Can. J. Math. Vol. **50** (3), 1998 pp. 595–604

MULTIPLIERS OF FRACTIONAL CAUCHY TRANSFORMS AND SMOOTHNESS CONDITIONS

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ABSTRACT. This paper studies conditions on an analytic function that imply it belongs to $\mathcal{M}_\alpha$, the set of multipliers of the family of functions given by $f(z) = \int_{|\zeta|=1} \frac{1}{(1-\bar{\zeta}z)^\alpha}\, d\mu(\zeta)$ ($|z| < 1$) where μ is a complex Borel measure on the unit circle and $\alpha > 0$. There are two main theorems. The first asserts that if $0 < \alpha < 1$ and $\sup_{|\zeta|=1} \int_0^1 |f'(r\zeta)|(1-r)^{\alpha-1}\, dr < \infty$ then $f \in \mathcal{M}_\alpha$. The second asserts that if $0 < \alpha \le 1$, $f \in H^\infty$ and $\sup_t \int_0^\pi \frac{|f(e^{i(t+s)})-2f(e^{it})+f(e^{i(t-s)})|}{s^{2-\alpha}}\, ds < \infty$ then $f \in \mathcal{M}_\alpha$. The conditions in these theorems are shown to relate to a number of smoothness conditions on the unit circle for a function analytic in the open unit disk and continuous in its closure.

1. **Introduction.** Let $\Delta = \{z \in \mathbf{C} : |z| < 1\}$ and $\Gamma = \{z \in \mathbf{C} : |z| = 1\}$. Let $\mathcal{M}$ denote the set of complex-valued Borel measures on Γ, and let $\|\mu\|$ denote the total variation of $\mu \in \mathcal{M}$. For $\alpha > 0$ let $\mathcal{F}_\alpha$ denote the set of functions f for which there exists $\mu \in \mathcal{M}$ such that

$$f(z) = \int_\Gamma \frac{1}{(1-\zeta z)^\alpha}\, d\mu(\zeta) \tag{1}$$

for $|z| < 1$. The power function in (1) is the principal branch. $\mathcal{F}_\alpha$ is a Banach space with respect to the norm defined by $\|f\|_{\mathcal{F}_\alpha} = \inf \|\mu\|$, where μ varies over the subset of $\mathcal{M}$ for which (1) holds.

A function f is called a multiplier of $\mathcal{F}_\alpha$ provided that $fg \in \mathcal{F}_\alpha$ for every $g \in \mathcal{F}_\alpha$. Let $\mathcal{M}_\alpha$ denote the set of multipliers of $\mathcal{F}_\alpha$. If $f \in \mathcal{M}_\alpha$ then the map $\mathcal{F}_\alpha \longmapsto \mathcal{F}_\alpha$ defined by $g \longmapsto fg$ is a bounded linear operator. $\mathcal{M}_\alpha$ is a Banach space with the natural norm defined by

$$\|f\|_{\mathcal{M}_\alpha} = \sup\{\|fg\|_{\mathcal{F}_\alpha} : g \in \mathcal{F}_\alpha, \|g\|_{\mathcal{F}_\alpha} \le 1\}. \tag{2}$$

We are interested in conditions on an analytic function which imply the function belongs to $\mathcal{M}_\alpha$. The main result in this paper is the following theorem.

THEOREM 1. *Let $0 < \alpha < 1$ and let dA denote two-dimensional Lebesgue measure. If $f \in H^\infty$ and*

$$I_\alpha(f) \equiv \sup_{|\zeta|=1} \iint_\Delta \frac{|f'(z)|(1-|z|)^{\alpha-1}}{|z-\zeta|^\alpha}\, dA(z) < \infty \tag{3}$$

Received by the editors August 29, 1995.
AMS subject classification: 30E20, 30D50.

then $f \in M_\alpha$. *There is a positive constant* A *depending only on* α *such that*

$$\|f\|_{M_\alpha} \le A\big(I_\alpha(f) + \|f\|_{H^\infty}\big) \tag{4}$$

for all such functions f.

Theorem 1 gives a broad sufficient condition for membership in M_α when $0 < \alpha < 1$. It implies a number of other results which primarily deal with radial variations and which relate to Lipschitz and Zygmund types of smoothness on Γ.

For $\alpha > 0$ each function in M_α has finite radial variations. In fact there is a constant A depending only on α such that if $f \in M_\alpha$ then

$$\int_0^1 |f'(r\zeta)|\, dr \le A\|f\|_{M_\alpha} \tag{5}$$

for $|\zeta| = 1$ [4, Theorem 2.6; 7, p. 14]. Since $\sup_{|\zeta|=1} \int_0^1 |f'(r\zeta)|\, dr < \infty$ we infer that $f \in H^\infty$ and $f(\zeta) \equiv \lim_{r\to 1-} f(r\zeta)$ exists for all $\zeta \in \Gamma$.

Theorem 2 is stated below and it shows that the boundedness of a certain weighted radial variation of f implies $f \in M_\alpha$. It holds for $0 < \alpha < 1$ and will be proved as a simple consequence of Theorem 1.

THEOREM 2. *Let* $0 < \alpha < 1$ *and suppose that the function* f *is analytic in* Δ. *If*

$$J_\alpha(f) \equiv \sup_{|\zeta|=1} \int_0^1 |f'(r\zeta)|(1-r)^{\alpha-1}\, dr < \infty \tag{6}$$

then $f \in M_\alpha$. *There is a positive constant* A *depending only on* α *such that*

$$\|f\|_{M_\alpha} \le A\big(J_\alpha(f) + \|f\|_{H^\infty}\big) \tag{7}$$

for all such functions f.

Since (6) implies that $\sup_{|\zeta|=1} \int_0^1 |f'(r\zeta)|\, dr < \infty$, the assumptions of Theorem 2 imply that $f \in H^\infty$ and $f(\zeta)$ exists for all $\zeta \in \Gamma$. In fact these assumptions imply that f extends continuously to $\bar{\Delta}$ and on Γ satisfies a Lipschitz condition of order $1-\alpha$. This was proved by Richard O'Neil in [6]. The result of O'Neil can be stated in the following way. Let $0 < \beta < 1$ and let $F\colon [-\pi, \pi] \longrightarrow \mathbf{C}$ be a periodic function with period 2π. A necessary and sufficient condition that F satisfies a Lipschitz condition of order β is that there is a positive constant A (depending on F) such that $|u(r,t) - F(t)| \le A(1-r)^\beta$ for $0 \le r < 1$ and $|t| \le \pi$, where $u(r,t)$ is the harmonic extension of F to Δ. O'Neil's result is applicable because the assumptions in Theorem 2 imply that $|f(re^{it}) - f(e^{it})| \le A(1-r)^{1-\alpha}$ for some constant A.

Theorem 2 directly relates to a number of earlier results about M_α. Theorem A stated below was proved in [1, 3] and Theorem B was proved in [3].

THEOREM A. *If* $f(z) = \sum_{n=0}^{\infty} a_n z^n$ *and* $\sum_{n=1}^{\infty} n^{1-\alpha}|a_n| < \infty$ *for some* α $(0 < \alpha < 1)$, *then* $f \in \mathcal{M}_\alpha$.

THEOREM B. *Suppose that the function* f *is analytic in* Δ *and continuous in* $\bar{\Delta}$. *If* $f(e^{it})$ *satisfies a Lipschitz condition of order* α *and* $0 < \alpha < 1$ *then* $f \in \mathcal{M}_\beta$ *for* $\beta > 1 - \alpha$.

It is easy to show that the assumptions in Theorem A as well as those in Theorem B imply (6). Thus Theorem 2 also yields Theorem A and Theorem B. In general the applicability of Theorem 2 derives from the fact that (6) relates to a number of other conditions.

Theorem 3, which is stated below, concerns second differences. For each function $f\colon \Gamma \longrightarrow \mathbf{C}$ and for each pair of real numbers t and s let

$$D(f;t,s) = f(e^{i(t+s)}) - 2f(e^{it}) + f(e^{i(t-s)}). \tag{8}$$

THEOREM 3. *Let* $0 < \alpha \le 1$ *and suppose that* $f \in H^\infty$. *If*

$$K_\alpha(f) \equiv \sup_t \int_0^\pi \frac{|D(f;t,s)|}{s^{2-\alpha}}\, ds < \infty \tag{9}$$

then $f \in \mathcal{M}_\alpha$. *There is a positive constant* A *depending only on* α *such that*

$$\|f\|_{\mathcal{M}_\alpha} \le A\big(K_\alpha(f) + \|f\|_{H^\infty}\big) \tag{10}$$

for all such functions f.

When $0 < \alpha < 1$ Theorem 3 is proved as a consequence of Theorem 2. When $\alpha = 1$ our argument depends on using Toeplitz operators.

We recall some facts about Toeplitz operators. Let P denote the orthogonal projection of $L^2(\Gamma)$ onto H^2 defined by $P(h) = \sum_{n=0}^{\infty} a_n z^n$ where $h(t) = \sum_{n=-\infty}^{\infty} a_n e^{int} \in L^2(\Gamma)$. For $\phi \in L^\infty(\Gamma)$ the Toeplitz operator with symbol ϕ is the operator on H^2 defined by $T_\phi(g) = P(\phi g)$. The duality between the disk algebra $\mathcal{A}$ and $\mathcal{M}$ shows that when $T_{\bar{f}}$ is restricted to $\mathcal{A}$ it gives the multiplication operator on $\mathcal{F}_1$ described earlier. Hence $f \in \mathcal{M}_1$ if and only if $T_{\bar{f}}$ is bounded on $\mathcal{A}$. Also we have $\|f\|_{\mathcal{M}_1} = \|T_{\bar{f}}\|_{\mathcal{A}} = \|T_{\bar{f}}\|_{H^\infty}$ and $T_{\bar{f}}$ is given by

$$T_{\bar{f}}(h)(z) = \frac{1}{2\pi i}\int_\Gamma \frac{\overline{f(\zeta)}h(\zeta)}{\zeta - z}\, d\zeta. \tag{11}$$

Toeplitz operators have been used by authors studying $\mathcal{M}_1$, especially in [7].

Theorem 3 extends the following result proved for $\alpha = 1$ in [8] and for $0 < \alpha < 1$ in [3].

THEOREM C. *Let* $0 < \alpha \le 1$ *and assume that* $f \in H^\infty$. *If*

$$\sup_t \int_{-\pi}^{\pi} \frac{|f(e^{i(t+s)}) - f(e^{it})|}{|s|^{2-\alpha}}\, ds < \infty \tag{12}$$

then $f \in \mathcal{M}_\alpha$.

We thank Fedor Nazarov for his remarks in [5], where a number of detailed comments are made about this paper. Nazarov suggested the present formulation of Theorem 1. He

gave a different proof of this result beginning with the Cauchy-Green formula and he showed alternative ways to deduce a number of results about M_α. One of the new facts which he proved is that if $f \in M_\alpha$ for some α $(0 < \alpha < 1)$ then $I_\beta(f) < \infty$ for each β such that $\alpha < \beta \le 1$. Nazarov gives credit to E. M. Dynkin for the main ideas described in [5].

2. **Preliminary Lemmas.** This section consists of seven lemmas which are used later on. Lemmas 1–4 are easy to prove but we do not include the arguments here. Lemma 5 is in [4] and Lemma 6 is in [3]. Lemma 7 is known and a proof depends on the Banach-Alaoglu theorem.

LEMMA 1. *If $z = re^{it}$, $0 \le r < 1$ and $|t| \le \pi$, then $|1-z| \ge \frac{1}{\pi}|t|$.*

LEMMA 2. *Let $\alpha > 1$. There is a positive constant A depending only on α such that*

$$\int_\varphi^\pi \frac{1}{|1-re^{it}|^\alpha}\,dt \le \frac{A}{|1-re^{i\varphi}|^{\alpha-1}} \tag{13}$$

for $0 < \varphi < \pi$ and $0 \le r < 1$.

Lemma 2 implies the known estimate that

$$\int_{-\pi}^\pi \frac{1}{|1-re^{it}|^\alpha}\,dt \le \frac{A}{(1-r)^{\alpha-1}} \tag{14}$$

for $0 \le r < 1$ and $\alpha > 1$, where the constant A depends only on α.

LEMMA 3. *Let $\alpha > 1$. There is a positive constant B depending only on α such that*

$$\int_0^r \frac{1}{|1-\rho e^{i\varphi}|^\alpha}\,d\rho \le \frac{B}{|1-re^{i\varphi}|^{\alpha-1}} \tag{15}$$

for $0 \le r < 1$ and $|\varphi| \le \pi$.

LEMMA 4. *Let $\beta > -1$ and let $\gamma \ge \beta + 1$. There is a positive constant C depending only on β and γ such that*

$$I(\beta,\gamma) \equiv \int_0^1 \frac{(1-r)^\beta}{|1-re^{it}|^{\gamma+1}}\,dr \le \frac{C}{|t|^{\gamma-\beta}} \tag{16}$$

for $0 < |t| \le \pi$.

LEMMA 5. *Let $\alpha > 0$ and assume that the function f is analytic in Δ. Then $f \in M_\alpha$ if and only if $f(z)\frac{1}{(1-\zeta z)^\alpha} \in F_\alpha$ for $|\zeta| = 1$ and there is a constant A such that $\|f(z)\frac{1}{(1-\zeta z)^\alpha}\|_{F_\alpha} \le A$ for $|\zeta| = 1$. Moreover, we have $\|f\|_{M_\alpha} = \sup_{|\zeta|=1} \|f(z)\frac{1}{(1-\zeta z)^\alpha}\|_{F_\alpha}$.*

LEMMA 6. *Let $\alpha > 0$ and assume that the function f is analytic in Δ. If*

$$L_\alpha(f) \equiv \int_0^1 \int_{-\pi}^\pi |f'(r\,e^{it})|(1-r)^{\alpha-1}\,dt\,dr < \infty$$

then $f \in F_\alpha$. There is a constant A depending only on α such that $\|f\|_{F_\alpha} \le |f(0)| + A\,L_\alpha(f)$ for all such functions f.

LEMMA 7. *Let $\alpha > 0$ and suppose that $f_n \in F_\alpha$ for $n = 1, 2, \ldots$ and $f_n(z) \longrightarrow f(z)$ as $n \longrightarrow \infty$ for each z in Δ. If there exists a constant $M > 0$ such that $\|f_n\|_{F_\alpha} \le M$ for $n = 1, 2, \ldots$, then $f \in F_\alpha$ and $\|f\|_{F_\alpha} \le M$.*

https://doi.org/10.4153/CJM-1998-033-3 Published online by Cambridge University Press

3. **Proof of Theorem 1.** Since $f \in H^\infty$ implies the uniform bound $|f'(z)| \leq \frac{\|f\|_{H^\infty}}{1-|z|^2}$ and $dA(z) = r\,dr\,dt$ ($z = re^{it}$), Theorem 1 is equivalent to showing that if $f \in H^\infty$ and

$$I^*_\alpha(f) = \sup_{|\zeta|=1} \int_0^1 \int_{-\pi}^{\pi} \frac{|f'(z)|\,(1-|z|)^{\alpha-1}}{|z-\zeta|^\alpha}\,dr\,dt < \infty \tag{17}$$

then $f \in \mathcal{M}_\alpha$ and

$$\|f\|_{\mathcal{M}_\alpha} \leq A\big(I^*_\alpha(f) + \|f\|_{H^\infty}\big) \tag{18}$$

where A depends only on α.

Let $0 < \alpha < 1$ and suppose that $f \in H^\infty$ and (17) holds. By considering the functions $f_n(z) = f(r_n z)$ where $0 < r_n < 1$ and $r_n \longrightarrow 1$ as $n \longrightarrow \infty$ we can assume that f is analytic in $\bar{\Delta}$. This is a consequence of Lemmas 5 and 7.

Let $|\zeta| = 1$. Then

$$f(z)\frac{1}{(1-\bar{\zeta}z)^\alpha} = \frac{f(\zeta)}{(1-\bar{\zeta}z)^\alpha} + g_\zeta(z),$$

where

$$g_\zeta(z) = \frac{f(z)-f(\zeta)}{(1-\bar{\zeta}z)^\alpha}. \tag{19}$$

Since $\|\frac{f(\zeta)}{(1-\bar{\zeta}z)^\alpha}\|_{\mathcal{F}_\alpha} = |f(\zeta)| \leq \|f\|_{H^\infty}$, Lemma 5 implies that it suffices to show that $g_\zeta \in \mathcal{F}_\alpha$ and

$$\|g_\zeta\|_{\mathcal{F}_\alpha} \leq A\big(I^*_\alpha(f) + \|f\|_{H^\infty}\big) \tag{20}$$

for $|\zeta| = 1$, where A depends only on α. Because of Lemma 6, this follows if we show that

$$\int_0^1 \int_{-\pi}^{\pi} |g'_\zeta(re^{it})|\,(1-r)^{\alpha-1}\,dt\,dr \leq B I^*_\alpha(f) \tag{21}$$

for $|\zeta| = 1$, and B depends only on α.

From (19) we obtain

$$g'_\zeta(z) = \frac{f'(z)}{(1-\bar{\zeta}z)^\alpha} + \alpha\bar{\zeta}\frac{f(z)-f(e^{it})}{(1-\bar{\zeta}z)^{\alpha+1}} + \alpha\bar{\zeta}\frac{f(e^{it})-f(\zeta)}{(1-\bar{\zeta}z)^{\alpha+1}}$$

where $z = re^{it}$ ($0 \leq r < 1$, $|t| \leq \pi$). Hence it suffices to show that

$$P(\zeta) \equiv \int_0^1 \int_{-\pi}^{\pi} \frac{|f'(re^{it})|}{|1-\bar{\zeta}re^{it}|^\alpha}(1-r)^{\alpha-1}\,dt\,dr \leq C I^*_\alpha(f), \tag{22}$$

$$Q(\zeta) \equiv \int_0^1 \int_{-\pi}^{\pi} \frac{|f(re^{it})-f(e^{it})|}{|1-\bar{\zeta}re^{it}|^{\alpha+1}}(1-r)^{\alpha-1}\,dt\,dr \leq D I^*_\alpha(f) \tag{23}$$

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and

$$R(\zeta) \equiv \int_0^1 \int_{-\pi}^{\pi} \frac{|f(e^{it}) - f(\zeta)|}{|1 - \bar{\zeta} r e^{it}|^{\alpha+1}} (1-r)^{\alpha-1}\, dt\, dr \le E I_\alpha^*(f) \tag{24}$$

for $|\zeta| = 1$ and C, D and E depend only on α.

Clearly we have $P(\zeta) \le I_\alpha^*(f)$ for $|\zeta| = 1$. To estimate $Q(\zeta)$ note that $|f(re^{it}) - f(e^{it})| \le \int_r^1 |f'(\rho e^{it})|\, d\rho$. Hence

$$\begin{aligned} Q(\zeta) &\le \int_0^1 \int_{-\pi}^{\pi} \int_r^1 \frac{|f'(\rho e^{it})|}{|1 - \bar{\zeta} r e^{it}|^{\alpha+1}} (1-r)^{\alpha-1}\, d\rho\, dt\, dr \\ &= \int_{-\pi}^{\pi} \left\{ \int_0^1 \int_0^\rho \frac{(1-r)^{\alpha-1}}{|1 - \bar{\zeta} r e^{it}|^{\alpha+1}}\, dr |f'(\rho e^{it})|\, d\rho \right\} dt \\ &\le \int_{-\pi}^{\pi} \left\{ \int_0^1 \int_0^\rho \frac{1}{|1 - \bar{\zeta} r e^{it}|^{\alpha+1}}\, dr (1-\rho)^{\alpha-1} |f'(\rho e^{it})|\, d\rho \right\} dt. \end{aligned}$$

Lemma 3 yields

$$Q(\zeta) \le \int_{-\pi}^{\pi} \int_0^1 \frac{B}{|1 - \rho \bar{\zeta} e^{it}|^{\alpha}} (1-\rho)^{\alpha-1} |f'(\rho e^{it})|\, d\rho\, dt,$$

and hence $Q(\zeta) \le B I_\alpha^*(f)$ for $|\zeta| = 1$.

Let $\zeta = e^{i\eta}$ $(-\pi < \eta \le \pi)$. Using periodicity we can write $R(\zeta) = S(\zeta) + T(\zeta)$, where

$$S(\zeta) = \int_0^1 \int_{\eta-\pi}^{\eta} \frac{|f(e^{it}) - f(e^{i\eta})|}{|1 - re^{i(t-\eta)}|^{\alpha+1}} (1-r)^{\alpha-1}\, dt\, dr \tag{25}$$

and

$$T(\zeta) = \int_0^1 \int_{\eta}^{\eta+\pi} \frac{|f(e^{it}) - f(e^{i\eta})|}{|1 - re^{i(t-\eta)}|^{\alpha+1}} (1-r)^{\alpha-1}\, dt\, dr. \tag{26}$$

Then $T(\zeta) = \int_0^1 \int_0^\pi \frac{|f(e^{i(s+\eta)}) - f(e^{i\eta})|}{|1 - re^{is}|^{\alpha+1}} (1-r)^{\alpha-1}\, ds\, dr$. Since $|f(e^{i(s+\eta)}) - f(e^{i\eta})| \le \int_\eta^{\eta+s} |f'(e^{i\varphi})|\, d\varphi$ for $0 < s \le 2\pi$ this gives

$$\begin{aligned} T(\zeta) &\le \int_0^1 \int_0^\pi \int_\eta^{\eta+s} \frac{|f'(e^{i\varphi})|\, (1-r)^{\alpha-1}}{|1 - re^{is}|^{\alpha+1}}\, d\varphi\, ds\, dr \\ &= \int_0^1 \left\{ \int_\eta^{\eta+\pi} \int_{\varphi-\eta}^{\pi} \frac{|f'(e^{i\varphi})|\, (1-r)^{\alpha-1}}{|1 - re^{is}|^{\alpha+1}}\, ds\, d\varphi \right\} dr. \end{aligned}$$

Hence (13) yields

$$\begin{aligned} T(\zeta) &\le \int_0^1 \int_\eta^{\eta+\pi} \frac{A |f'(e^{i\varphi})|\, (1-r)^{\alpha-1}}{|1 - re^{i(\varphi-\eta)}|^{\alpha}}\, d\varphi\, dr \\ &\le A \int_0^1 \int_{-\pi}^{\pi} \frac{|f'(e^{i\varphi})|\, (1-r)^{\alpha-1}}{|1 - re^{i\varphi}\bar{\zeta}|^{\alpha}}\, d\varphi\, dr \\ &\le A\, I_\alpha^*(f). \end{aligned}$$

The same estimate also can be obtained for $S(\zeta)$.

4. **Proof of Theorem 2.** Let $0 < \alpha < 1$ and suppose that the function f is analytic in Δ and satisfies (6). As noted earlier this implies $f \in H^\infty$.

Let $|\zeta| = 1$ and set $\zeta = e^{i\eta}$. Then

$$\begin{aligned}
&\int_0^1 \int_{-\pi}^{\pi} \frac{|f'(re^{it})|\,(1-r)^{\alpha-1}}{|re^{it}-\zeta|^\alpha}\,dt\,dr \\
&= \int_0^1 \int_{\eta-\pi}^{\eta+\pi} \frac{|f'(re^{it})|\,(1-r)^{\alpha-1}}{|re^{it}-e^{i\eta}|^\alpha}\,dt\,dr \\
&= \int_0^1 \int_{-\pi}^{\pi} \frac{|f'(re^{i(s+\eta)})|\,(1-r)^{\alpha-1}}{|1-re^{is}|^\alpha}\,ds\,dr.
\end{aligned}$$

Hence Lemma 1 implies

$$\begin{aligned}
&\int_0^1 \int_{-\pi}^{\pi} \frac{|f'(re^{it})|\,(1-r)^{\alpha-1}}{|re^{it}-\zeta|^\alpha}\,dt\,dr \\
&\le \int_0^1 \int_{-\pi}^{\pi} \frac{\pi^\alpha}{|s|^\alpha} |f'(re^{i(s+\eta)})|\,(1-r)^{\alpha-1}\,ds\,dr \\
&= \pi^\alpha \int_{-\pi}^{\pi} \frac{1}{|s|^\alpha} \left\{ \int_0^1 |f'(re^{i(s+\eta)})|\,(1-r)^{\alpha-1}\,dr \right\} ds \\
&\le \pi^\alpha \int_{-\pi}^{\pi} \frac{1}{|s|^\alpha} J_\alpha(f)\,ds \\
&= \frac{2\pi}{1-\alpha} J_\alpha(f).
\end{aligned}$$

Thus $I_\alpha(f) \le \frac{2\pi}{1-\alpha} J_\alpha(f) < \infty$. Therefore Theorem 1 implies $f \in M_\alpha$. Also (4) yields (7).

5. **Proof of Theorem 3.** We first prove Theorem 3 when $0 < \alpha < 1$. Let $0 < \alpha < 1$ and suppose that $f \in H^\infty$ and (9) is satisfied.

Since $f \in H^\infty$ the Poisson formula gives

$$f(re^{it}) = \frac{1}{2\pi} \int_{-\pi}^{\pi} P(r, s-t)\, f(e^{is})\,ds \tag{27}$$

where

$$P(r,s) = \frac{1-r^2}{1-2r\cos s + r^2} \tag{28}$$

$(0 \le r < 1,\ |s| \le \pi)$. By differentiating (27) with respect to r, we find that

$$e^{it} f'(re^{it}) = \frac{1}{\pi} \int_{-\pi}^{\pi} Q(r, s-t)\, f(e^{is})\,ds \tag{29}$$

where

$$Q(r,s) = \frac{(1-r^2)\cos s - 2r}{(1-2r\cos s + r^2)^2}. \tag{30}$$

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Q is an even function of s, has period 2π and $\int_0^\pi Q(r,s)ds = 0$. Hence (29) implies

$$\begin{aligned} e^{it}f'(re^{it}) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} Q(r,s)\big\{f(e^{i(t+s)}) - f(e^{i(t-s)})\big\}\,ds \\ &= \frac{1}{\pi}\int_0^{\pi} Q(r,s)\big\{f(e^{i(t+s)}) - f(e^{i(t-s)})\big\}\,ds \\ &= \frac{1}{\pi}\int_0^{\pi} Q(r,s)\big\{f(e^{i(t+s)}) - 2f(e^{it}) + f(e^{i(t-s)})\big\}\,ds. \end{aligned}$$

Therefore

$$|f'(re^{it})| \le \frac{1}{\pi}\int_0^{\pi} |Q(r,s)||D(f;t,s)|\,ds. \tag{31}$$

Since $(1+r^2)\cos s - 2r = (1-r)^2 - 2(1+r^2)\sin^2\frac{s}{2}$ and $1 - 2r\cos s + r^2 = |1 - r\,e^{is}|^2$, we have $|Q(r,s)| \le \frac{(1-r)^2+s^2}{|1-re^{is}|^4}$. Hence (31) yields

$$\int_0^1 |f'(re^{it})|(1-r)^{\alpha-1}\,dr \le \frac{1}{\pi}\int_0^{\pi} F(s)|D(f;t,s)|\,ds + \frac{1}{\pi}\int_0^{\pi} G(s)|D(f;t,s)|\,ds$$

where

$$F(s) = \int_0^1 \frac{(1-r)^{\alpha+1}}{|1-re^{is}|^4}\,dr \tag{32}$$

and

$$G(s) = s^2\int_0^1 \frac{(1-r)^{\alpha-1}}{|1-r\,e^{is}|^4}\,dr. \tag{33}$$

Lemma 4 implies that $F(s) \le \frac{B}{s^{2-\alpha}}$ and $G(s) \le \frac{C}{s^{2-\alpha}}$ for $0 < s \le \pi$, where B and C depend only on α. Therefore

$$\int_0^1 |f'(re^{it})|(1-r)^{\alpha-1}\,dr \le \frac{B+C}{\pi}\int_0^{\pi} \frac{|D(f;t,s)|}{s^{2-\alpha}}\,ds$$

Since $K_\alpha(f) < \infty$ we conclude that $\sup_t \int_0^1 |f'(re^{it})|(1-r)^{\alpha-1}\,dr < \infty$. Hence Theorem 2 implies $f \in \mathcal{M}_\alpha$. The argument also yields (10).

Next we prove Theorem 3 in the case $\alpha = 1$. Suppose that $f \in H^\infty$ and

$$K_1(f) \equiv \sup_t \int_0^{\pi} \frac{|D(f;t,s)|}{s}\,ds < \infty. \tag{34}$$

Let $T_{\bar f}: H^\infty \longrightarrow H^\infty$ denote the Toeplitz operator. Then

$$\begin{aligned} \|T_{\bar f}\|_{H^\infty} &= \sup\left\{\frac{1}{2\pi}\left|\int_\Gamma \frac{\overline{f(\zeta)}h(\zeta)}{\zeta - z}\,d\zeta\right| : \|h\|_{H^\infty} \le 1,\ |z| < 1\right\} \\ &= \sup\left\{\frac{1}{2\pi}\left|\int_\Gamma \frac{\overline{f(\zeta)}h(\zeta)}{\zeta - r\sigma}\,d\zeta\right| : \|h\|_{H^\infty} \le 1,\ 0 \le r < 1,\ |\sigma| = 1\right\} \\ &= \sup\left\{\frac{1}{2\pi}\left|\int_\Gamma \frac{\overline{f(\zeta)}h(\zeta)}{1 - r\sigma\bar\zeta}\frac{1}{\zeta}\,d\zeta\right| : \|h\|_{H^\infty} \le 1,\ 0 \le r < 1,\ |\sigma| = 1\right\} \\ &= \sup\left\{\frac{1}{2\pi}\left|\int_\Gamma \frac{\overline{f(\sigma\zeta)}h(\sigma\zeta)}{1 - r\bar\zeta}\frac{1}{\zeta}\,d\zeta\right| : \|h\|_{H^\infty} \le 1,\ 0 \le r < 1,\ |\sigma| = 1\right\}. \end{aligned}$$

By writing $\overline{f(\sigma\zeta)} = \left[\overline{f(\sigma\zeta)} - 2\overline{f(\sigma)} + \overline{f(\sigma\bar\zeta)}\right] + 2\overline{f(\sigma)} - \overline{f(\sigma\bar\zeta)}$ we see that $\|T_{\bar f}\|_{H^\infty} \leq I + J + K$, where

$$I = \sup\left\{\frac{1}{2\pi}\left|\int_\Gamma \frac{\overline{f(\sigma\zeta)} - 2\overline{f(\sigma)} + \overline{f(\sigma\bar\zeta)}}{1 - r\zeta}\frac{h(\sigma\zeta)}{\zeta}\,d\zeta\right|\right\},$$

$$J = \sup\left\{\frac{1}{2\pi}\left|\int_\Gamma \frac{2\overline{f(\sigma)}}{1 - r\zeta}\frac{h(\sigma\zeta)}{\zeta}\,d\zeta\right|\right\},$$

and

$$K = \sup\left\{\frac{1}{2\pi}\left|\int_\Gamma \frac{\overline{f(\sigma\bar\zeta)}}{1 - r\zeta}\frac{h(\sigma\zeta)}{\zeta}\,d\zeta\right|\right\},$$

again where $\|h\|_{H^\infty} \leq 1$, $0 \leq r < 1$ and $|\sigma| = 1$.

Let $\zeta = e^{is}$ and $\sigma = e^{it}$. We use Lemma 1 as follows.

$$\begin{aligned} I &\leq \sup\left\{\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{|f(e^{i(t+s)}) - 2f(e^{it}) + f(e^{i(t-s)})|}{|1 - re^{-is}|}\,ds : 0 \leq r < 1,\ |t| \leq \pi\right\} \\ &\leq \sup\left\{\frac{1}{\pi}\int_0^{\pi}\frac{|D(f;t,s)|}{|1 - re^{-is}|}\,ds : 0 \leq r < 1,\ |t| \leq \pi\right\} \\ &\leq \sup\left\{\int_0^{\pi}\frac{|D(f;t,s)|}{s}\,ds : |t| \leq \pi\right\} \\ &= K_1(f) \end{aligned}$$

Also we have

$$\begin{aligned} J &\leq 2\|f\|_{H^\infty}\sup\left\{\frac{1}{2\pi}\left|\int_\Gamma \frac{h(\sigma\zeta)}{(1 - r\zeta)\zeta}\,d\zeta\right| : \|h\|_{H^\infty} \leq 1,\ 0 \leq r < 1,\ |\sigma| = 1\right\} \\ &= 2\|f\|_{H^\infty}\sup\left\{\left|\frac{1}{2\pi i}\int_\Gamma \frac{h(w)}{w - r\sigma}\,dw\right| : \|h\|_{H^\infty} \leq 1,\ 0 \leq r < 1, |\sigma| = 1\right\}. \end{aligned}$$

Hence Cauchy's formula implies that

$$\begin{aligned} J &\leq 2\|f\|_{H^\infty}\sup\left\{|h(r\sigma)| : \|h\|_{H^\infty} \leq 1,\ 0 \leq r < 1,\ |\sigma| = 1\right\} \\ &= 2\|f\|_{H^\infty}. \end{aligned}$$

We also use Cauchy's formula to estimate K as follows. Note that the function g defined by $g(w) = \overline{f(\bar w)}$ for $|w| < 1$ belongs to H^∞. Hence the change of variables $w = \bar\sigma\zeta$ gives

$$\begin{aligned} \frac{1}{2\pi i}\int_\Gamma \frac{\overline{f(\sigma\bar\zeta)}h(\sigma\zeta)}{(1 - r\zeta)\zeta}\,d\zeta &= \frac{1}{2\pi i}\int_\Gamma \frac{g(w)h(\sigma^2 w)}{w - r\bar\sigma}\,dw \\ &= g(r\bar\sigma)h(r\sigma) = \overline{f(r\sigma)}h(r\sigma). \end{aligned}$$

Therefore $K \leq \|f\|_{H^\infty}$.

Combining the inequalities for I, J and K derived above we obtain $\|T_{\bar f}\|_{H^\infty} \leq K_1(f) + 3\|f\|_{H^\infty}$. Hence $\|f\|_{M_1} \leq K_1(f) + 3\|f\|_{H^\infty} < \infty$, and therefore $f \in \mathcal{M}_1$. This completes the proof of Theorem 3.

https://doi.org/10.4153/CJM-1998-033-3 Published online by Cambridge University Press

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https://doi.org/10.4153/CJM-1998-033-3 Published online by Cambridge University Press