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AN ESSENTIAL INTEGRAL DOMAIN WITH A NON-ESSENTIAL LOCALIZATION

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An integral domain D is said to be an *essential domain* if D is an intersection of valuation rings that are localizations of D. D is called a v-multiplication ring if the finite divisorial ideals of D form a group. Griffin has shown [**2**, pp. 717–718] that every v-multiplication ring is an essential domain, and that an essential domain having a defining family of valuation rings $\{V_\alpha\}$ which is of finite character (i.e., every nonzero element of D is a non-unit in at most finitely many V_α) is necessarily a v-multiplication ring. It is noted in [**4**, p. 860] that any localization of a v-multiplication ring is again a v-multiplication ring. In this vein, Joe Mott has asked whether a localization of an essential domain must again be an essential domain. An example of an essential domain that is not a v-multiplication ring is given in [**4**], however it can be seen for this example that each localization is again an essential domain [**6**]. Our purpose here is to construct an essential integral domain D having a prime ideal P such that D_P is not essential.

In constructing the example we will use some properties of Kronecker function rings that we briefly review, see for example [**1**]. If A is an integrally closed domain with quotient field F, t is an indeterminate over F, and $\{B_\alpha\}$ is the set of valuation rings of F that contain A, then for each $B \in \{B_\alpha\}$ the ring $B(t)$ gotten by localizing the polynomial ring $B[t]$ with respect to the multiplication system of polynomials having a unit coefficient is a valuation ring of $F(t)$ extending B, and the ring $C = \bigcap_\alpha B_\alpha(t)$ is called *the Kronecker function ring of A with respect to t*. It is well-known that C is a Bezout domain, so in particular, each localization of C at a prime ideal is a valuation ring. Moreover, each such valuation ring has the form $B(t)$ where $B \in \{B_\alpha\}$. We note that this implies that each nonzero prime ideal of C has a nonzero contraction to A. We also have $C \cap F = A$, and t is a unit in C.

The idea in our construction is as follows: We begin with a 1-dimensional quasi-local integrally closed domain V such that V is not a valuation ring. Then V is certainly not an essential domain. We obtain such a V on a field K such that there exists an integral domain $D \subset V$ with V having center a maximal ideal P of D, $D_P = V$, and such that for each prime $Q \neq P$ of D, D_Q is a valuation ring. In order that D be

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essential we then also want D to have the property that

$$D = \cap \{D_Q | Q \neq P, Q \text{ a prime of } D\}.$$

We obtain this last property in stages, making use of Kronecker function rings, and expressing D as an infinite union of subrings.

Construction of the example. Let k be a field, let $y, z, x_1, x_2, \ldots$ be indeterminates over k, and let K denote the field $k(y, z, x_1, x_2, \ldots)$. Let W be the rank one discrete valuation ring $k(z, x_1, x_2, \ldots)[y]_{(y)}$, and let M denote the maximal ideal of W. Then $W = k(z, x_1, x_2, \ldots) + M$, and $V = k(x_1, x_2, \ldots) + M$ is a 1-dimensional quasi-local integrally closed domain with quotient field K such that W is the unique rank one valuation ring on K that contains V. Let K_n denote the subfield $k(y, z, x_1, \ldots, x_n)$ of K, and let $V_n = V \cap K_n$. We note that $V_n = k(x_1, \ldots, x_n) + (M \cap K_n)$ is a 1-dimensional quasi-local integrally closed domain with quotient field K_n, and that $W_n = W \cap K_n$ is the unique rank one valuation ring of the field K_n that contains V_n. To begin the construction we wish to define on K_2 an appropriate Kronecker function ring extension of V_1. Let

$$t_2 = (1 + yx_2)/y,$$

and note that

$$K_1(t_2) = K_1(x_2) = K_2.$$

We set R_2 to be the Kronecker function ring of V_1 defined with respect to t_2, and let $D_2 = R_2 \cap V_2$. Since $1 + yx_2$ is a unit of V_2 and y is in the maximal ideal of V_2, $1/t_2$ is a unit of R_2 that is in the maximal ideal of V_2. Hence $1/t_2 \in D_2$, and $D_2[t_2] = R_2$, so that R_2 is a localization of D_2. Moreover, y is in the maximal ideal of V_1 implies that yt_2 is in every nonzero prime of R_2. Since $yt_2 = 1 + yx_2$ is a unit of V_2, we see that

$$D_2[1/yt_2] = V_2$$

cf. [**5**, Lemma 1.1, p. 292]. Also from the fact that yt_2 is in every nonzero prime ideal of R_2 and $1 - yt_2$ is in the center P_2 of V_2 on D_2, we have that P_2 is a maximal ideal of D_2 [**3**, Corollary 1.20, p. 113]. It follows that if Q is any prime ideal of D_2 other than P_2 then $R_2 \subset (D_2)_Q$. Hence each such $(D_2)_Q$ is a valuation ring, and we have

$$R_2 = \cap \{(D_2)_Q | Q \text{ is a prime of } D_2 \text{ distinct from } P_2\}.$$

We define R_3 to be the Kronecker function ring of D_2 with respect to

$$t_3 = (1 + yx_3)/y$$

on the field

$$K_2(t_3) = K_2(x_3) = K_3.$$

We set $D_3 = R_3 \cap V_3$. Since y is in every nonzero prime of D_2, yt_3 is in every nonzero prime of R_3. Moreover, $yt_3 = 1 + yx_3$ is a unit of V_3, so as above we have that R_3 and V_3 are both localizations of D_3, the center P_3 of V_3 on D_3 is a maximal ideal, and

$$R_3 = \cap \{(D_3)_Q | Q \text{ is a prime of } D_3 \text{ distinct from } P_3\}.$$

An easy induction argument yields for each integer $n > 2$ integral domains R_n and $D_n = R_n \cap V_n$ on the field K_n such that

(i) R_n is the Kronecker function ring of D_{n-1} with respect to $t_n = (1 + yx_n)/y$, and $D_{n-1} \subset D_n$.

(ii) R_n and V_n are both localizations of D_n, the center P_n of V_n on D_n is a maximal ideal of D_n, and $P_n \cap D_{n-1} = P_{n-1}$.

(iii) $R_n = \cap \{(D_n)_Q | Q$ is a prime of D_n distinct from $P_n\}$.

Moreover, between D_{n-1} and D_n we have the following

(iv) For each prime $Q' \neq P_{n-1}$ of D_{n-1} there is a unique prime Q of D_n such that $Q \cap D_{n-1} = Q'$. This prime Q is such that

$$(D_n)_Q = (D_{n-1})_{Q'}(t_n).$$

Proof. By (iii), $R_{n-1} \subset (D_{n-1})_{Q'}$ so $(D_{n-1})_{Q'}$ is a valuation ring of K_{n-1} containing D_{n-1}. Since R_n is the Kronecker function ring of D_{n-1}, $(D_{n-1})_{Q'}$ has a unique extension $(D_{n-1})_{Q'}(t_n)$ to a valuation ring of K_n containing R_n. In view of (iii), the center Q of $(D_{n-1})_{Q'}(t_n)$ on D_n is the unique prime of D_n lying over Q' in D_{n-1}, and

$$(D_n)_Q = (D_{n-1})_{Q'}(t_n).$$

We set $D = \bigcup_{n=2}^{\infty} D_n$ and $P = \bigcup_{n=2}^{\infty} P_n$. Since $V = \bigcup_{n=2}^{\infty} V_n$, and $(D_n)_{P_n} = V_n$ for each n, we have $V = D_P$. Moreover, if Q is a prime of D distinct from P, then $Q \cap D_n = Q_n \neq P_n$ for some n. Hence $R_n \subset (D_n)_{Q_n'}$ and $(D_n)_{Q_n}$ is a valuation ring on K_n. Condition (iv) implies that

$$D_Q = (D_n)_{Q_n}(t_{n+1}, t_{n+2}, \ldots),$$

so that D_Q is a valuation ring. Finally we wish to show that

$$D = \cap \{D_Q | Q \text{ is a prime of } D \text{ distinct from } P\}.$$

Let $\theta \in K$, $\theta \notin D$. We wish to show that $\theta \notin D_Q$ for some prime $Q \neq P$ of D. Since $K = \bigcup_{n=2}^{\infty} K_n$, $\theta \in K_{n-1}$ for some integer $n > 2$. Since $R_n \cap K_{n-1} = D_{n-1}$, we have that $\theta \notin R_n$. Hence, by condition (iii), $\theta \notin (D_n)_{Q_n}$ for some prime $Q_n \neq P_n$ of D_n. It follows from condition (iv) that there exists a prime Q of D such that

$$Q \cap D_n = Q_n \quad \text{and} \quad D_Q = (D_n)_{Q_n}(t_{n+1}, t_{n+2}, \ldots).$$

Since $\theta \notin (D_n)_{Q_n'}$ we have that $\theta \notin D_Q$. This completes our verification of the example.

Remark. It would be interesting to know if there exists a 1-dimensional essential domain with a non-essential localization. Since the 1-dimensional quasi-local domains V_n in the above construction have valuative dimension 2, the Kronecker function rings R_n have dimension 2, and it can be seen that D is 2-dimensional.

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