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SUFFICIENTLY HOMOGENEOUS CLOSED EMBEDDINGS OF $\mathbb{A}^{n-1}$ INTO $\mathbb{A}^n$ ARE LINEAR

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ABSTRACT. We show that over a field k of characteristic zero an affine $(n-1)$-space $\mathbb{A}_k^{n-1}$ embedded as a closed subvariety in affine n-space $\mathbb{A}_k^n$ and homogeneous for a codimension two linear torus action on $\mathbb{A}_k^n$ is defined by the vanishing of a variable.

To determine the nature of closed embeddings of affine m-space $\mathbb{A}^m$ into affine n-space $\mathbb{A}^n$ is a central problem in the geometry of affine spaces. The famous "epimorphism theorem" of Abhyankar-Moh [AM] and Suzuki [S] settles the case $m = 1, n = 2$ over a field k of characteristic zero: A "line" $\mathbb{A}_k^1$ embedded in the "plane" $\mathbb{A}_k^2$ is defined by the vanishing of a variable. (A *variable* on $\mathbb{A}^n$ is a function $y = y_1$ that extends to a global coordinate system $y_1, y_2, \ldots, y_n$ for $\mathbb{A}^n$.) In most other cases, only very partial results have been obtained so far, even when, or better particularly when $m = n-1$ (see [RS], for instance). With the epimorphism theorem as a starting point we will show that if $\operatorname{char} k = 0$ and if $Y \simeq \mathbb{A}_k^{n-1}$ is closed in $X \simeq \mathbb{A}_k^n$ and *sufficiently homogeneous* in the sense that it is invariant under a $(n-2)$-torus $T \simeq \mathbb{G}_{m,k}^{n-2}$ acting linearly and effectively on X, then again Y is defined by a variable. (Special cases of this result with $n = 3$ were considered in [PTD]). Our proof, an induction taking off from $n = 2$ and the epimorphism theorem, quite naturally forces us to consider certain finite group actions in addition to those of tori. We show:

MAIN THEOREM. *Let k be a field of characteristic zero and suppose*

$$\mathbb{A}_k^{n-1} \simeq Y \subset X \simeq \mathbb{A}_k^n$$

with Y closed in X and invariant under an effective linear action of

$$G = T \times F$$

where $T \simeq \mathbb{G}_{m,k}^r$ is a torus of dimension $r \geq n-2$ and F a finite abelian group. Then there exists a system of G-semi-invariant variables $y_1, \ldots, y_n$ for X such that Y is the zero locus of y_1.

In a sense this is only a modest contribution to the embedding problem since it begs the main question for our homogeneous situation, namely: "*Is every effective action of a $(n-2)$-torus on affine n-space linearizable*?" We were motivated to undertake the present

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study largely to gain some understanding of the difficulties one is likely to encounter trying to resolve this question by an inductive procedure along the lines of [KR1], starting with the expected positive answer for $n = 3$ (see [KR3], [KR4]).

1. **Notation and preliminary results.** We fix a field k of characteristic zero and set

$$A = k^{[n]} \text{ (the polynomial ring in } n \text{ variables over } k),$$
$$X = \mathbb{A}^n_k = \operatorname{Spec} A,$$
$$T = \mathbb{G}^r_{m,k} \text{ the } r\text{-dimensional torus over } k).$$

Let F be a finite abelian group and suppose

$$G \simeq T \times F$$

acts linearly and effectively on X. We call $n - r$ the codimension of the action. Let

$$\hat{G} \simeq \hat{T} \times \hat{F}$$

be the character group of G. We note that T is canonically determined as a subgroup of G and $\hat{T}$ as a quotient of $\hat{G}$ and that $r = \operatorname{rank} \hat{G}$. The action of G can be diagonalized, that is

$$A = k[x_1, \dots, x_n] \tag{1.1}$$

such that

$$t \cdot x_i = \chi_i(t) x_i$$

with $\chi_i \in \hat{G}$. Moreover, $\chi_1, \dots, \chi_n$ generate $\hat{G}$ (since the action is effective).

Hence, with respect to a fixed diagonalization, the action of G is given by a surjective homomorphism

$$\begin{aligned} \varphi \colon \mathbb{Z}^n &\longrightarrow \hat{G} \\ \alpha = (a_1, \dots, a_n) &\longmapsto \chi^\alpha = \chi_1^{a_1} \dots \chi_n^{a_n} \end{aligned} \tag{1.2}$$

and the action of G is in fact determined by

$$E = \operatorname{Ker} \varphi \subset \mathbb{Z}^n.$$

1.3. Definition - Lemma. Let

$$f = \sum f_\alpha x^\alpha \in A = k[x_1, \dots, x_n].$$

(We use the shorthand notation $\alpha = (a_1, \dots, a_n)$ and $x^\alpha = x_1^{a_1} \cdots x_n^{a_n}$). Then f is a semi-invariant for G if and only if

$$\operatorname{Supp} f = \{\alpha \mid f_\alpha \neq 0\}$$

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is contained in a coset

$$\alpha^* + E$$

of E. We call $\chi = \varphi(\alpha^*)$ the weight of f.

1.4. Suppose χ_n is of infinite order in $\hat{G}$. The stabilizer G' of $(0, \ldots, 0, 1)$ then acts on $X' = x_n^{-1}(1) \simeq \mathbb{A}_k^{n-1}$, and there is an essentially bijective correspondence between G-homogeneous functions on X and G'-homogeneous functions on X' which we now describe.

Let $p: \mathbb{Z}^n \longrightarrow \mathbb{Z}^{n-1}, p(a_1, \ldots, a_n) = (a_1, \ldots, a_{n-1})$ be the projection on the first $n-1$ factors. We obtain a commutative diagram with exact rows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & E & \longrightarrow & \mathbb{Z}^n & \xrightarrow{\varphi} & \hat{G} & \longrightarrow & 0 \\ & & {\scriptstyle\rho}\downarrow & & {\scriptstyle p}\downarrow & & \downarrow & & \\ 0 & \longrightarrow & E' & \longrightarrow & \mathbb{Z}^{n-1} & \xrightarrow{\varphi'} & \hat{G}/\langle\chi_n\rangle & \longrightarrow & 0 \end{array}$$

where, by definition,

$$\varphi'(a_1, \ldots, a_{n-1}) = \chi_1^{a_1} \cdots \chi_{n-1}^{a_{n-1}} \pmod{\chi_n}$$

and

$$E' = \operatorname{Ker} \varphi'.$$

Since χ_n has infinite order, a_n is uniquely determined by $(a_1, \ldots, a_{n-1})$ for any $(a_1, \ldots, a_n) \in E$. Hence

$$\rho: E \longrightarrow E'$$

is an isomorphism and there is a unique homomorphism

$$u: E' \longrightarrow \mathbb{Z}$$

such that

$$\rho^{-1}(a_1, \ldots, a_{n-1}) = \big(a_1, \ldots, a_{n-1}, -u(a_1, \ldots, a_{n-1})\big)$$

for $(a_1, \ldots, a_{n-1}) \in E'$.

$\hat{G}/\langle\chi_n\rangle$ has rank $r-1$ and hence is the character group of $G' = T' \times F'$, where T' is a $(r-1)$-torus and F' is finite abelian. It is clear that G' is the stabilizer of $(0, \ldots, 0, 1) \in X$ and that φ', or E', describes the action of G' on $X' = x_n^{-1}(1)$ in the sense of 1.2. Moreover, if

$$H(x_1, \ldots, x_{n-1}, x_n) \in k[x_1, \ldots, x_{n-1}, x_n, x_n^{-1}]$$

is G-homogeneous of weight $\chi \in \hat{G}$, then $H(x_1, \ldots, x_{n-1}, 1)$ is G'-homogeneous of weight $\bar{\chi} = \chi (\operatorname{mod} \chi_n) \in \hat{G}'$. Conversely, let

$$h(x_1, \ldots, x_{n-1}) = \sum h_\beta x^\beta \in k[x_1, \ldots, x_{n-1}]$$

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be G'-homogeneous of weight $\bar{\chi} \in \hat{G}'$. Pick $\alpha^* = (a_1^*, \ldots, a_{n-1}^*)$ such that $\operatorname{Supp} h \subset \alpha^* + E'$. Then

$$H(x_1, \ldots, x_{n-1}, x_n) = \sum h_\beta x^\beta x_n^{-u(\beta-\alpha^*)} \in k[x_1, \ldots, x_{n-1}, x_n, x_n^{-1}]$$

is G-homogeneous of weight $\varphi(a_1^*, \ldots, a_{n-1}^*, 0)$.

1.5. REMARK. (i) We have on $X' \times G$

$$H(x_1, \ldots, x_{n-1}, \chi_n) = \chi_1^{-a_1^*} \cdots \chi_{n-1}^{-a_{n-1}^*} h(\chi_1 x_1, \ldots, \chi_{n-1} x_{n-1})$$

and H is uniquely determined by this relation.

(ii) E is a submodule of finite index in $\tilde{E}$, the kernel of $\mathbb{Z}^n \longrightarrow \hat{G} \longrightarrow \hat{T} \simeq \mathbb{Z}^r$. Let γ_i be the image of χ_i in $\mathbb{Z}^r$. Then for $\beta = (b_1, \ldots, b_{n-1}) \in \operatorname{Supp} h$ the unique corresponding term in H is

$$h_\beta x^\beta x_n^{a_n} \text{ with } -a_n\gamma_n = (b_1 - a_1^*)\gamma_1 + \cdots + (b_{n-1} - a_{n-1}^*)\gamma_{n-1}.$$

(iii) We have established a bijective correspondence between G-homogeneous elements H of $k[x_1, \ldots, x_{n-1}, x_n, x_n^{-1}]$, up to multiplication by a power of x_n, on the one hand and G'-homogeneous elements h of $k[x_1, \ldots, x_{n-1}]$ on the other.

(iv) In the passage from h to H there is a (generally non unique) choice of α^* that will lead to $H \in k[x_1, \ldots, x_{n-1}, x_n]$ and not divisible by x_n, and the resulting H is unique.

Let the notation be as in 1.1 and 1.5. In general, for $\beta_1, \ldots, \beta_s \in \mathbb{Z}^r$, we denote by $\langle \beta_1, \ldots, \beta_s \rangle_+$ the subsemigroup they generate. We write Γ for the semigroup of the action of T on X and note ([KbR])

$$\Gamma = \langle \gamma_1, \ldots, \gamma_n \rangle_+.$$

We recall that the following two conditions are equivalent ([KbR], [BH]).

1.6. (i) Γ is *unmixed*, that is

$$0 \neq \gamma \in \Gamma \Longrightarrow -\gamma \notin \Gamma,$$

(ii) the action of T on X is *fixpointed*, that is, every T-orbit has a fixpoint in its closure.

1.7. DEFINITION - LEMMA. Let the notation be as in 1.1 and 1.5. We denote by M the set of nonconstant T-invariant monomials in $x_1, \ldots, x_n$ and let

$$\pi: X \longrightarrow X//T = \operatorname{Spec} A^T$$

be the canonical map.

(i) $A^T = k[M]$.

(ii) The *nullcone* (at 0) is defined to be

$$X_0 = \pi^{-1}\big(\pi(0)\big).$$

X_0 is the union of all T-orbits with 0 in their closure, and the ideal of X_0 in A is generated by M. Any irreducible component of X_0 is a linear subspace of X with unmixed T-action.

(iii) An irreducible $x \in A$ is called a *nullvariable* if $x^{-1}(0) \subset X_0$ or, equivalently, if x divides every element of M. In particular, x is a variable and in fact is part of any T-homogeneous system of variables for X. Moreover, a nullvariable is G-homogeneous.

(iv) Let $Y \subset X$ be closed and T-invariant. Then $Y//T$ is naturally identified with a closed subset of $X//T$ and $Y_0 = X_0 \cap Y$.

PROOF. Only the G-homogeneity of a nullvariable needs commenting upon. It follows from: (a) The finite group F permutes the codimension 1 components of X_0. (b) Two nonproportional nullvariables have different T-weights. If not, one could replace the first by the second wherever it occurs in an invariant monomial and thereby construct an invariant monomial not divisible by the first.

1.8. REMARK. Suppose T acts with codimension ≤ 2 and let $Y \simeq \mathbb{A}_k^{n-1}$ be closed in X with $Y \not\subset x_i^{-1}(0), i = 1, \ldots, n$. Then T acts effectively on Y with codimension ≤ 1, and hence this action is linearizable by [BB]. In particular, the action has a fixpoint, which we may assume to be the origin of X.

Our next two statements are certainly well known to the specialists. We include them for lack of proper references.

1.9. PROPOSITION ("EQUIVARIANT EPIMORPHISM THEOREM"). *Let the group G act linearly on $k[x,y]$ and suppose $f \in k[x,y]$ is G-semi-invariant and defines a line (i.e. $k[x,y]/f \simeq k^{[1]}$). Then there exists a G-semi-invariant $g \in k[x,y]$ such that $k[x,y] = k[f,g]$.*

PROOF. The proof of the epimorphism theorem given in [A] shows: We may assume f is monic in y and $\deg_y f = \deg f$. Then there is a divisor d of $\deg_y f$ such that $k[x,y] = k[f,g]$, where g is the d-approximate root *w.r.t.* y of f. By the uniqueness properties of approximate roots, g is G-semi-invariant.

1.10. LEMMA. *Let B be a factorial affine k-domain and K its field of quotients. Let A be a B-algebra and $u_1, \ldots, u_r \in A$ such that*

$$K \otimes_B A = K[u_1, \ldots, u_r].$$

Suppose that $u_1, \ldots, u_r$ are algebraically independent mod mA for every maximal ideal m of B. Then

$$A = B[u_1, \ldots, u_r].$$

PROOF. For each m,

$$(B/m)[u_1, \ldots, u_r] \longrightarrow A/mA$$

is injective by assumption, that is, $mA \cap B[u_1, \ldots, u_r] = mB[u_1, \ldots, u_r]$. Given $x \in A$, there exists $p \in B$ such that $px \in B[u_1, \ldots, u_r]$. Hence if $m \ni p, px \in B[u_1, \ldots, u_r] \cap pA \subset B[u_1, \ldots, u_r] \cap mA = mB[u_1, \ldots, u_r]$. It follows that $px \in \bigcap_{m \ni p} mB[u_1, \ldots, u_r] = \tilde{p}B[u_1, \ldots, u_r]$, where $\tilde{p}$ is the product of the distinct prime divisors of p. If we choose p with a minimal number of prime divisors, we have $p = 1$.

2. **The cases** $\dim X//T = 0$ **and** $r + \dim X^T \geq n - 1$. We give part of the proof of the main theorem in this section. We note that

$$\dim X^T \leq \dim X//T \leq n - r \leq 2.$$

We keep the notation of 1.1 and 1.7. We will assume tacitly that $0 \in Y$ and that none of $x_1, \ldots, x_n$ vanishes identically on Y.

2.1. If $\dim X//T = 0$, then the action of T on X is unmixed and no T-weight γ_i is 0. Hence if x_1, say, appears as a monomial in the equation H of Y, we have, up to multiplication by a constant, $H = x_1 + \tilde{H}$ with $\tilde{H} \in k[x_2, \ldots, x_n]$ and G-homogeneous.

2.2. LEMMA. *(i) Let $Y \subset X$ be an irreducible, G-stable closed hypersurface such that $Y^T = X^T$ and Y is smooth along X^T. If*

$$r + \dim X^T \geq n - 1,$$

then the defining equation for Y is part of a G-homogeneous system of variables for X.

(ii) Suppose $Y \simeq \mathbb{A}^{n-1}$, $r = n - 2$ and $\dim Y^T \geq 1$. Then the defining equation for Y is part of a G- homogeneous system of variables for X.

PROOF. In [KR1], Lemma 1.1 and Lemma 1.2, T-homogeneous variables defining Y are constructed, and it remains to verify only that they are part of a G-homogeneous system of variables. As to (i), we observe that if $f = \alpha x_{r+1} - \beta x_r$ is the equation for Y as in the proof of [KR1], Lemma 1.1, then α and β are G-homogeneous (and coprime), and we can choose a and b G-homogenous such that $a\alpha + b\beta = 1$. Then $g = bx_{r+1} + ax_r$ is G-homogeneous as well. As to (ii), we note that in the case (i) of the proof of [KR1], Lemma 1.2, $\mathbb{A}^1 \simeq Y^T \subset X^T \simeq \mathbb{A}^2$ is G-stable for the natural action of G on X^T and hence, by 1.9, defined by a variable that is part of a G-homogeneous system of variables for X^T.

2.3. By 2.2, the main theorem is proven in the following cases:

(i) $r \geq n - 1$,

(ii) $\dim X^T = 2$,

(iii) $\dim X^T = \dim Y^T = 1$.

2.4. Proof of the main theorem in case $\dim X^T = 1$ and $\dim Y^T = 0$. There is a unique variable, x_1 say, with T-weight 0, and we have a G-equivariant decomposition $X = X^T \times V$ with $X^T = \operatorname{Spec} k[x_1]$, $V = \operatorname{Spec} k[x_2, \ldots, x_n]$.

(α) Suppose $\dim X//T = 1$. Then $X^T = X//T$ and $Y = 0 \times V$.

(β) Suppose $\dim X//T = 2$. Then $X//T = X^T \times V//T$ with $V//T = \operatorname{Spec} k[m]$, where $m = x_2^{a_2} \cdots x_n^{a_n}$ is T-invariant. Now $\dim Y//T \leq 1$ since T acts effectively with codimension 1 on Y, and $\dim Y//T = 0$ is not possible since otherwise $Y \subset X_0$, which is of dimension $\leq n - 2$. Hence $Y//T \simeq \mathbb{A}^1$. Moreover, since Y is smooth and Y^T is connected, $Y//T$ and X^T meet normally in one point in $X//T = \operatorname{Spec} k[x_1, m]$, with $m = 0$ as equation for X^T. Hence the equation for $Y//T$ in $X//T$ is of the form $x_1 + \phi(m)$ with $\phi(m) \in k[m]$, and the equation for Y in X is $x_1 + \phi(x_2^{a_2} \cdots x_n^{a_n})$, so clearly part of G-homogeneous system of variables for X.

3. **The main induction.** We continue the proof of the main theorem. By the results of Section 2 it remains to consider the case

$$\dim X//T > 0 \text{ and } \dim X^T = 0.$$

Note that now $X^T = \{0\}$ and $0 \in Y$. We keep in force the assumption $Y \not\subset x_i^{-1}(0)$, $i = 1, \dots, n$.

3.1. LEMMA. *Suppose x_n is a nullvariable on X and*

$$H = x_n + \tilde{H}$$

is irreducible and G-homogeneous with $\tilde{H}(0) = 0$ and $(0, \dots, 0, 1) \notin \operatorname{Supp} \tilde{H}$. Then $\tilde{H} = 0$.

PROOF. Since x_n is a nullvariable, there exists

$$(*) \qquad \alpha = (a_1, \dots, a_n) \in E, \quad a_1, \dots, a_{n-1} \geq 0, \quad a_n > 0.$$

On the other hand, if $\tilde{H} \neq 0$, $\tilde{H}$ has a monomial not divisible by x_n. This gives a relation $\chi_n = \chi_1^{c_1} \cdots \chi_{n-1}^{c_{n-1}}$ and hence an element

$$(**) \qquad \gamma = (c_1, \dots, c_{n-1}, -1) \in E, \quad c_1, \dots, c_{n-1} \geq 0.$$

Since x_n is a nullvariable, the action of T on $\{x_n = 0\}$ is unmixed (see 1.7 (ii)), that is, the semi group $\Gamma' = \langle \gamma_1, \dots, \gamma_{n-1} \rangle_+$ is unmixed. But $(*)$ and $(**)$ give $-a_n\gamma_n \in \Gamma'$ and $a_n\gamma_n \in \Gamma'$, and this is not possible. So $\tilde{H} = 0$.

3.2. LEMMA. *If $x \in A$ is a nullvariable for X, then $y = x|Y$ is a nullvariable for Y.*

PROOF. By 1.7(iii), x is one of the x_i. Say $x = x_n$. By assumption, $Y \not\subset X_n = x_n^{-1}(0)$ and hence $Z = X_n \cap Y$ consists of codimension one components of the nullcone Y_0 of Y, all passing through the origin (see 1.7(iv)). If L is the linear part of the equation H of Y, then L is linearly independent of x_n by 3.1, that is, Y and X_n meet normally in the origin in one irreducible component. It follows that y is irreducible on Y and hence a nullvariable by 1.7(iii).

3.3. LEMMA. *There exists $x \in A$, x a nullvariable for X.*

PROOF. (i) Suppose $\dim X//T = 1$. Then $X//T = k[m]$, where m is an invariant monomial, and any x_i dividing m is a nullvariable on X.

(ii) Suppose $\dim X//T = 2$. Then $E \subset \mathbb{Z}^n$, the module of relations among the G-characters of the action (see 1.2) is of rank 2 and contains two linearly independent elements $\alpha = (a_1, \dots, a_n)$ and $\beta = (b_1, \dots, b_n)$ with non-negative coefficients, *i.e.*

$$\alpha, \beta \in E_+ = E \cap (\mathbb{Z}_+)^n.$$

For $\gamma = (c_1, \dots, c_n) \in \mathbb{Z}^n$, put $\operatorname{supp} \gamma = \{i \mid c_i \neq 0\}$. Choose $j \in \operatorname{supp} \beta$ such that $a_j/b_j = \min_i \{a_i/b_i\}$. Then $\alpha' = b_j\alpha - a_j\beta \in E_+$, α' and β are linearly independent, and $j \notin \operatorname{supp}\alpha'$. So we may assume to begin with that $\operatorname{supp}\beta \not\subset \operatorname{supp}\alpha$ and, repeating the argument if necessary, that $\operatorname{supp}\alpha \not\subset \operatorname{supp}\beta$ as well. Then every element of E_+ is a linear combination with non-negative rational coefficients of α and β and hence x_i is a nullvariable for X if and only if $i \in \operatorname{supp}\alpha \cap \operatorname{supp}\beta$.

So assume $\operatorname{supp}\alpha \cap \operatorname{supp}\beta = \emptyset$. We may assume that the equation of Y is of the form

$$H = x_n + \tilde{H}$$

where $\tilde{H} \neq 0$ has a monomial not divisible by x_n. As in the Proof of 3.1 we obtain an element

$$\gamma = (c_1, \ldots, c_{n-1}, -1) \in E$$

with $c_i \geq 0$, $i = 1, \ldots, n-1$. We claim that α, β, γ are linearly independent, in contradiction to rank $E = 2$. In fact, this is obvious if $n \notin \operatorname{supp}\alpha \cup \operatorname{supp}\beta$. So suppose $n \in \operatorname{supp}\alpha$, say. Now $\operatorname{supp}\alpha$ has at least two elements (since $X^T = \{0\}$) and we may assume $n-1 \in \operatorname{supp}\alpha$ as well, that is, $\alpha = (a_1, \ldots, a_{n-1}, a_n)$ with $a_{n-1} > 0$, $a_n > 0$. A nontrivial relation $a\alpha + b\beta + c\gamma = 0$ is then not compatible with $\operatorname{supp}\alpha \cap \operatorname{supp}\beta = \emptyset$.

3.1. *The induction.* Suppose now $n \geq 3$. In view of 3.2 and 3.3 we may assume that x_n, say, is a nullvariable on X and Y and that the linear part of the equation H for Y does not depend on x_n. Put $X' = x_n^{-1}(1)$ and $Y' = X' \cap Y$. We are then in the situation of 1.4. By 1.8, the action of G on Y is linearizable and by 1.7(iii), x_n restricts to a variable on Y. Hence $Y' \simeq \mathbb{A}^{n-2}$. By induction on n (for $n = 2$ we invoke the equivariant epimorphism theorem) we may assume that

$$h(x_1, \ldots, x_{n-1}) = H(x_1, \ldots, x_{n-1}, 1)$$

is part of a G'-homogeneous system of variables

$$h_1 = h, h_2, \ldots, h_{n-1}$$

for X'. Say x_1 appears in the linear part of H. Free to modify x_1 by x_1-free terms in H, we may assume to begin with that no term in H is of the form $x_i x_n^c$, $c \geq 0$, $i \geq 2$. Also, a term $x_1 x_n^c$, $c \geq 1$, cannot appear since otherwise $\gamma_n = 0$. Hence $h_1 = x_1 + \tilde{h}_1$ with $\operatorname{ord}_0 \tilde{h}_1 > 1$. After a G'-homogeneous linear change of variables we may then assume that $h_i = x_i + \tilde{h}_i$ with $\operatorname{ord}_0 \tilde{h}_i > 1$ for $i = 2, \ldots, n-1$ as well.

Now pick $\alpha_i^* = (0, \ldots, 1, \ldots, 0)$ (a 1 in place i) and let H_i be defined by h_i and α_i^* as in 1.4. Then $H_1 = H$ (see 1.5(iv)). Let $i \geq 2$ and consider $\beta = (b_1, \ldots, b_{n-1}) \in \operatorname{Supp} h_i$. By 1.5(ii), the corresponding monomial in H_i is $x^\beta x_n^{b_n}$ with

(*) $$-b_n\gamma_n = b_1\gamma_1 + \cdots + (b_i - 1)\gamma_i + \cdots + b_{n-1}\gamma_{n-1}.$$

Since $\langle \gamma_1, \ldots, \gamma_{n-1} \rangle_+ = \Gamma'$ is unmixed and

(**) $$-c_n\gamma_n \in \Gamma' \text{ with } c_n > 0$$

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(see the Proof of 3.1), $b_n < 0$ can occur in $(*)$ for $b_i = 0$ only. Suppose it does, for $i = n-1$, say. We then have

$$-b_n\gamma_n \in -\gamma_{n-1} + \Gamma',$$

and in view of $(**)$ this can occur for finitely many $b_n < 0$ only. Fix one of them.

We then have "relations"

$$\varepsilon_1 = (b_1, \ldots, b_{n-2}, -1, b_n) \in E,$$
$$\varepsilon_2 = (c_1, \ldots, c_{n-2}, c_{n-1}, c_n) \in E,$$

with $b_1, \ldots, b_{n-2} \geq 0$, $b_n < 0$, $c_1, \ldots, c_{n-1} \geq 0$ and $c_n > 0$. Suppose we have $\varepsilon_1' = (b_1', \ldots, b_{n-2}', -1, b_n) \in E$ as well with $b_1', \ldots, b_{n-2}' \geq 0$ and $\varepsilon_1' \neq \varepsilon_1$. Then ε_1' is not in the $\mathbb{Q}$-span of ε_1 and ε_2, and since rank $E = 2$, ε_1 and ε_2 are linearly dependent and hence $b_1 = \cdots = b_{n-2} = 0, -\gamma_{n-1} + b_n\gamma_n = 0$ and $b_1'\gamma_1 + \cdots + b_{n-2}'\gamma_{n-2} = 0$ with $b_i' > 0$ for at least one i. We therefore have an invariant monomial $x_1^{b_1'} \cdots x_{n-2}^{b_{n-2}'}$ not divisible by x_n, and x_n is not a nullvariable.

Hence for each $i = 2, \ldots, n-1$ there are only finitely many $\beta = (b_1, \ldots, b_{n-1}) \in (\mathbb{Z}_+)^{n-1}$ such that $(*)$ holds with $b_n < 0$. Moreover, $b_i = 0$ in that case. We can therefore modify h_2 by a G'-homogeneous polynomial in $h_1, h_3 \ldots, h_{n-1}$ to make sure this case does not occur, then do the same successively for $h_3, \ldots, h_{n-1}$. We will then have for $i = 1, \ldots, n-1$

$$(***) \qquad H_i = x_i + \tilde{H}_i \in k[x_1, \ldots, x_{n-1}, x_n]$$

with $\operatorname{ord}_0 \tilde{H}_i > 1$.

Now in $k[G][x_1, \ldots, x_{n-1}]$, the homomorphism $\eta\colon x_i \longmapsto H_i$ is just conjugation by the automorphism $x_i \longmapsto \chi_i x_i$, and it is defined over $k[x_n, x_n^{-1}] \subset k[G]$, with $x_n = \chi_n$. Hence

$$k[x_1, \ldots, x_{n-1}, x_n, x_n^{-1}] = k[H_1, \ldots, H_{n-1}, x_n, x_n^{-1}].$$

Moreover, by $(***)$, $H_1, \ldots, H_{n-1}$ are algebraically independent mod x_n. By 1.10, applied to $B = k[x_n] \subset A$, we have

$$A = k[H_1, \ldots, H_{n-1}, x_n].$$

This finishes the induction.

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