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Recurrence of Cosine Operator Functions on Groups

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Abstract. In this note, we study the recurrence and topologically multiple recurrence of a sequence of operators on Banach spaces. In particular, we give a sufficient and necessary condition for a cosine operator function, induced by a sequence of operators on the Lebesgue space of a locally compact group, to be topologically multiply recurrent.

1 Introduction

In [10, 11], Costakis, Manoussos, and Parissis studied the notions of recurrence and topologically multiple recurrence for linear operators acting on Banach spaces, and characterized recurrence and topologically multiple recurrence for several classes of linear operators, including weighted shifts on $\ell^p(\mathbb{Z})$, which are a special case of weighted translation operators on the Lebesgue space of a locally compact group. Inspired by the works in [10,11], we recently [8] characterized topologically multiply recurrent weighted translations on locally compact groups, generalizing some results in [10,11]. Based on previous works about cosine operator functions [4–7], the purpose of this note is to give a sufficient and necessary condition for a sequence of operators generated by weighted translations to be topologically multiply recurrent in terms of the weight function, the Haar measure, and the group element. Such a sequence of operators can be regarded as a cosine operator function.

In topological dynamics, an operator T on a Banach space X is called *topologically multiply recurrent* if for every positive integer N and every nonempty open set U in X, there is some $n \in \mathbb{N}$ such that $U \cap T^{-n}U \cap T^{-2n}U \cap \cdots \cap T^{-Nn}U \neq \varnothing$. If $N = 1$, then T is called *recurrent*; that is, T satisfies the condition $U \cap T^{-n}U \neq \varnothing$. It is easy to see that an operator T is recurrent if T is topologically transitive or mixing, which follows immediately from the definitions of topological transitivity and mixing. An operator T is *topologically transitive* if given two nonempty open subsets $U, V \subset X$, there is some $n \in \mathbb{N}$ such that $T^nU \cap V \neq \varnothing$. If $T^nU \cap V \neq \varnothing$ from some n onward, then T is called *topologically mixing*. For a single operator on a separable Banach space, topological transitivity is equivalent to hypercyclicity. An operator T on a Banach space X is called *hypercyclic* if there exists a vector $x \in X$ such that its orbit under T, denoted by $\mathrm{Orb}(x, T) := \{x, Tx, T^2x, \dots\}$, is dense in X, in which case x is said to

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be a *hypercyclic vector* of T. Hypercyclicity arises from the invariant subset problem in analysis, and is related to topological dynamics.

In linear dynamics and in topological dynamics, several authors extend the study of hypercyclicity and topological transitivity to the setting of a sequence of operators. A sequence of operators $(T_n)_{n\in\mathbb{N}_0}$ on a separable Banach space X is called *hypercyclic* if there exists an element $x \in X$ such that the orbit $\{T_n x : n \in \mathbb{N}_0\}$ is dense in X, where $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$ and $T_0 := I$ is the identity map. Also, $(T_n)_{n\in\mathbb{N}_0}$ is *topologically transitive* if given nonempty open sets $U, V \subset X$, we have $T_n(U) \cap V \neq \varnothing$ for some $n \in \mathbb{N}$. Similarly, $(T_n)_{n\in\mathbb{N}_0}$ is said to be *topologically mixing* if $(T_n)_{n\in\mathbb{N}_0}$ satisfies $T_n(U) \cap V \neq \varnothing$ from some n onward. Hypercyclicity and transitivity for a single operator and a sequence of operators have been studied intensely in the past three decades. We refer the reader to [1, 12] for a survey.

Likewise, it is natural to extend the study of recurrence for a single operator to the setting of a sequence of operators.

Definition 1.1 A sequence $(T_n)_{n\in\mathbb{N}_0}$ of operators on a Banach space X is called *recurrent* if for every nonempty open set $U \subset X$, there exists some $n \in \mathbb{N}$ such that

$$U \cap T_n^{-1}(U) \neq \varnothing.$$

A vector $x \in X$ is called *recurrent* for $(T_n)_{n\in\mathbb{N}_0}$ if there exists a strictly increasing sequence of positive integers $(n_k)_{k\in\mathbb{N}}$ such that $T_{n_k} x \to x$ as $k \to \infty$.

We note that transitivity implies recurrence for a sequence $(T_n)_{n\in\mathbb{N}_0}$.

Definition 1.2 A sequence $(T_n)_{n\in\mathbb{N}_0}$ of operators on a Banach space X is called *topologically multiply recurrent* if for every positive integer N and every nonempty open set U in X, there is some $n \in \mathbb{N}$ such that $U \cap T_n^{-1}U \cap T_{2n}^{-1}U \cap \cdots \cap T_{Nn}^{-1}U \neq \varnothing$.

It should be noted that some recurrent results for the case of a single operator in [10, Propositions 2.1 and 2.6] and [11, Proposition 3.6] can be generalized without difficulty to the case of a sequence of operators.

2 Recurrence of Cosine Operator Functions

Our study of the dynamics of cosine operator functions is motivated by the work in [3, 13, 14]. A *cosine operator function* on a Banach space X is a mapping $\mathcal{C}$ from the real line into the space of continuous operators on X satisfying $\mathcal{C}(0) = I$, and $2\mathcal{C}(t)\mathcal{C}(s) = \mathcal{C}(t+s) + \mathcal{C}(t-s)$ for all $s, t \in \mathbb{R}$. The latter equality is called the d'Alembert functional equation, which implies $\mathcal{C}(t) = \mathcal{C}(-t)$ for all $t \in \mathbb{R}$. In [3], Bonilla and Miana obtained a sufficient condition for a cosine operator function $\mathcal{C}(t)$ defined by

$$\mathcal{C}(t) = \frac{1}{2}\big(T(t) + T(-t)\big)$$

to be topologically transitive, where T is a strongly continuous translation group on some weighted Lebesgue space $L^p(\mathbb{R})$. Also, for a Borel measure μ and $\Omega \subset \mathbb{R}^d$, Kalmes characterized in [13] transitive cosine operator functions, generated by second

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order partial differential operators on $L^p(\Omega,\mu)$. Moreover, Kostić showed the main structural properties of hypercyclic and chaotic integrated C-cosine functions in [14].

In this section, we will continue our study in [4–7] of a sequence of special operators, generated by weighted translations on groups, and characterize recurrence for such a sequence of operators, which can be viewed as a cosine operator function.

In what follows, let G be a locally compact group with a right-invariant Haar measure λ, and denote by $L^p(G)$ $(1 \le p < \infty)$ the complex Lebesgue space with respect to λ.

A bounded function $w : G \to (0,\infty)$ is called a *weight* on G. Let $a \in G$ and let δ_a be the unit point mass at a. A *weighted translation operator* on G is a weighted convolution operator $T_{a,w}: L^p(G) \longrightarrow L^p(G)$ defined by

$$T_{a,w}(f) := wT_a(f) \qquad \big(f \in L^p(G)\big),$$

where w is a weight on G and $T_a(f) = f * \delta_a \in L^p(G)$ is the convolution:

$$(f*\delta_a)(x) := \int_G f(xy^{-1})\,d\delta_a(y) = f(xa^{-1}) \qquad (x \in G).$$

If $w^{-1} \in L^\infty(G)$, then the inverse of $T_{a,w}$ is $T_{a^{-1},w^{-1}*\delta_{a^{-1}}}$, which is also a weighted translation. To simplify notation, we write $S_{a,w}$ for $T_{a^{-1},w^{-1}*\delta_{a^{-1}}}$. We assume throughout that $w, w^{-1} \in L^\infty(G)$ and then define a sequence of bounded linear operators $C_n: L^p(G) \to L^p(G)$ by

$$C_n = \frac{1}{2}(T^n_{a,w} + S^n_{a,w})$$

for all $n \in \mathbb{Z}$, where $T^{-n}_{a,w} := (T^{-1}_{a,w})^n = S^n_{a,w}$. Then $(C_n)_{n\in\mathbb{Z}}$ is a cosine operator function by letting $\mathcal{C}(n) = C_n$. Using the fact that $C_n = C_{-n}$ for all $n \in \mathbb{Z}$, we will only consider the sequence of operators $(C_n)_{n\in\mathbb{N}_0}$, and characterize topologically multiple recurrence of the sequence $(C_n)_{n\in\mathbb{N}_0}$ in terms of the weight function w, the element a and the Haar measure λ.

We recall that an element a in a group G is called a *torsion element* if it is of finite order. In a locally compact group G, an element $a \in G$ is called *periodic* [9] if the closed subgroup $G(a)$ generated by a is compact. We call an element in G *aperiodic* if it is not periodic. For discrete groups, periodic and torsion elements are identical; in other words, aperiodic elements are the non-torsion elements.

We will make use of the property of aperiodicity to achieve our results. It was shown in [9, Lemma 2.1] that an element a in a locally compact group G is aperiodic if, and only if, for any compact subset $K \subset G$, there exists $N \in \mathbb{N}$ such that $K \cap Ka^n = \varnothing$ (equivalently, $K \cap Ka^{-n} = \varnothing$) for $n > N$. We note that [9, Remark 2.2] in many familiar non-discrete groups, including the additive group $\mathbb{R}^d$, the Heisenberg group, and the affine group, all elements except the identity are aperiodic.

We are ready to characterize topologically multiply recurrent cosine operator functions $(C_n)_{n\in\mathbb{N}_0}$, generated by an aperiodic group element $a \in G$. For $n \in \mathbb{N}$, let

$$\varphi_n = \prod_{s=1}^{n} w * \delta^s_{a^{-1}} \qquad \text{and} \qquad \widetilde{\varphi}_n = \Big(\prod_{s=0}^{n-1} w * \delta^s_a\Big)^{-1},$$

and for a Borel set $E \subset G$, define

$$\nu_n(E) = \int_E \varphi^p_n(x)\,d\lambda(x) \qquad \text{and} \qquad \widetilde{\nu}_n(E) = \int_E \widetilde{\varphi}^p_n(x)\,d\lambda(x).$$

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We are now ready to prove the main result.

Theorem 2.1 *Let G be a locally compact group and let a be an aperiodic element in G. Let $1 \le p < \infty$ and $w, w^{-1} \in L^\infty(G)$. Let $T_{a,w}$ be a weighted translation on $L^p(G)$ with inverse $S_{a,w}$, and let $C_n = \frac{1}{2}(T^n_{a,w} + S^n_{a,w})$. Then the following conditions are equivalent.*

(i) *$(C_n)_{n\in\mathbb{N}_0}$ is topologically multiply recurrent.*

(ii) *For each $N \in \mathbb{N}$ and each compact subset $K \subset G$ with $\lambda(K) > 0$, there are sequences of Borel sets $(E^+_{l,k})$ and $(E^-_{l,k})$ in K, and a sequence (n_k) of positive numbers such that for $E_k = E^+_{l,k} \cup E^-_{l,k}$, we have*

$$\lambda(K) = \lim_{k\to\infty} \lambda(E_k), \quad \lim_{k\to\infty} \nu_{ln_k}(E_k) = \lim_{k\to\infty} \widetilde{\nu}_{ln_k}(E_k) = 0,$$
$$\lim_{k\to\infty} \nu_{(N+l)n_k}(E^+_{l,k}) = \lim_{k\to\infty} \widetilde{\nu}_{(N+l)n_k}(E^-_{l,k}) = 0$$

for all $1 \le l \le N$.

Proof (ii) ⇒ (i). We show that (C_n) is topologically multiply recurrent for some N. Let U be a nonempty open subset of $L^p(G)$. Since the space $C_c(G)$ of continuous functions on G with compact support is dense in $L^p(G)$, we can pick $f \in C_c(G)$ with $f \in U$. Let K be the compact support of f. By aperiodicity of a, there exists $M \in \mathbb{N}$ such that $K \cap Ka^{\pm n} = \varnothing$ for all $n > M$. Now let $E_k \subset K$ and the subsequences (ν_{ln_k}) and $(\widetilde{\nu}_{ln_k})$ satisfy condition (ii). First, we show that

$$\left\| T^{ln_k}_{a,w}(f\chi_{E_k})\right\|_p \to 0 \quad \text{and} \quad \left\| T^{(N+l)n_k}_{a,w}(f\chi_{E^+_{l,k}})\right\|_p \to 0$$

as $k \to \infty$, for $1 \le l \le N$.

Let $\varepsilon > 0$. Then there exists some $M' \in \mathbb{N}$ such that $n_k > M$, $\nu_{ln_k}(E_k) < \varepsilon/\|f\|^p_\infty$, and $\nu_{(N+l)n_k}(E^+_{l,k}) < \varepsilon/\|f\|^p_\infty$ for $k > M'$. Hence,

$$\begin{aligned}\|T^{ln_k}_{a,w}(f\chi_{E_k})\|^p_p &= \int_{E_k a^{ln_k}} |w(x)w(xa^{-1})\cdots w(xa^{-(ln_k-1)})|^p |f(xa^{-ln_k})|^p\, d\lambda(x)\\ &= \int_{E_k} |w(xa^{ln_k})w(xa^{ln_k-1})\cdots w(xa)|^p|f(x)|^p\, d\lambda(x)\\ &= \int_{E_k} \varphi^p_{ln_k}(x)|f(x)|^p\, d\lambda(x) = \nu_{ln_k}(E_k)\|f\|^p_\infty < \varepsilon\end{aligned}$$

and

$$\begin{aligned}&\left\| T^{(N+l)n_k}_{a,w}(f\chi_{E^+_{l,k}})\right\|^p_p\\ &\quad= \int_{E^+_{l,k}a^{(N+l)n_k}} \left|w(x)w(xa^{-1})\cdots w(xa^{-((N+l)n_k-1)})\right|^p \left|f(xa^{-(N+l)n_k})\right|^p d\lambda(x)\\ &\quad= \int_{E^+_{l,k}} \left|w(xa^{(N+l)n_k})w(xa^{(N+l)n_k-1})\cdots w(xa)\right|^p \left|f(x)\right|^p d\lambda(x)\\ &\quad= \int_{E^+_{l,k}} \varphi^p_{(N+l)n_k}(x)\left|f(x)\right|^p d\lambda(x) = \nu_{(N+l)n_k}(E^+_{l,k})\|f\|^p_\infty < \varepsilon\end{aligned}$$

for $k > N$.

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Applying similar arguments to $S_{a,w}^{ln_k}$ and $S_{a,w}^{(N+l)n_k}$ and using the sequences $(\widetilde{v}_{ln_k})$ and $(\widetilde{v}_{(N+l)n_k})$, yield

$$\lim_{k\to\infty}\|S_{a,w}^{ln_k}(f\chi_{E_k})\|_p^p = \lim_{k\to\infty}\int_{E_k a^{-ln_k}}\frac{1}{|w(xa)w(xa^2)\cdots w(xa^{ln_k})|^p}|f(xa^{ln_k})|^p d\lambda(x) = 0$$

and

$$\begin{aligned}&\lim_{k\to\infty}\|S_{a,w}^{(N+l)n_k}(f\chi_{E_{l,k}^-})\|_p^p\\&=\lim_{k\to\infty}\int_{E_{l,k}^- a^{-(N+l)n_k}}\frac{1}{|w(xa)w(xa^2)\cdots w(xa^{(N+l)n_k})|^p}|f(xa^{(N+l)n_k})|^p d\lambda(x)\\&=0.\end{aligned}$$

For each $k\in\mathbb{N}$, we let

$$\begin{aligned}v_k = f\chi_{E_k} + T_{a,w}^{n_k}(f\chi_{E_{1,k}^+}) + T_{a,w}^{2n_k}(f\chi_{E_{2,k}^+}) + \cdots + T_{a,w}^{Nn_k}(f\chi_{E_{N,k}^+})\\ + S_{a,w}^{n_k}(f\chi_{E_{1,k}^-}) + S_{a,w}^{2n_k}(f\chi_{E_{2,k}^-}) + \cdots + S_{a,w}^{Nn_k}(f\chi_{E_{N,k}^-}).\end{aligned}$$

Since $Ka^{rn_k}\cap Ka^{sn_k}=\varnothing$ for $r,s\in\mathbb{Z}$ with $r\neq s$, we have

$$\|v_k - f\|_p^p \le \|f\|_\infty^p\lambda(K\smallsetminus E_k) + \sum_{s=1}^N\|T_{a,w}^{sn_k}(f\chi_{E_{s,k}^+})\|_p^p + \sum_{s=1}^N\|S_{a,w}^{sn_k}(f\chi_{E_{s,k}^-})\|_p^p$$

and, by the inequality $\|f+g\|_p^p\le 2^p\|f\|_p^p + 2^p\|g\|_p^p$,

$$\begin{aligned}&\|C_{ln_k}v_k - f\|_p^p\\&\quad\le \|T_{a,w}^{ln_k}v_k - f\|_p^p + \|S_{a,w}^{ln_k}v_k - f\|_p^p\\&\quad\le \|f\|_\infty^p\lambda(K\smallsetminus E_k) + \sum_{s=1}^N\|T_{a,w}^{sn_k}(f\chi_{E_k})\|_p^p + \sum_{s=N+1}^{N+l}\|T_{a,w}^{sn_k}(f\chi_{E_{s-l,k}^+})\|_p^p\\&\qquad+\sum_{s=1}^{N-l}\|S_{a,w}^{sn_k}(f\chi_{E_k})\|_p^p + \|f\|_\infty^p\lambda(K\smallsetminus E_k)\\&\qquad+\sum_{s=1}^{N}\|S_{a,w}^{sn_k}(f\chi_{E_k})\|_p^p + \sum_{s=N+1}^{N+l}\|S_{a,w}^{sn_k}(f\chi_{E_{s-l,k}^-})\|_p^p + \sum_{s=1}^{N-l}\|T_{a,w}^{sn_k}(f\chi_{E_k})\|_p^p.\end{aligned}$$

Hence, $\lim_{k\to\infty}v_k = f$ and $\lim_{k\to\infty}C_{ln_k}v_k = f$, which imply

$$U\cap C_{n_k}^{-1}(U)\cap C_{2n_k}^{-1}(U)\cap\cdots\cap C_{Nn_k}^{-1}(U)\neq\varnothing$$

for some k.

(i) ⇒ (ii). Let (C_n) be topologically multiply recurrent for some N. Let $K\subset G$ be a compact set with $\lambda(K)>0$. Let $\chi_K\in L^p(G)$ be the characteristic function of K. By aperiodicity of a, there is some M such that $K\cap Ka^{\pm n}=\varnothing$ for $n>M$.

Let $\varepsilon\in(0,1)$, and let $U=\{g\in L^p(G):\|g-\chi_K\|_p<\varepsilon^2\}$. Then, by the topologically multiply recurrent assumption, there exists $m>M$ such that

$$U\cap C_m^{-1}(U)\cap C_{2m}^{-1}(U)\cap\cdots\cap C_{Nm}^{-1}(U)\neq\varnothing.$$

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Hence, there exists a vector $f \in L^p(G)$ such that

$$\|f - \chi_K\|_p < \varepsilon^2 \quad \text{and} \quad \|C_{lm} f - \chi_K\|_p < \varepsilon^2$$

for $1 \le l \le N$. By the continuity of the mapping $h \in L^p(G, \mathbb{C}) \mapsto Re\, h \in L^p(G, \mathbb{R})$ and the fact that $T_{a,w}$ and $S_{a,w}$ commute with it, we can assume without loss of generality that f is real-valued.

Let $A = \{x \in K : |f(x) - 1| \ge \varepsilon\}$. Then

$$\varepsilon^{2p} > \|f - \chi_K\|_p^p \ge \int_A |f(x) - 1|^p d\lambda(x) \ge \varepsilon^p \lambda(A),$$

giving $\lambda(A) < \varepsilon^p$. Similarly, one has $\lambda(\{x \in K : |C_{lm} f(x) - 1| \ge \varepsilon\}) < \varepsilon^p$. Now setting $E := \cap_{l=1}^N \{x \in K : |C_{lm} f(x) - 1| < \varepsilon\} \bigcap \{x \in K : |f(x) - 1| < \varepsilon\}$, it follows that

$$\lambda(K \setminus E) < (N+1)\varepsilon^p, \; f(x) > 1 - \varepsilon > 0, \text{ and } C_{lm} f(x) > 1 - \varepsilon > 0 \; (1 \le l \le N, x \in E).$$

Combining the above, the invariance of Haar measure λ, and $K \cap Ka^{\pm m} = \varnothing$, we arrive at

$$\begin{aligned}
2^p \varepsilon^{2p} > \|2C_{lm} f - 2\chi_K\|_p^p &\ge \|T_{a,w}^{lm} f \chi_K + S_{a,w}^{lm} f \chi_K - 2\chi_K\|_p^p \\
&\ge \int_{Ka^{lm}} \big| T_{a,w}^{lm} f \chi_K(x) + S_{a,w}^{lm} f \chi_K(x) \big|^p d\lambda(x) \\
&= \int_K \big| T_{a,w}^{lm} f \chi_K(xa^{lm}) + S_{a,w}^{lm} f \chi_K(xa^{lm}) \big|^p d\lambda(x) \\
&= \int_K \big| \varphi_{lm}(x) f \chi_K(x) + \varphi_{lm}^{-1}(xa^{lm}) f \chi_K(xa^{2lm}) \big|^p d\lambda(x) \\
&\ge \int_E \big| \varphi_{lm}(x) f \chi_K(x) + \varphi_{lm}^{-1}(xa^{lm}) f \chi_K(xa^{2lm}) \big|^p d\lambda(x) \\
&= \int_E \big| \varphi_{lm}(x) f \chi_K(x) \big|^p d\lambda(x) > (1-\varepsilon)^p \int_E \varphi_{lm}^p(x) d\lambda(x) \\
&= (1-\varepsilon)^p \nu_{lm}(E),
\end{aligned}$$

which implies $\lim_{k\to\infty} \nu_{ln_k}(E_k) = 0$. By the similar argument, one can obtain $\lim_{k\to\infty} \widetilde{\nu}_{ln_k}(E_k) = 0$.

Now define $E_l^- = \{x \in E : T_{a,w}^{lm} f(x) > 1 - \varepsilon\}$ and $E_l^+ = E \setminus E_l^-$. Then for $x \in E_l^+$, one has $S_{a,w}^{lm} f(x) > 1 - \varepsilon$, which follows from the fact

$$1 - \varepsilon < C_{lm} f(x) = \frac{1}{2} T_{a,w}^{lm} f(x) + \frac{1}{2} S_{a,w}^{lm} f(x) \le \frac{1}{2}(1-\varepsilon) + \frac{1}{2} S_{a,w}^{lm} f(x).$$

For a Borel subset F of G, we have $\|C_n h \chi_F\|_p \le \|C_n h\|_p$ for arbitrary n and $h \in L^p(G, \mathbb{R})$. Obviously, the mapping $h \in L^p(G, \mathbb{R}) \mapsto h^- \in L^p(G, \mathbb{R})$ satisfies $\|(h+g)^-\|_p \le \|h^- + g^-\|_p$ and commutes with $T_{a,w}$ and $S_{a,w}$, where $h^- := \max\{0, -h\}$. Therefore, we have

$$\begin{aligned}
\|(C_{lm} f^-)\chi_F)\|_p \le \|(C_{lm} f)^-\|_p &= \|(C_{lm} f - \chi_K + \chi_K)^-\|_p \\
&\le \|(C_{lm} f - \chi_K)^-\|_p + \|\chi_K^-\|_p \\
&= \|(C_{lm} f - \chi_K)^-\|_p \le \|C_{lm} f - \chi_K\|_p < \varepsilon^2.
\end{aligned}$$

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By $K \cap Ka^m = \varnothing$, the invariance of Haar measure λ, the positivity of $T_{a,w}^{Nm} f^+$ and $S_{a,w}^{Nm} f^+$, and the inequality $\|f+g\|_p^p \le 2^p \|f\|_p^p + 2^p \|g\|_p^p$, we have

$$
\begin{aligned}
&(1-\varepsilon)^p \nu_{(N+l)m}(E_l^+) \\
&= (1-\varepsilon)^p \int_{E_l^+} \varphi_{(N+l)m}^p(x)\, d\lambda(x) \\
&< \int_{E_l^+} |w(xa^{(N+l)m}) w(xa^{(N+l)m-1}) \cdots w(xa)|^p |S_{a,w}^{lm} f^+(x)|^p\, d\lambda(x) \\
&= \int_{E_l^+ a^{(N+l)m}} |w(x) w(xa^{-1}) \cdots w(xa^{-((N+l)m-1)})|^p |S_{a,w}^{lm} f^+(xa^{-(N+l)m})|^p\, d\lambda(x) \\
&= \int_{E_l^+ a^{(N+l)m}} |T_{a,w}^{(N+l)m} S_{a,w}^{lm} f^+(x)|^p\, d\lambda(x) \\
&= \int_{E_l^+ a^{(N+l)m}} |T_{a,w}^{Nm} f^+(x)|^p\, d\lambda(x) \le 2^p \int_{E_l^+ a^{(N+l)m}} |C_{Nm} f^+(x)|^p\, d\lambda(x) \\
&= 2^p \|(C_{Nm} f^+) \chi_{E_l^+ a^{(N+l)m}}\|_p^p = 2^p \|(C_{Nm}(f^- + f)) \chi_{E_l^+ a^{(N+l)m}}\|_p^p \\
&= 2^p \|(C_{Nm} f^-) \chi_{E_l^+ a^{(N+l)m}} + (C_{Nm} f - \chi_K) \chi_{E_l^+ a^{(N+l)m}} + \chi_{K \cap E_l^+ a^{(N+l)m}}\|_p^p \\
&\le 2^p (2^{2p} \|(C_{Nm} f^-) \chi_{E_l^+ a^{(N+l)m}}\|_p^p + 2^{2p} \|C_{Nm} f - \chi_K\|_p^p + 2^{2p} \|\chi_{K \cap E_l^+ a^{(N+l)m}}\|_p^p) \\
&\le 2^{3p} (2\varepsilon^{2p} + 0)
\end{aligned}
$$

which implies $\lim_{k\to\infty} \nu_{(N+l)m}(E_l^+) = 0$. Similarly, one has

$$\lim_{k\to\infty} \widetilde{\nu}_{(N+l)m}(E_l^-) = 0. \qquad \blacksquare$$

It should be noted that the Haar measure is the counting measure if the group G is discrete. Therefore, $K = E_k$ for all $k \in \mathbb{N}$ sufficiently large in the proof of Theorem 2.1. Hence, we have the characterization below for discrete groups.

Corollary 2.2 *Let G be a discrete group and let a be a non-torsion element in G. Let $1 \le p < \infty$ and $w, w^{-1} \in \ell^\infty(G)$. Let $T_{a,w}$ be a weighted translation on $\ell^p(G)$ with inverse $S_{a,w}$, and let $C_n = \frac{1}{2}(T_{a,w}^n + S_{a,w}^n)$. Then the following conditions are equivalent.*

- (i) *$(C_n)_{n\in\mathbb{N}_0}$ is topologically multiply recurrent.*
- (ii) *For each $N \in \mathbb{N}$ and each non-empty finite subset $K \subset G$, there are sequences $(E_{l,k}^+)$ and $(E_{l,k}^-)$ of subsets of K, and a sequence (n_k) of positive numbers such that for $K = E_{l,k}^+ \cup E_{l,k}^-$, we have*

$$\lim_{k\to\infty} \|\varphi_{ln_k}|_K\|_\infty = \lim_{k\to\infty} \|\widetilde{\varphi}_{ln_k}|_K\|_\infty = 0,$$

$$\lim_{k\to\infty} \|\varphi_{(N+l)n_k}|_{E_{l,k}^+}\|_\infty = \lim_{k\to\infty} \|\widetilde{\varphi}_{(N+l)n_k}|_{E_{l,k}^-}\|_\infty = 0$$

for all $1 \le l \le N$.

Proof (ii) $\Rightarrow$ (i). One can show that $(C_n)_{n\in\mathbb{N}_0}$ is topologically multiply recurrent by using $(\varphi_{ln_k}), (\widetilde{\varphi}_{ln_k})$ instead of $(\nu_{ln_k}), (\widetilde{\nu}_{ln_k})$ in the proof of Theorem 2.1.

For the converse, the proof is identical to that in Theorem 2.1. Indeed, by the fact that K is a finite subset of a discrete group G whose Haar measure is the counting

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measure, we have the inequality

$$\begin{aligned}2^p\varepsilon^{2p} > \|2C_{lm}f - 2\chi_K\|_p^p &\geq \|T_{a,w}^{lm}f\chi_K + S_{a,w}^{lm}f\chi_K - 2\chi_K\|_p^p \\ &\geq \sum_{Ka^{lm}} |T_{a,w}^{lm}f\chi_K(x) + S_{a,w}^{lm}f\chi_K(x)|^p \\ &= \sum_K |T_{a,w}^{lm}f\chi_K(xa^{lm}) + S_{a,w}^{lm}f\chi_K(xa^{lm})|^p \\ &= \sum_K |\varphi_{lm}(x)f\chi_K(x) + \varphi_{lm}^{-1}(xa^{lm})f\chi_K(xa^{2lm})|^p \\ &= \sum_K |\varphi_{lm}(x)f\chi_K(x)|^p d\lambda(x) > (1-\varepsilon)^p \sum_K \varphi_{lm}^p(x),\end{aligned}$$

which says that

$$\varphi_{lm}(x) < \frac{2\varepsilon^2}{1-\varepsilon} \qquad \text{for} \qquad x \in K.$$

The similar argument can be applied to the sequences $(\widetilde{\varphi}_{ln_k})$, $(\varphi_{(N+l)n_k}|_{E_k^+})$ and $(\widetilde{\varphi}_{(N+l)n_k}|_{E_k^-})$. Therefore, the weight conditions in (ii) are satisfied. ■

Remark 2.3 We note that if $(C_n)_{n\in\mathbb{N}_0}$ is topologically multiply recurrent on $\ell^p(G)$, then $T_{a,w}$ is also multiply recurrent on $\ell^p(G)$. This follows from Corollary 2.2 and [8, Theorem 2.1] directly.

Example 2.4 Let $G = \mathbb{Z}$, $a = -1 \in \mathbb{Z}$, which is torsion free. Let $w * \delta_{-1}$ be a weight on $\mathbb{Z}$ with $w^{-1} \in \ell^\infty(\mathbb{Z})$. Then the weighted translation operator $T_{-1,w*\delta_{-1}}$ is given by

$$T_{-1,w*\delta_{-1}}f(j) = w(j+1)f(j+1) \qquad \big(f \in \ell^p(\mathbb{Z})\big).$$

In fact, the operator $T_{-1,w*\delta_{-1}}$ is just the bilateral backward weighted shift T, defined by $Te_j = w_j e_{j-1}$ with $w_j = w(j)$. Here, $(e_j)_{j\in\mathbb{Z}}$ is the canonical basis of $\ell^p(\mathbb{Z})$ and $(w_j)_{j\in\mathbb{Z}}$ is a sequence of positive real numbers. Let $S = T^{-1}$ and $C_n = \frac{1}{2}(T^n + S^n)$. Then by Corollary 2.2, both $(C_n)_{n\in\mathbb{N}_0}$ and T are topologically multiply recurrent if given $N, q \in \mathbb{N}$ and $\varepsilon > 0$, there exists a positive integer n such that for all $|j| < q$, we have

$$\varphi_{ln}(j) = \prod_{s=1}^{ln}(w * \delta_{-1}) * \delta_1^s(j) = \prod_{s=0}^{ln-1} w(j-s) < \varepsilon,$$

$$\widetilde{\varphi}_{ln}^{-1}(j) = \prod_{s=0}^{ln-1}(w * \delta_{-1}) * \delta_{-1}^s(j) = \prod_{s=1}^{ln} w(j+s) > \frac{1}{\varepsilon}$$

for all $1 \leq l \leq 2N$. If we define $w: \mathbb{Z} \to (0,\infty)$ by

$$w(j) = \begin{cases} \frac{1}{4} & \text{if } j < 0, \\ 4 & \text{if } j \geq 0, \end{cases}$$

then w satisfies the weight condition above.

Example 2.5 Let $G = \mathbb{R}$, $a = 4$, which is an aperiodic element of $\mathbb{R}$. Let w be a weight on $\mathbb{R}$ with $w^{-1} \in L^\infty(\mathbb{R})$. Then the weighted translation $T_{4,w}$ on $L^p(\mathbb{R})$ is defined by

$$T_{4,w}f(x) = w(x)f(x-4) \qquad \big(f \in L^p(\mathbb{R})\big).$$

Let $C_n = \frac{1}{2}(T^n_{4,w} + S^n_{4,w})$, where $S_{4,w}$ is the inverse of $T_{4,w}$. Then, by Theorem 2.1, a sequence of operators $(C_n)_{n\in\mathbb{N}_0}$ is topologically multiply recurrent if given some $N \in \mathbb{N}$ and a compact subset K of $\mathbb{R}$, there exists a sequence (n_k) of positive integers such that for all $1 \le l \le 2N$, we have

$$\int_K \Big(\prod_{s=1}^{ln_k} w(x+4s)\Big)^p d\lambda(x) \longrightarrow 0 \quad \text{and} \quad \int_K \Bigg(\frac{1}{\prod_{s=0}^{ln_k-1} w(x-4s)}\Bigg)^p d\lambda(x) \longrightarrow 0$$

as $k \to \infty$.

Example 2.6 Let

$$G = \mathbb{H} := \left\{ \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} : x, y, z \in \mathbb{R} \right\}$$

be the Heisenberg group. It is known that the group $\mathbb{H}$ is neither abelian nor compact. We write an element in $\mathbb{H}$ as (x, y, z) for convenience. Let $(x, y, z), (x', y', z') \in \mathbb{H}$. Then the multiplication is defined by

$$(x, y, z) \cdot (x', y', z') = (x + x', y + y', z + z' + xy'),$$
$$(x, y, z)^{-1} = (-x, -y, xy - z).$$

Let $a = (2, 0, 4)$, and let $w, w^{-1} \in L^\infty(\mathbb{H})$. Then $a^{-1} = (-2, 0, -4)$, and the weighted translation $T_{(2,0,4),w}$ on $L^p(\mathbb{H})$ is given by

$$T_{(2,0,4),w} f(x, y, z) = w(x, y, z) f(x-2, y, z-4) \qquad \big(f \in L^p(\mathbb{H})\big).$$

Let $C_n = \frac{1}{2}(T^n_{(2,0,4),w} + S^n_{(2,0,4),w})$ where $S_{(2,0,4),w} = T^{-1}_{(2,0,4),w}$. Then, by Theorem 2.1, $(C_n)_{n\in\mathbb{N}_0}$ is topologically multiply recurrent if given some $N \in \mathbb{N}$ and a compact subset K of $\mathbb{H}$, there exists a sequence (n_k) of positive integers such that for all $1 \le l \le 2N$, we have

$$\int_K \Big(\prod_{s=1}^{ln_k} w(x+2s, y, z+4s)\Big)^p d\lambda(x) \to 0,$$
$$\int_K \Bigg(\frac{1}{\prod_{s=0}^{ln_k-1} w(x-2s, y, z-4s)}\Bigg)^p d\lambda(x) \to 0,$$

as $k \to \infty$.

Next, we turn our attention to the connection between recurrence and dynamics for the cosine operator function $(C_n)_{n\in\mathbb{N}_0}$. First, we recall a result in [7].

Theorem 2.7 ([7, Theorem 2.1]) *Let G be a locally compact group and let a be an aperiodic element in G. Let $1 \le p < \infty$ and $w, w^{-1} \in L^\infty(G)$. Let $T_{a,w}$ be a weighted translation on $L^p(G)$ with inverse $S_{a,w}$, and let $C_n = \frac{1}{2}(T^n_{a,w} + S^n_{a,w})$. Then the following conditions are equivalent.*

(i) *$(C_n)_{n\in\mathbb{N}_0}$ is topologically transitive.*

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(ii) *For each compact subset $K \subset G$ with $\lambda(K) > 0$, there exist sequences (E_k^+) and (E_k^-) of Borel sets in K, and a sequence (n_k) of positive numbers such that for $E_k = E_k^+ \cup E_k^-$, we have*

$$\lambda(K) = \lim_{k\to\infty} \lambda(E_k), \quad \lim_{k\to\infty} \nu_{n_k}(E_k) = \lim_{k\to\infty} \widetilde{\nu}_{n_k}(E_k) = 0,$$
$$\lim_{k\to\infty} \nu_{2n_k}(E_k^+) = \lim_{k\to\infty} \widetilde{\nu}_{2n_k}(E_k^-) = 0.$$

By the result above, we observe that recurrence and topological transitivity are equivalent in this case.

Corollary 2.8 *Let G be a locally compact group and let a be an aperiodic element in G. Let $1 \le p < \infty$ and let $T_{a,w}$ be a weighted translation on $L^p(G)$. Let $C_n = \frac{1}{2}(T_{a,w}^n + S_{a,w}^n)$. Then $(C_n)_{n\in\mathbb{N}_0}$ is topologically transitive if and only if $(C_n)_{n\in\mathbb{N}_0}$ is recurrent.*

Proof Since topological transitivity implies recurrence, we only need to show that if $(C_n)_{n\in\mathbb{N}_0}$ is recurrent, then it is transitive.

Let $N = 1$ in Theorem 2.1. Then Theorem 2.1(ii) implies transitivity of $(C_n)_{n\in\mathbb{N}_0}$ by Theorem 2.7. ∎

Example 2.9 Let $G = \mathbb{Z}$, $a = -1 \in \mathbb{Z}$ and let $w * \delta_{-1}$ be a weight on $\mathbb{Z}$ with $w^{-1} \in \ell^\infty(\mathbb{Z})$. Then $T_{-1,w*\delta_{-1}}$ is an invertible bilateral backward weighted shift on $\ell^p(\mathbb{Z})$ with inverse $T_{1,w^{-1}}$. Let $C_n = \frac{1}{2}(T_{-1,w*\delta_{-1}}^n + T_{1,w^{-1}}^n)$. Then, by Corollary 2.8, transitivity and recurrence occur on $(C_n)_{n\in\mathbb{N}_0}$ simultaneously.

Another result in [7] reveals that topological mixing implies topologically multiple recurrence on $(C_n)_{n\in\mathbb{N}_0}$.

Proposition 2.10 ([7, Corollary 2.5]) *Let G be a locally compact group and let a be an aperiodic element in G. Let $1 \le p < \infty$ and $w, w^{-1} \in L^\infty(G)$. Let $T_{a,w}$ be a weighted translation on $L^p(G)$ with inverse $S_{a,w}$, and let $C_n = \frac{1}{2}(T_{a,w}^n + S_{a,w}^n)$. Then the following conditions are equivalent.*

(i) *$(C_n)_{n\in\mathbb{N}_0}$ is topologically mixing.*
(ii) *For each compact subset $K \subset G$ with $\lambda(K) > 0$, there is a sequence of Borel sets (E_n) in K such that*

$$\lambda(K) = \lim_{n\to\infty} \lambda(E_n) \quad \text{and} \quad \lim_{n\to\infty} \nu_n(E_n) = \lim_{n\to\infty} \widetilde{\nu}_n(E_n) = 0.$$

This result gives the following corollary readily.

Corollary 2.11 *Let G be a locally compact group and let a be an aperiodic element in G. Let $1 \le p < \infty$ and $w, w^{-1} \in L^\infty(G)$. Let $T_{a,w}$ be a weighted translation on $L^p(G)$ with inverse $S_{a,w}$, and let $C_n = \frac{1}{2}(T_{a,w}^n + S_{a,w}^n)$. If $(C_n)_{n\in\mathbb{N}_0}$ is topologically mixing, then $(C_n)_{n\in\mathbb{N}_0}$ is topologically multiply recurrent on $L^p(G)$.*

Finally, we make a remark to illustrate that chaos also implies topologically multiple recurrence in the case of $(C_n)_{n\in\mathbb{N}_0}$. We recall that an operator T on a Banach space X is *chaotic* in the sense of Devaney's definition if T is topologically transitive and the set of periodic elements, $\{x \in X : \exists\, n \in \mathbb{N} \text{ with } T^n x = x\}$, is dense in X. According to this definition, a sequence of bounded linear operators $(T_n)_{n\in\mathbb{N}_0}$ on a Banach space X is called *chaotic* in the successive way in [15] if $(T_n)_{n\in\mathbb{N}_0}$ is topologically transitive and the set of periodic elements, denoted by

$$\mathcal{P}\big((T_n)_{n\in\mathbb{N}_0}\big) = \{x \in X : \exists\, m \in \mathbb{N} \text{ with } T_{km}x = x,\ k = 1,2,3,\dots\},$$

is dense in X. In [6], we give the characterization for the cosine operator function $(C_n)_{n\in\mathbb{N}_0}$ to be chaotic.

Theorem 2.12 ([6, Theorem 2.1]) *Let G be a locally compact group and let a be an aperiodic element in G. Let $1 \le p < \infty$ and $T_{a,w}$ be a weighted translation on $L^p(G)$ with inverse $S_{a,w}$. Let $C_n = \frac{1}{2}(T^n_{a,w} + S^n_{a,w})$, and let $\mathcal{P}((C_n)_{n\in\mathbb{N}_0})$ be the set of periodic elements. Then the following conditions are equivalent.*

(i) *$(C_n)_{n\in\mathbb{N}_0}$ is chaotic.*
(ii) *$\mathcal{P}((C_n)_{n\in\mathbb{N}_0})$ is dense in $L^p(G)$.*
(iii) *For each compact subset $K \subset G$ with $\lambda(K) > 0$, there is a sequence of Borel sets (E_k) in K such that $\lambda(K) = \lim_{k\to\infty} \lambda(E_k)$ and both sequences*

$$\varphi_n := \prod_{s=1}^{n} w * \delta^s_{a^{-1}} \quad \text{and} \quad \widetilde{\varphi}_n := \Big(\prod_{s=0}^{n-1} w * \delta^s_a\Big)^{-1}$$

admit respectively subsequences (φ_{n_k}) and $(\widetilde{\varphi}_{n_k})$ satisfying

$$\lim_{k\to\infty}\bigg(\sum_{l=1}^{\infty}\int_{E_k}\varphi^p_{ln_k}(x)\,d\lambda(x) + \sum_{l=1}^{\infty}\int_{E_k}\widetilde{\varphi}^p_{ln_k}(x)\,d\lambda(x)\bigg) = 0.$$

Based on the result above, we have the following implication since the last equality in Theorem 2.12 can be rewritten as

$$\lim_{k\to\infty}\bigg(\sum_{l=1}^{\infty} \nu_{ln_k}(E_k) + \sum_{l=1}^{\infty}\widetilde{\nu}_{ln_k}(E_k)\bigg) = 0.$$

Corollary 2.13 *Let G be a locally compact group and let a be an aperiodic element in G. Let $1 \le p < \infty$ and $w, w^{-1} \in L^\infty(G)$. Let $T_{a,w}$ be a weighted translation on $L^p(G)$ with inverse $S_{a,w}$, and let $C_n = \frac{1}{2}(T^n_{a,w} + S^n_{a,w})$. If $(C_n)_{n\in\mathbb{N}_0}$ is chaotic, then $(C_n)_{n\in\mathbb{N}_0}$ is topologically multiply recurrent on $L^p(G)$.*

We note that there does exist a weighted shift that is topologically multiply recurrent, but is neither chaotic nor topologically mixing in [2].

Example 2.14 Let $G = \mathbb{Z}$, $a = -1 \in \mathbb{Z}$ and let $w * \delta_{-1}$ be a weight on $\mathbb{Z}$ with $w^{-1} \in \ell^\infty(\mathbb{Z})$. Then $T_{-1,w*\delta_{-1}}$ is an invertible bilateral backward weighted shift on $\ell^p(\mathbb{Z})$ with inverse $T_{1,w^{-1}}$. Let $C_n = \frac{1}{2}(T^n_{-1,w*\delta_{-1}} + T^n_{1,w^{-1}})$. Then, by Corollaries 2.11 and 2.13, $(C_n)_{n\in\mathbb{N}_0}$ is topologically multiply recurrent if $(C_n)_{n\in\mathbb{N}_0}$ is topologically mixing or chaotic.

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