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DIFFEOMORPHISMS WITH THE SHADOWING PROPERTY

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Abstract

It is proved that for every diffeomorphism f on a surface satisfying Axiom A, f is in the C^2-interior of the set of all diffeomorphisms having the shadowing property if and only if f satisfies the strong transversality condition.

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The shadowing property, which is also well known as the pseudo orbit tracing property, is closely related to the stability of a diffeomorphism satisfying Axiom A. In [6] it is stated that for a diffeomorphism f satisfying Axiom A, if f satisfies the strong transversality condition, then f has the shadowing property. Conversely, the strong transversality condition for f was proved in [2] and [7] when f is in the C^1-interior of the set of all diffeomorphisms having the shadowing property. It is also proved in [2] that every diffeomorphism in the C^r-interior of the set of all diffeomorphisms having the shadowing property satisfies Axiom A when $r = 1$. However it is unknown whether the conclusion also holds for the case when $r \geq 2$.

In this paper, in the context of C^2 topology, by using a result stated in [8] the relationship between the shadowing property and the transversality of the stable manifolds and the unstable manifolds of a C^2 diffeomorphism on a surface satisfying Axiom A was discussed.

Let M be a C^∞ closed manifold and $\mathrm{Diff}^r(M)$ ($r \geq 1$) be the space of C^r diffeomorphisms of M endowed with C^r topology. In the following results let M be a surface.

THEOREM. *Let $f \in \mathrm{Diff}^2(M)$ satisfy Axiom A. Then f is in the C^2-interior of the* *set of all diffeomorphisms having the shadowing property if and only if f satisfies the strong transversality condition.*

Let $f \in \mathrm{Diff}^2(M)$ satisfy Axiom A. If f satisfies the strong transversality condition, then f is structurally stable (see [5]). Thus f is in the C^2-interior of the set of all diffeomorphisms having the shadowing property (because f has the shadowing property and which is invariant under a conjugacy). Since the non-wandering set of f is a disjoint union of basic sets, our theorem will be obtained from the following

PROPOSITION. *Let Λ_i $(i = 1, 2)$ be basic sets of $f \in \mathrm{Diff}^2(M)$ and suppose $x \in W^s(\Lambda_1) \cap W^u(\Lambda_2) \setminus \Lambda_1 \cup \Lambda_2$. If there is a C^2 neighborhood $\mathscr{U}(f)$ of f such that every $g \in \mathscr{U}(f)$ has the shadowing property, then $T_xM = T_xW^s(x) + T_xW^u(x)$.*

Let d be a metric on M induced from a Riemannian metric $\|\cdot\|$ on TM. A sequence $\{x_k\}_{k=a}^b$ $(-\infty \le a < b \le \infty)$ of points is called a *δ-pseudo-orbit* of $f \in \mathrm{Diff}^r(M)$ $(r \ge 1)$ if $d(f(x_k), x_{k+1}) < \delta$ for $a \le k \le b-1$. Given $\varepsilon > 0$, $\{x_k\}_{k=a}^b$ is said to be *ε-shadowed* by $x \in M$ if $d(f^k(x), x_k) < \varepsilon$ for $a \le k \le b$. We say that f has the *shadowing property* if for $\varepsilon > 0$ there is $\delta > 0$ such that every δ-pseudo-orbit of f can be ε-shadowed by some point.

A hyperbolic set Λ is called a *basic set* if there is a compact neighborhood U of Λ in M such that $\bigcap_{n\in\mathbb{Z}} f^n(U) = \Lambda$ and $f_{|\Lambda}$ has a dense orbit. The *local stable* and the *unstable manifolds* are denoted by $W^s_{\varepsilon_0}(x)$ and $W^u_{\varepsilon_0}(x)$ $(x \in \Lambda)$ respectively for some $\varepsilon_0 > 0$. The *stable manifold*, $W^s(x)$, and the *unstable manifold*, $W^u(x)$, of $x \in \Lambda$ are defined in the usual way, and we put $W^\sigma(\Lambda) = \bigcup_{x\in\Lambda} W^\sigma(x)$ $(\sigma = s, u)$. A basic set Λ is called of *saddle type* if $0 < \dim W^s(x) < \dim M$ for $x \in \Lambda$.

Hereafter let M be a surface. The notion of C^0-transversality between stable and unstable manifolds of basic sets Λ_i and Λ_j was introduced in [8] as follows. If there exists $x \in W^s(\Lambda_i) \cap W^u(\Lambda_j) \setminus \Lambda_i \cup \Lambda_j$, then for $\varepsilon > 0$ we denote by $C^\sigma_\varepsilon(x)$ the connected component of x in $W^\sigma(x) \cap B_\varepsilon(x)$ $(\sigma = s, u)$ and let $B^+_\varepsilon(x)$ and $B^-_\varepsilon(x)$ be the compoments of $B_\varepsilon(x) \setminus C^s_\varepsilon(x)$. Here $B_\varepsilon(x) = \{y \in M \mid d(x, y) \le \varepsilon\}$. We say that *$W^s(x)$ and $W^u(x)$ meet C^0-transversely at x* if $\dim W^\sigma(x) = 1$ $(\sigma = s, u)$, $B^+_\varepsilon(x) \cap C^u_\varepsilon(x) \neq \emptyset$ and $B^-_\varepsilon(x) \cap C^u_\varepsilon(x) \neq \emptyset$ for every $\varepsilon > 0$.

Let Λ be a basic set of $f \in \mathrm{Diff}^r(M)$ $(r \ge 1)$. Since $\dim M = 2$, there is a locally f-invariant C^0-foliation with C^1-leaves defined in some neighborhood of Λ (see [1]). This foliation plays an essential role in the proof of the following lemma.

LEMMA 1 ([8, Proposition A]). *Let Λ_i $(i = 1, 2)$ be basic sets of $f \in \mathrm{Diff}^r(M)$ $(r \ge 1)$, and suppose that $x \in W^s(p) \cap W^u(q) \setminus \Lambda_1 \cup \Lambda_2$ $(p \in \Lambda_1, q \in \Lambda_2)$. If f has the shadowing property, then $W^s(p)$ and $W^u(q)$ meet C^0 transversely at x.*

REMARK. Let $x \in W^s(p) \cap W^u(q)$ be as in Lemma 1. If $W^s(p)$ and $W^u(q)$ meet C^0-transversely at x, then they do not have a non-degenerate tangency at x (for the definition of a non-degenerate tangency see [3, p. 104]). Thus, if Λ is a Newhouse wild hyperbolic set of $f \in \mathrm{Diff}^2(M)$ ([3]), and if we put $\Lambda_1 = \Lambda_2 = \Lambda$, then, by Newhouse's result and Lemma 1, there exists a non-empty C^2-open set $\mathscr{O}$ such that every $g \in \mathscr{O}$ does not have the shadowing property.

To prove our proposition we shall use the following basic fact.

LEMMA 2. *Let Λ_i $(i = 1, 2)$ be basic sets of $f \in \mathrm{Diff}^r(M)$ $(r \geq 1)$, and suppose that $x \in W^s(p) \cap W^u(q) \setminus \Lambda_1 \cup \Lambda_2$ $(p \in \Lambda_1, q \in \Lambda_2)$. Then there are $\varepsilon > 0$ and a C^r diffeomorphism $\tilde{\varphi}_x : B_\varepsilon(x) \to \mathbb{R}^2 = \{(v, w) \mid v, w \in \mathbb{R}\}$ such that $\tilde{\varphi}_x(x) = (0, 0)$ and $\tilde{\varphi}(C^s_\varepsilon(x)) \subset v$-axis.*

PROOF. Let $x \in W^s(p) \cap W^u(q)$ be as above. Since $T_{\Lambda_1}M = E^s \oplus E^u$ is hyperbolic, there are $\delta > 0$ and C^r maps $\varphi_s : E^s_p(\delta) \to E^u_p$ and $\varphi_u : E^u_p(\delta) \to E^s_p$ such that $W^s_{\varepsilon_0}(p) = \exp_p(E^s_p(\delta), \varphi_s(E^s_p(\delta)))$ and $W^u_{\varepsilon_0}(p) = \exp_p(\varphi_u(E^u_p(\delta)), E^u_p(\delta))$. Here $E^\sigma_p(\varepsilon) = \{v \in E^\sigma_p : \|v\| \leq \varepsilon\}$. Since f is a diffeomorphism, (iterating x by f if necessary) we may assume that $x \in W^s_{\varepsilon_0}(p) \cap B_{\delta/2}(p)$. Let us denote the natural projection from $E^s_p \oplus E^u_p$ to E^σ_p by $\bar{\pi}^\sigma$ $(\sigma = s, u)$ and define a C^r-diffeomorphism $\varphi : B_\delta(p) \to T_pM = E^s_p \oplus E^u_p$ by

$$\varphi(y) = \left(\bar{\pi}^s(\exp_p^{-1} y) - \varphi_u(\bar{\pi}^u(\exp_p^{-1} y)), \bar{\pi}^u(\exp_p^{-1} y) - \varphi_s(\bar{\pi}^s(\exp_p^{-1} y))\right)$$

for $y \in B_\delta(p)$. Then $\varphi(W^s_{\varepsilon_0}(p)) \subset E^s_p$ (see [4, p. 81]). Since $x \in W^s_{\varepsilon_0}(p) \cap B_{\delta/2}(p)$, if we put $\varepsilon = \delta/2$, then $\varphi(C^s_\varepsilon(x)) \subset E^s_p(\delta)$. Let $\eta : T_{\varphi(x)}(T_pM) \to T_pM$ be the parallel transformation. Then $\tilde{\varphi}_x = \eta \circ \varphi : B_\varepsilon(x) \to \mathbb{R}^2$ satisfies the conclusion of this lemma.

PROOF OF PROPOSITION. Let Λ_i $(i = 1, 2)$ be basic sets of $f \in \mathrm{Diff}^2(M)$ and suppose $x \in W^s(\Lambda_1) \cap W^u(\Lambda_2) \setminus \Lambda_1 \cup \Lambda_2$. We shall prove that if there is a C^2-neighborhood $\mathscr{U}(f)$ of f such that every $g \in \mathscr{U}(f)$ has the shadowing property, then $T_xM = T_xW^s(x) + T_xW^u(x)$.

By Lemma 2 there are $\delta > 0$ and a C^2-diffeomorphism $\tilde{\varphi}_x : B_{\varepsilon_0}(x) \to \mathbb{R}^2$ such that $\tilde{\varphi}_x(C^s_\delta(x)) \subset v$*-axis* and $\tilde{\varphi}_x(x) = (0, 0)$. If $T_{(0,0)}\tilde{\varphi}_x(C^u_\delta(x)) \neq v$*-axis*, then we have $T_xM = T_xW^s(x) + T_xW^u(x)$ that is; $W^s(x)$ and $W^u(x)$ meet transversely at x. Thus we assume that $T_{(0,0)}\tilde{\varphi}_x(C^u_\delta(x)) = v$*-axis*. It is easy to see that there are $\varepsilon > 0$ and a C^2-function $\gamma : [-\varepsilon, \varepsilon] \to \mathbb{R}$ such that $\mathrm{graph}(\gamma) \subset \tilde{\varphi}_x(C^u_\delta(x))$ and $(0, \gamma(0)) = \tilde{\varphi}_x(x) = (0, 0)$. If $\gamma''(0) \neq 0$ then, since $\gamma'(0) = 0$, $W^s(x)$ and $W^u(x)$ do not meet C^0-transversely at x. This is inconsistent with Lemma 1 and so $\gamma''(0) = 0$. If we denote a C^2-metric as ρ_{C^2}, then for every δ', there exists $0 < \varepsilon' < \varepsilon$ such that

$$\rho_{C^2}\left(\tilde{\varphi}_x^{-1}(\mathrm{graph}(\gamma(-\varepsilon', \varepsilon'))), C^s_{\varepsilon'}(x)\right) < \delta'$$

since $\gamma'(0) = 0$ and $\gamma''(0) = 0$. Thus, by using a standard procedure, for every $\nu > 0$ and every C^2-neighborhood $\mathcal{U}(f)$ of f such that every $g \in \mathcal{U}(f)$ has the shadowing property, we can construct a C^2-diffeomorphism $\psi : M \to M$ such that

$$\begin{cases} \psi(x) = x \\ \psi_{|M \setminus B_\nu(x)} = \text{id} \\ \psi(W^s(x) \cap B_{\nu'}(x)) \subset W^u(x) \\ \tilde{f} = \psi^{-1} \circ f \in \mathcal{U}(f), \end{cases}$$

where $0 < \nu' < \nu$ is sufficiently small. From this we have

$$W^s(x, \tilde{f}) \cap B_{\nu'}(x) = W^u(x, \tilde{f}) \cap B_{\nu'}(x).$$

Here $W^\sigma(x, \tilde{f})$ $(\sigma = s, u)$ are the stable and the unstable manifolds of $\tilde{f}$ at x. By Lemma 1, this is a contradiction since $\tilde{f}$ has the shadowing property and so the proof is completed.

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