory of Banach algebras* (D. Van Nostrand, New York, 1960).
[9] R. Schatten, *Norm ideals of completely continuous operators* (Springer-Verlag, 1960).
[10] B. J. Tomiuk, 'Structure theory of complemented Banach algebras', *Canadian J. of Math.* 14 (1962), 651–659.

University of Ottawa
Ottawa, Canada