THERMAL INSTABILITY OF COUETTE FLOW

S. C. AGRAWAL and J. C SAINI

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Summary

The present paper investigates the thermal instability of a non-homogeneous fluid rotating between two co-axial cylinders when the inner cylinder is being heated uniformly. The conditions are established under which the oscillatory and non-oscillatory modes exist and further it has been shown that the oscillatory modes are amplified due to the adverse temperature gradient. In the case of non-oscillatory modes, sufficient conditions for stability and instability are obtained.

1. Introduction

Hydrodynamic stability of Couette flow is a well understood problem and after Lord Rayleigh showed that the necessary and sufficient condition for stability of an incompressible non-viscous fluid rotating between two co-axial cylinders is that $\Phi(r)$ where $\Phi(r) = (2/r)\Omega(d/dr)(r^2\Omega)$, and Ω is the angular velocity of the fluid at a distance r from the axis of rotation, should be everywhere positive, many authors contributed to its further investigations. Howard and Gupta (1962) have investigated the hydromagnetic stability of heterogeneous, incompressible and non-viscous fluid between two co-axial cylinders. Agrawal (1969) re-investigated the results of Howard and Gupta and modified some of their results. Rudraiah (1970) carried out the stability analysis of axial flow of heterogeneous, incompressible and electrically conducting fluid between two fixed co-axial cylinders. DiPrima (1961), Krueger and DiPrima (1962) and Agrawal (1970) have gone further into the analysis by allowing asymmetric perturbations. The onset of thermal instability in a static horizontal layer of homogeneous fluid kept under a uniform temperature gradient has been investigated by many authors.

The assumption, as taken by these authors, of homogeneity and uniformity in temperature of fluid in every physical system is not encouraging. In fact it is not necessary that every fluid be homogeneous and of uniform temperature.

With these motivations, Banerjee (1971) investigated the onset of thermal instability of a horizontal fluid layer with a basic density stratification.

In the present paper we have investigated the stability of an incompressible, heterogeneous and non-viscous fluid rotating between two co-axial cylinders maintained at different constant temperatures. The temperature of the inner cylinder is assumed to be greater than that of the outer one and gravitational force is acting in the direction opposite to the radial direction. The basic density is of the exponential type decreasing with increasing r. The investigations are restricted to axisymmetric perturbations and the normal mode technique is used. Throughout the present paper, the narrow gap approximation is utilized to simplify the equations governing the stability problem.

2. The physical problem

An incompressible, non-viscous, thermally conducting and heterogeneous fluid fills the gap between two co-axial infinitely long circular cylinders maintained at constant temperatures. Let T_1 and T_2 be the temperatures of inner and outer cylinders and R'_1, R'_2 be their respective radii. The inner cylinder is being heated uniformly so $T_1 > T_2$. The fluid is rotating with angular velocity which depends upon r only.

The basic non-homogeneity is of the exponential type, i.e. $e^{-\beta r}$, $\beta > 0$ being a constant so that the density decreases in the increasing r-direction. The gravitational force is acting in the direction opposite to the radial direction. The initial state we intend to investigate is, therefore, one in which the velocity, temperature, density and pressure at any point in the fluid in cylindrical polar co-ordinates (r, θ, z) are respectively given by

$$\begin{aligned} \underline{v} &= (0, V(r), 0) \\ T &= T(r) \\ \rho &= \rho(r) \\ p &= p(r) \end{aligned} \tag{1}$$

where

(i) Euler's equations of motion show that $V(r)$ is an arbitrary function of r,
(ii) $T(r) = -Ar + B$,

where

$$A = \frac{T_1 - T_2}{R'_2 - R'_1} > 0,$$

and

$$B = \frac{R'_1 T_2 - R'_2 T_1}{R'_1 - R'_2}$$

as given by the heat conduction equation,

(iii) following Banerjee (1971), the resultant density distribution arising due to the interaction of basic and thermal stratification is taken to be

$$\rho(r) = \rho_0(e^{-\beta r} + \alpha[T_1 - T]),$$

where α is the coefficient of volume expansion and

(iv) the pressure $p(r)$ is given by

$$p(r) = \int (\rho_0 V^2(r)/r - g\rho)\, dr.$$

Throughout the present analysis the equations of motion are simplified by assuming the Boussinesq's approximation which says that in most of the cases of practical importance the density variations can be neglected everywhere in the equations of motion except in its association with the external forces. Also we have assumed the gap between the cylinders to be small in comparison to their mean radii so that the terms of order (d/R_0), where

$$R_0 = \frac{R_1' + R_2'}{2}, \qquad d = R_2' - R_1'$$

can be safely neglected and the two operators D and $D*$ become identical, i.e, $D \equiv D*$.

Let the basic state characterized by (1) be slightly perturbed so that the perturbed state is given by

$$\begin{aligned} \underset{\sim}{v}' &= (u_r', V(r) + u_\theta', u_z') \\ T' &= T(r) + \Theta' \\ \rho' &= \rho_0\{e^{-\beta r} + \delta\rho'/\rho_0 + \alpha(T_1 - T - \Theta')\} \\ p' &= p(r) + \delta p' \end{aligned} \tag{2}$$

where (u_r', u_θ', u_z'), Θ', $\delta p'$ are respectively the perturbations in velocity field, temperature and pressure while $\delta\rho'$ is the perturbation in the density because of the basic heterogeneity assumed and Θ' accounts for the perturbation in density due to the perturbation in temperature. Perturbations are taken to be axisymmetric and any disturbance is decomposed in normal modes where the dependence of any perturbation quantity $f'(r, z, t)$ on r, z and t is taken to be of the form

$$f(r)e^{i(nt + kz)}$$

where κ is real and n, in general, is complex. The linearised equations governing the system are

$$\rho_0(inu_r - 2\frac{V}{r}u_\theta) = -D\tilde{\omega} + g\alpha\rho_0\Theta - g\delta\rho \tag{3}$$

$$\rho_0(inu_\theta + DV\lambda . u_r) = 0 \tag{4}$$

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(5) $$\rho_0 inu_z = -i\kappa\bar{w}$$

(6) $$Du_r + i\kappa u_z = 0$$

(7) $$in\delta\rho - \beta u_r \rho_0 e^{-\beta r} = 0$$

and

(8) $$in\Theta - Au_r = k(D^2 - \kappa^2)\Theta$$

where k is the thermometric conductivity and $\bar{w}$ is the amplitude function of perturbation in pressure.

On eliminating u_0, u_z and $\bar{w}$ from equations (3)–(7) and making the quantities non-dimensional, the final equations governing the stability problem are

(9) $$\sigma^2(D^2 - a^2)u_r + a^2F(r)u_r = ia^2\sigma\Theta$$

(10) $$(D^2 - a^2 - i\sigma)\Theta = -R_2 u_r$$

subjected to the boundary conditions

(11) $$u_r = \Theta = 0 \text{ at } r = R_1' \text{ and } r = R_2'$$

where

$$F(r) = \Phi^*(r) + R_1 e^{-\beta r},$$

$$(D^*, a, \beta^*) = d(D, \kappa, \beta),$$

$$(r^*, R_1'^*, R_2'^*) = \frac{1}{d}(r, R_1', R_2'),$$

$$\sigma = nd^2 | k,$$

$$\Phi^* = \frac{d^4\Phi(r)}{k^2},$$

$$u_r^* = u_r k / g\alpha d^2$$

$$R_1 = \frac{g\beta d^4}{k^2} > 0$$

and $$R_2 = Ag\alpha d^4/k^2 > 0.$$

In (9), (10), and (11) we have omitted the asterisks.

3. Variational principle for σ

In this section we shall prove that the characteristic value problem formulated in §2 can be expressed in terms of a variational principle. Multiply equation (9) by u_r integrate over the range of r and make use of equation (10). This gives

$$\sigma^2\left(I_1 - a^2\frac{I_4}{R_2}\right) + i\sigma\frac{a^2}{R_2}I_3 - a^2\int_{R'_1}^{R'_2} F(r)u_r^2dr = 0, \tag{12}$$

where

$$\begin{aligned} I_1 &= \int_{R'_1}^{R'_2}[(Du_r)^2 + a^2u_r^2]dr \\ I_3 &= \int_{R'_1}^{R'_2}[(D\Theta)^2 + a^2\Theta^2]dr \\ I_4 &= \int_{R'_1}^{R'_2}\Theta^2dr. \end{aligned} \tag{13}$$

Equation (12) provides the basis for a variational principle. To see this consider the effect $\delta\sigma$ on σ of arbitrary variations δu_r and $\delta\Theta$ in u_r and Θ respectively in accordance with the equation (12) which are arbitrary except for the requirement that these variations satisfy the boundary conditions. We have to the first order in variations

$$\begin{aligned} \sigma^2\left(\delta I_1 - \frac{a^2}{R_2}\delta I_4\right) + i\sigma\frac{a^2}{R_2}\delta I_3 - 2a^2\int_{R'_1}^{R'_2} F(r)u_r\delta u_rdr \\ + \delta\sigma\left\{2\sigma\left(I_1 - \frac{a^2}{R_2}I_4\right) + i\frac{a^2}{R_2}I_3\right\} = 0 \end{aligned} \tag{14}$$

where

$$\delta I_1 = \delta\int_{R'_1}^{R'_2}[(Du_r)^2 + a^2u_r^2]dr = -2\int_{R'_1}^{R'_2}(D^2 - a^2)u_r\cdot\delta u_rdr$$

Similarly,

$$\delta I_3 = -2\int_{R_1}^{R'_2}(D^2 - a^2)\Theta\cdot\delta\Theta dr$$

and

$$\delta I_4 = 2\int_{R_1'}^{R_2'}\Theta\delta\Theta dr.$$

Inserting these expressions in equation (14) we have after some simplifications

$$\begin{aligned} -2\int_{R'_1}^{R'_2}[\sigma^2(D^2 - a^2)u_r + a^2F(r)u_r]\delta u_r\cdot dr + \delta\sigma\left\{2\sigma\left(I_1 - \frac{a^2I_4}{R_2}\right) + ia^2\frac{I_3}{R_2}\right\} \\ - 2i\sigma\frac{a^2}{R_2}\int_{R'_1}^{R'_2}(D^2 - a^2 - i\sigma)\delta\Theta\cdot\Theta dr. \end{aligned} \tag{15}$$

Further, the variations $\delta\sigma$, $\delta\Theta$ and δu_r are connected by the equation

$$(D^2 - a^2 - i\sigma)\delta\Theta = i\Theta\delta\sigma - R_2\delta u_r. \tag{16}$$

With the help of equation (16), equation (15) reduces to

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$$\delta\sigma\left[2\sigma I_1 + \frac{ia^2}{R_2} I_3\right] = 2\int_{R'_1}^{R'_2} \delta u_r[\sigma^2(D^2 - a^2)u_r + a^2 F(r)u_r - i\sigma a^2 \Theta]dr. \tag{17}$$

Equation (17) shows that the quantity which appears as a factor of δu_r under the integral sign vanishes if the equation (9) is satisfied. Hence a necessary and sufficient condition for $\delta\sigma$ to vanish identically to the first order for all small variations in u_r and Θ subjected only to the boundary conditions, is that u_r and Θ be the solutions of the characteristic value problem.

4. Existence of oscillatory and non-oscillatory modes

The form of the perturbations shows that for an oscillatory mode to exist σ_r, the real part of σ, must be non-zero. In the present section we shall be proving two theorems which will show that both types of modes-oscillatory and non-oscillatory ones-exist in the present problem.

THEOREM 1. *If the function $F(r)$ is negative everywhere in the flow domain, the modes whether stable or unstable are non-oscillatory.*

PROOF. We multiply equation (9) by u_r^*, the complex conjugate of u_r, integrate over the range of r and make use of equation (10). This yields

$$-\sigma^2 I_1 + a^2 I_2 = i\sigma\frac{a^2}{R_2}(I_3 - i\sigma^* I_4) \tag{18}$$

where

$$I_1 = \int_{R'_1}^{R'_2} (|Du_r|^2 + a^2|u_r|^2)dr > 0$$

$$I_2 = \int_{R'_1}^{R'_2} F(r)|u_r|^2 dr < 0$$

$$I_3 = \int_{R'_1}^{R'_2} (|D\Theta|^2 + a^2|\Theta|^2)dr > 0$$

$$I_4 = \int_{R'_1}^{R'_2} |\Theta|^2 dr > 0$$

and σ^* is the complex conjugate of σ.

Now assume on the contrary that oscillatory modes exist when $F(r)$ is negative everywhere, so that $\sigma_r \neq 0$ which in term implies that $\sigma \neq 0$. Equation (18) can be divided by σ and then we have

$$-\sigma I_1 + a^2\frac{\sigma^*}{|\sigma|^2} I_2 = i\frac{a^2}{R_2}(I_3 - i\sigma^* I_4). \tag{19}$$

Equating the real and imaginary parts in (19) we have respectively

(20) $$\sigma_r\left[-I_1 + a^2\frac{I_2}{|\sigma|^2} - a^2\frac{I_4}{R_2}\right] = 0$$

and

(21) $$-\sigma_i I_1 - \frac{a^2}{|\sigma|^2}\sigma_i I_2 = a^2\frac{I_3}{R_2} - a^2\frac{I_4}{R_2}\sigma_i .$$

Since $\sigma_r \neq 0$, equation (20) implies that

(22) $$-I_1 + a^2\frac{I_2}{|\sigma|^2} = a^2\frac{I_4}{R_2}.$$

This is impossible since $I_2 < 0$. Therefore, if $F(r) < 0$ everywhere, the system is necessarily non-oscillatory.

Alternative proof. Now we shall handle equations (21) and (22) to arrive at a contradiction. If we eliminate I_2 from equations (21) and (22), it gives

(23) $$\sigma_i I_1 = -a^2\frac{I_3}{2R_2}$$

implying thereby that $\sigma_i < 0$. If we further eliminate I_1 between (21) and (22), it gives

(24) $$2\sigma_i\left[\frac{a^2}{R_2}I_4 - a^2\frac{I_2}{|\sigma|^2}\right] = \frac{a^2}{R_2}I_3$$

and since $I_2 < 0$, it gives $\sigma_i > 0$.

Thus, if $F(r)$ is everywhere negative, equation (23) yields instability while equation (24) yields stability of the system and hence a contradiction to our assumption that modes are oscillatory. Therefore if $F(r)$ is negative everywhere in the flow domain, then the modes are non-oscillatory ones. Hence the theorem.

REMARK. The theorem utilizes the fact that I_2 should be negative. Even if the function $F(r)$ is not everywhere negative, I_2 can be negative in some cases, when $F(r)$ changes sign and the conclusion of the theorem will be valid.

THEOREM 2. *If the function $F(r)$ is everywhere positive then given any wave number, say a_0, the necessary and sufficient condition for the modes to be oscillatory is that R_2 must exceed some quantity P depending upon a_0, the basic density stratification and the rotational velocity $V(r)$.*

PROOF. In §3 it has been shown that the characteristic value problem can be expressed in terms of a variational principle and hence we assume solutions for u_r and Θ satisfying the boundary conditions only, namely,

$$u_r = \sin \pi r' \text{ and } \Theta = \sin \pi r'$$

where

$$r' = \frac{N(r - R_1')}{R'_2 - R'_1}, \qquad N = 1, 2, \cdots, (0 \leqq r' \leqq N), \tag{25}$$

$N = 1$ being the most critical value.

Putting this in equation (12) we have

$$\sigma^2 \left(\pi^2 + a^2 - \frac{a^2}{R_2}\right) + i\frac{a^2}{R_2}(\pi^2 + a^2)\sigma - a^2 L = 0, \tag{26}$$

where
$$L = \int_0^1 F(r') \sin^2 \pi r' dr'$$

and is positive from the given hypothesis. Extracting the roots for σ from equation (26), one gets

$$\sigma = \frac{-i\dfrac{a^2}{R_2}(\pi^2 + a^2) \pm \sqrt{-\dfrac{a^4}{R_2^2}(\pi^2 + a^2)^2 + 4a^2 L\left(\pi^2 + a^2 - \dfrac{a^2}{R_2}\right)}}{2\left(\pi^2 + a^2 - \dfrac{a^2}{R_2}\right)}. \tag{27}$$

Obviously for a given wave number a_0, $\sigma_r \neq 0$ if and only if the quantity inside the square root sign is positive. Now for any given wave number a_0 the quantity inside the square root sign is

$$\begin{aligned} &-\frac{a_0^4}{R_2^2}(\pi^2 + a_0^2)^2 + 4a_0^2 L\left(\pi^2 + a_0^2 - \frac{a_0^2}{R_2}\right) \\ &= \frac{4La_0^2(\pi^2 + a_0^2)}{R_2^2}\left\{R_2^2 - R_2\frac{a_0^2}{(\pi^2 + a_0^2)} - a_0^2\frac{(\pi^2 + a_0^2)}{4L}\right\} \\ &= \frac{4La_0^2(\pi \;\; + a_0^2)}{R_2^2}(R_2 - P)(R_2 + Q), \end{aligned} \tag{28}$$

where

$$\begin{aligned} P &= \frac{1}{2}\frac{a_0^2}{(\pi^2 + a_0^2)}\left[\sqrt{1 + \frac{(\pi^2 + a_0^2)^3}{La_0^2}} + 1\right] > 0 \\ Q &= \frac{1}{2}\frac{a_0^2}{(\pi^2 + a_0^2)}\left[\sqrt{1 + \frac{(\pi^2 + a_0^2)^3}{La_0^2}} - 1\right] > 0. \end{aligned} \tag{29}$$

Equation (28) clearly shows that $\sigma_r \neq 0$ if and only of $R_2 > P$. In other words, given any wave number say a_0, the necessary and sufficient condition for the existence of oscillatory modes is that R_2 should exceed the quantity P. Given the wave number, basic density distribution and the rotational velocity, P can be calculated from (29). Physically the condition $R_2 > P$, means that the temperature gradient A exceeds some critical temperature gradient A_c say. The theorem then states that, for a given wave number, the modes whether stable or unstable remain non-oscillatory so long as the temperature gradient A does not exceed A_c and it becomes oscillatory if A exceeds A_c.

In particular, if the rotational velocity $V(r)$ is zero identically, then

$$F(r) = R_1 e^{-\beta r} > 0$$

and the conclusions of the theorem 2 are still valid. Banerjee (1971) has investigated the stability of a continuously stratified layer of an incompressible, non-viscous fluid statically confined between two horizontal boundaries and heated underside. We note that there is a perfect analogy between the problem investigated by Banerjee and the present problem when $V(r) = 0$ (see equations (17) and (18) in Banerjee (1971) and equations (9) and (10) in the present paper when $V(r) = 0$), His doubt that oscillatory modes may not exist in the present situation, therefore, is not correct but he has rightly remarked that such modes when they exist are amplified due to the adverse temperature gradient.

5. Instability of oscillatory modes

That the oscillatory modes are unstable immediately follows from equation (23) which has been obtained under the assumption that $\sigma_r \neq 0$. In fact, equation (23) yields that

$$\sigma_i < 0$$

implying thereby the instability of the system.

The same conclusion can also be obtained in a different way. It has been shown from equation (27) that for a given wave number a_0, the modes are oscillatory if and only if $R_2 > P$. In the event of oscillatory modes, equation (27) shows that

$$\sigma_i = -\frac{a_0^2}{R_2^2} \frac{(\pi^2 + a_0^2)}{2\left(\pi^2 + a_0^2 - \dfrac{a_0^2}{R_2}\right)}$$

and since

$$R_2 > P \Rightarrow R_2 > \frac{a_0^2}{\pi^2 + a_0^2},$$

it follows that

$$\sigma_i < 0.$$

Thus the instability of the system. We therefore have the following theorem.

THEOREM 3. *The oscillatory modes exist in the present situation and are amplified due to the adverse temperature gradient.*

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6. The discussion of non-oscillatory modes

We eliminate Θ between equations (9) and (10) to get

$$\sigma^2[(D^2 - a^2)^2 - i\sigma(D^2 - a^2)]u_r + a^2(D^2 - a^2 - i\sigma)F(r)u_r = - i\sigma a^2 R_2 u_r \tag{30}$$

The vanishing of Θ and u_r at the boundary implies the vanishing of D^2u_r at the boundary. Hence the equation (30) has to be investigated together with

$$u_r = D^2u_r = 0 \text{ at } r = \frac{R_1'}{d}, \frac{R_2'}{d}. \tag{31}$$

Now multiply equation (30) by u_r^*, the complex conjugate of u_r and integrate over the range of r. This yields

$$\sigma^2 I_1 + i\sigma^3 I_2 + a^2 \int_{R'_1}^{R'_2} u_r^* D^2[F(r)u_r]dr - a^2(a^2 + i\sigma)\int_{R'_1}^{R'_2} F(r)|u_r|^2 dr = - i\sigma a^2 R_2 I_3, \tag{32}$$

where now

$$I_1 = \int_{R'_1}^{R'_2} (|D^2u_r|^2 + 2a^2|Du_r|^2 + a^4|u_r|^2)dr > 0$$

$$I_2 = \int_{R'_1}^{R'_2} (|Du_r|^2 + a^2(|u_r|^2)dr > 0$$

and

$$I_3 = \int_{R'_1}^{R'_2} |u_r|^2 dr > 0.$$

In the present section, we shall be interested in the investigation of non-oscillatory modes and hence $\sigma_r = 0$. The equation (32) reduces to

$$- \sigma_i^2 I_1 + \sigma_i^3 I_2 + a^2 \int_{R'_1}^{R'_2} u_r^* D^2[F(r)u_r]dr - a^2(a^2 - \sigma_i)\int_{R'_1}^{R'_2} F(r)|u_r|^2 dr = \sigma_i a^2 R_2 I_3. \tag{33}$$

It can easily be shown that

$$\operatorname{Re}\int_{R'_1}^{R'_2} u_r^* D^2[F(r)u_r]dr = -\int_{R'_1}^{R'_2} |Du_r|^2 F(r)dr + \frac{1}{2}\int_{R'_1}^{R'_2} D^2F(r)\cdot|u_r|^2 dr. \tag{34}$$

With the help of (34), the real part of equation (33) yields after some simplification that

$$\begin{aligned} &I_2\sigma_i^3 - I_1\sigma_i^2 + a^2\left\{\int_{R'1}^{R'2} [F(r) - R_2](|u_r|^2 dr\right\}\sigma_i \\ &-a^2\int_{R'_1}^{R'_2} [(|Du_r|^2 + a^2|u_r|^2)F(r) - \frac{1}{2}D^2F(r)\cdot|u_r|^2]dr = 0. \end{aligned} \tag{35}$$

From equation (35) we have

THEOREM 4. *The system is completely stabilized for all wave numbers provided the inequalities*

$$R_2 < F(r) \tag{36}$$
$$\text{and} \qquad D^2F(r) \leqq 0$$

hold everywhere in the flow domain.

PROOF. Under the conditions (36), Descartes' rule of sign says that equation (35) has at most 3 positive roots and no negative root. Either all the three roots are positive or else one is positive and the other two complex. In the later situation, we have to discard the complex roots since σ_i is purely real. In any case, since there is no negative value of σ_i allowed under the conditions (36), the system is completely stable for all wave numbers, and moreover each stable mode has degeneracy either one or three.

COROLLARY 1. *If $D^2F(r)$ is not everywhere negative, then the system is stabilized for all wave numbers given by*

$$a^2 > \frac{1}{2}\max\left\{\frac{|D^2(F(r))|}{F(r)}\right\}$$

in the flow domain provided $R_2 < F(r)$ everywhere.

Proof immediately follows from equation (35).

COROLLARY 2. *If $R_2 < F(r)$ everywhere and $D^2F(r)$ is not everywhere negative, then, if the system is unstable for the wave numbers*

$$a^2 < \frac{1}{2}\max\left\{\frac{|D^2(F(r))|}{F(r)}\right\},$$

their growth rate has an upper bound, namely,

$$|\sigma_i| < \frac{1}{2}\max\left\{\frac{|D^2(F(r))|}{F(r)-R_2}\right\}.$$

PROOF. The equation (35) can be written as

$$\sigma_i^3 I_2 - \sigma_i^2 I_1 + a^2 \int_{R'_1}^{R'_2}\left[\frac{1}{2}D^2(F(r)) - a^2F(r) + \sigma_i(F(r)-R_2)\right]|u_r|^2 dr$$
$$- a^2\int_{R'_1}^{R'_2} F(r)|u_r|^2 dr = 0$$

Clearly if $\sigma_i < 0$, we must have

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$$\frac{1}{2}D^2(F(r)) - a^2F(r) - |\sigma_i|(F(r) - R_2) > 0 \tag{37}$$

at least at one point in the flow region

$$\text{or} \qquad |\sigma_i| < \frac{1}{2}\max\left\{\frac{|D^2(F(r))|}{F(r) - R_2}\right\}.$$

THEOREM 5. *If* $F(r) < 0$ *everywhere and* $D^2F(r) > 0$ *everywhere, then the system is unstable.*

PROOF. It will suffice to show that under these conditions there is at least one negative value of σ_i allowed by equation (35). Let $\sigma_{i_1}, \sigma_{i_2}$ and σ_{i_3} be the roots of the cubic equation (35). Then the product

$$\sigma_{i_1}.\sigma_{i_2}.\sigma_{i_3} = \frac{a^2}{I_2}\int_{R'_1}^{R'_2}\left[(|Du_r|^2 + a^2|u_r|^2)F(r) - \frac{1}{2}D^2(F(r))|u_r|^2\right]dr < 0$$

(since $F(r) < 0$ everywhere and $D^2(F(r)) > 0$ everywhere).

Therefore, at least one root should be negative which implies the instability of the system.

COROLLARY 1. *If* $D^2F(r)$ *is not everywhere positive but instead either changes sign in the given interval or is negative everywhere then the short wave length perturbations given by*

$$a^2 > \frac{1}{2}\max\left\{\frac{|D^2(F(r))|}{|F(r)|}\right\}$$

are unstable provided $F(r)$ *is negative everywhere in the flow domain.*

PROOF. Immediately follows from equation (35).

THEOREM 6. *If* $F(r)$ *is everywhere negative, then the system is unstable provided* R_2 *exceeds* 1.

PROOF. The proof immediately follows from equation (27) which admits of a negative value of σ_i. In order that $F(r)$ is negative everywhere $\Phi(r)$ should be negative everywhere and moreover R_1 must not exceed the minimum of the quantity $|\Phi(r)|e^{\beta r}$ in the flow domain. Physically the theorem states that if the rotational velocity and the basic density stratification are prescribed in the above manner, the system becomes unstable provided the temperature gradient A exceeds some critical temperature gradient. Further it also follows from equation (27) that under these conditions the modes are non-oscillatory.

7. Discussion

The effect of heating on the stability of basically stratified Couette flow is investigated in the present paper. The study is devoted to the characterisation of oscillatory and non-oscillatory modes and an attempt has been to achieve some instability and stability criteria for these oscillatory and non-oscillatory modes. The heating of the inner cylinder has two effects.

(i) Oscillatory modes are introduced in the system which were otherwise non-oscillatory, and

(ii) It has a destabilizing effect on the stability introduced by the basic stabilizing density stratification.

Instability of the oscillatory modes is established in all situations and in the case of non-oscillatory modes, we have obtained the sufficient conditions of stability and instability.

The existence of oscillatory modes depends upon the adverse temperature gradient. Specifically the oscillatory modes exist in the present problem if and only if some critical temperature gradient is being exceeded. The conditions obtained clearly show a stabilizing character of basic density stratification and a destabilizing effect of adverse heating. The results depend both qualitatively and quantitatively on rotation, basic density stratification and the adverse heating.

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Department of Mathematics
Institute of Advanced Studies
Meerut University
Meerut, India
2500001

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