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Moduli spaces of framed logarithmic and parabolic connections on a Riemann surface

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Abstract

We construct moduli spaces of framed logarithmic connections and also moduli spaces of framed parabolic connections. It is shown that these moduli spaces possess a natural algebraic symplectic structure. We also give an upper bound of the transcendence degree of the algebra of regular functions on the moduli space of parabolic connections.

Contents

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1. Introduction

Let X be a compact connected Riemann surface. Since the fundamental group $\pi_1(X)$ of X is finitely presented, and $\mathrm{GL}(r,\mathbb{C})$ is an affine algebraic group defined over $\mathbb{C}$, the space of homomorphisms $\mathrm{Hom}(\pi_1(X),\ \mathrm{GL}(r,\mathbb{C}))$ is an affine complex algebraic variety. The adjoint action of $\mathrm{GL}(r,\mathbb{C})$ on itself produces an action of $\mathrm{GL}(r,\mathbb{C})$ on $\mathrm{Hom}(\pi_1(X),\ \mathrm{GL}(r,\mathbb{C}))$. The moduli space

$$\mathcal{M}_R(r) \ := \ \mathrm{Hom}(\pi_1(X),\ \mathrm{GL}(r,\mathbb{C}))/\!\!/\mathrm{GL}(r,\mathbb{C}) = \mathrm{Spec}\, \mathbb{C}[\mathrm{Hom}(\pi_1(X),\ \mathrm{GL}(r,\mathbb{C}))]^{\mathrm{GL}(r,\mathbb{C})}$$

of equivalence classes of representations has an algebraic symplectic structure which was constructed by Goldman [Gol] and Atiyah and Bott [AtBo]. Let $\mathcal{M}_C(r)$ be the moduli space of holomorphic connections on X of rank r. This moduli space also has an algebraic symplectic structure. The Riemann–Hilbert correspondence identifies $\mathcal{M}_R(r)$ with $\mathcal{M}_C(r)$. The Riemann–Hilbert correspondence is only complex analytic and not algebraic, and consequently the identification between $\mathcal{M}_R(r)$ and $\mathcal{M}_C(r)$ is complex analytic but not algebraic. However, the transport of the symplectic form on $\mathcal{M}_R(r)$ to $\mathcal{M}_C(r)$ by this complex analytic identification actually remains algebraic. This paper is divided into two parts. The first part is related to the fact that $\mathcal{M}_C(r)$ has an algebraic symplectic structure. The second part is related to the fact that the Riemann–Hilbert correspondence is not algebraic.

First, let us discuss on the first part. Fix finitely many distinct points $x_1, \ldots, x_n$ of X and denote the divisor $x_1 + \cdots + x_n$ on X by D. Consider logarithmic connections on X of rank r whose polar part is supported on D. The corresponding moduli space is known to have a Poisson structure. This Poisson structure is *not* symplectic if $n > 0$.

It is shown in Corollary 4.24 that the Poisson structure on the moduli space of logarithmic connections can be elevated to a symplectic structure by introducing frames, over the points of D, of the holomorphic vector bundle underlying the logarithmic connections. This entails construction of the moduli space of framed logarithmic connections that occupy a large fraction of the article. The key theorem in the first part of this paper is Theorem 4.21, which establishes the d-closedness of the canonical nondegenerate 2-form on the moduli space of framed connections. This produces a Poisson structure on the moduli space of logarithmic connections; a geometric invariant theoretic construction of this moduli space was given by Nitsure [Nit].

In [BLP1] and [BLP2], generalized Higgs bundles on X were considered where the Higgs fields are allowed to have poles along a fixed divisor D on X. The corresponding moduli spaces have a Poisson structure which was constructed independently by Bottacin [Bo] and Markman [Mark]. It was shown in [BLP1] and [BLP2] that by imposing frames of the vector bundles underlying the Higgs bundles, over D, these Poisson structures can be enhanced to symplectic structure. The present work is an analogue of [BLP2] for connections in place of Higgs fields.

The moduli space of logarithmic parabolic connections was constructed in [IIS] and [Ina]. If we fix eigenvalues of residues of logarithmic parabolic connections, then the moduli space of logarithmic parabolic connections with the fixed eigenvalues of residues has a canonical symplectic structure. In § 4.6, we discuss a relationship between the framed logarithmic connections and the logarithmic parabolic connections. As an outcome, it is proved that the moduli space of logarithmic parabolic connections has a canonical Poisson structure, whose restriction to the locus of fixed eigenvalues of residues induces the symplectic structure due to [IIS] and [Ina] (Corollary 4.25). Moreover, this Poisson structure satisfies the condition that the forgetful map to the moduli space of logarithmic connections, that forgets the parabolic structure, is a Poisson map. The restriction of this Poisson map to the loci of fixed eigenvalues of residues is an isomorphism if the eigenvalues are generic, and it produces a resolution of singularities if the eigenvalues are special.

Now let us discuss the second part. In this part, we focus on the algebraic moduli space of logarithmic parabolic connections such that eigenvalues of residues are fixed. We call this moduli space the *de Rham moduli space*. This moduli space is related to other moduli spaces having rich geometric structures. First, there is the moduli space of equivalence classes of representations of the fundamental group $\pi_1(X \setminus D)$ with fixed local monodromy data around the points of D, which is known as the *character variety*. The relationship between the moduli space of logarithmic parabolic connections and the character variety is given by the Riemann–Hilbert correspondence. In the framework of [IIS] and [Ina], the Riemann–Hilbert correspondence gives a simultaneous family of holomorphic maps from the de Rham moduli spaces to the character varieties over all the eigenvalues of residues. This Riemann–Hilbert morphism is biholomorphic when the eigenvalues of residues are generic, and it is an analytic resolution of singularities when the eigenvalues of residues are special. Note that the characteristic variety in [IIS] and [Ina] is not smooth for special eigenvalues of residues, but its singularities actually well explain the geometry of special solutions of the isomonodromy equation (see [SaTe]). Simpson introduced in [Sim1] the notion of a filtered local system which bijectively corresponds to the parabolic connections under the assumption that the eigenvalues of residues are fixed. In [Ya], Yamakawa constructed the algebraic moduli space of filtered local systems, which is actually nonsingular. We call it the *Betti moduli space*. Yamakawa proved in [Ya] that the Riemann–Hilbert morphism is a biholomorphism between the de Rham moduli space and the Betti moduli space. Secondly, there is the moduli space of logarithmic parabolic Higgs bundles with fixed eigenvalues of residues together with stability data. We call this moduli space the *Dolbeault moduli space*. The relation between these moduli spaces is given by the logarithmic version of the non-abelian Hodge theory constructed by Simpson in [Sim1].

In the case where the polar divisor D is empty, Simpson introduced in [Sim2] and [Sim3], the three moduli spaces in his framework: the *de Rham moduli space*, the *Dolbeault moduli space* and the *Betti moduli space*. These are *algebraic* moduli spaces and are related to each other by the non-abelian Hodge theory and the Riemann–Hilbert correspondence. However, the algebraic structures of these moduli spaces are very different. In this paper, we consider the logarithmic version of these three moduli spaces. First, our Betti moduli space is affine when the eigenvalues of the residues are generic. So the transcendence degree of its affine coordinate ring is equal to the dimension of the moduli space. On the other hand, the transcendence degree of the ring of global algebraic functions on the Dolbeault moduli space is exactly the half of the dimension of the moduli space, a fact which is deduced from the properness of the Hitchin map. In some cases, the global algebraic functions on the de Rham moduli spaces are simply the constant scalars [BiRa]. For general logarithmic connections, the coefficients of the characteristic polynomial of residue at each singular point give algebraic functions on the moduli space. The main theorem of the second part of this paper is Theorem 5.22, which states that the transcendence degree of the ring of global algebraic functions on our de Rham moduli space is less than or equal to that of our Dolbeault moduli space. In particular, our de Rham moduli space is not affine (this was announced in [BIKS]). To be precise, there was in fact an inadequate argument on finite generation of a graded ring in the outline of the proof of [BIKS, Theorem 10]. In this paper, we reconstruct a proof of it through a refinement of the statement (see Theorem 5.22). As a consequence of Theorem 5.22, the Riemann–Hilbert morphism, which appears in [IIS], [Ina], is not algebraic in the logarithmic case (see Corollary 5.25).

Regarding the above three moduli spaces, we are mostly interested in the case where X is defined over the field of complex numbers. However, it is also worth considering the case where the base field is of positive characteristic. When the base field is of positive characteristic, N. Katz introduced the notion of p-curvature in [Ka], from which Laszlo and Pauly derived the proper Hitchin map on a de Rham moduli space (see [LaPa]). By the investigation of the Hitchin map on a de Rham moduli space by Groechenig in [Groe], the ring of global algebraic functions on the de Rham moduli space of connections without pole has the same transcendence degree as that of the ring of global algebraic functions on the Dolbeault moduli space, when the characteristic of the base field is positive. So the similar inequality as in Theorem 5.22 for connections without pole becomes the equality for curves when the base field is of positive characteristic, while the inequality is strict for curves of higher genus defined over the field of complex numbers (see [BiRa]).

Analogous to the regular case in [Bi], we can also show in the logarithmic case that the pullback, via the Riemann–Hilbert morphism, of the canonical algebraic symplectic form on the Betti moduli space coincides with that on the de Rham moduli space. Although not stated explicitly, it can also be found in the proof of [Ina, Proposition 7.3]. This was also proved in the earlier work in the rank two case by Iwasaki [Iw1]. In fact, the main point of [Iw1] is the construction of the isomonodromic lift of the family of symplectic forms. A more conceptual construction of the isomonodromic lift of the family of symplectic forms was constructed by Komyo in [Ko], from the moduli theoretic point of view, by using the cohomological description of the isomonodromic deformation given in [BHH].

Boalch proved the following: The monodromy map between any moduli space of unramified irregular singular connections of any rank on a curve of genus zero and its corresponding wild character variety is symplectic structure preserving [Bo1, p. 182, Theorem 6.1] (see also [Bo2]). The algebraic moduli space of unramified irregular singular connections and its algebraic symplectic structure are constructed in [InSa].

Now we give a brief outline of the contents of this paper.

§ 2 provides general notions of framed principal G-bundles on a compact Riemann surface X and also of framed G-connections.

From § 3, we restrict to the case of $G = \mathrm{GL}(r, \mathbb{C})$. §§ 3.1 provides the formulation of moduli problem for framed connections. §§ 3.2 provides the construction of the moduli space of framed $\mathrm{GL}(r, \mathbb{C})$-connections as a Deligne–Mumford stack and also the irreducibility of its open substack where the underlying framed bundles are simple.

§ 4 is devoted to the construction of a canonical 2-form on the moduli space of framed connections and also to prove its d-closedness. The main technical part is §§ 4.3. Over the open subset where the underlying framed bundles are simple, the canonical 2-form on the moduli space of framed connections becomes d-closed (Propositions Proposition 4.7 and 4.17). Its proof is essentially reduced to the d-closedness of the canonical 2-form on the character variety constructed by Goldman in [Gol] when the genus of X is greater than 1. When the genus of X is zero or one, the proof of d-closedness is reduced to that for the form on the moduli space of parabolic connections given in [Ina]. In §§ 4.4, we prove the d-closedness of the canonical 2-form on the entire moduli space of simple framed connections (see Theorem 4.21), which is the main theorem of the first half. Its proof is reduced to Propositions 4.7 and 4.17 through an argument for extending the polar divisor. §§ 4.5 and 4.6 are immediate consequences of Theorem 4.21. We can see that the Poisson structure on several known moduli spaces of connections can be reconstructed from the symplectic structure on the moduli space of framed connections.

§ 5 is devoted to establishing an upper bound for the transcendence degree of the ring of global algebraic functions on the moduli space of parabolic connections. In §§ 5.1, we recall the notions of parabolic connections and parabolic Higgs bundles, which work over the base field of arbitrary characteristic. In §§ 5.2, we prove in Proposition 5.14 that the locus of non-simple underlying quasi-parabolic bundles has codimension at least 2 in the moduli space of parabolic connections. The proof is carried out by constructing a parameter space of non-simple quasi-parabolic bundles and a compatible connections on them. The essential part is to bound the dimension of the parameter space of non-simple quasi-parabolic bundles (see Propositions 5.10, 5.11, 5.12 and 5.13). Since we need to verify many cases, the proofs of these propositions contain a considerable amount of calculation, but each step is checked by relatively elementary arguments. By virtue of Proposition 5.14, the ring of global algebraic functions on the moduli space of parabolic connections can be replaced with that on the open loci where the underlying quasi-parabolic bundles are simple. §§ 5.3 provides the main estimate for the transcendence degree of the global algebraic functions on the moduli space of parabolic connections. Over the moduli space of simple quasi-parabolic bundles, we construct in Proposition 5.21 something like a relative compactification of a Deligne–Hitchin family, whose generic fiber is a relative compactification of the moduli space of compatible parabolic connections and whose special fiber is that of parabolic Higgs bundles. This family gives a family of sheaves of graded rings over the moduli space of simple quasi-parabolic bundles. A rough idea of the proof of Theorem 5.22 is to estimate the transcendence degree of the ring of global sections of this sheaf of graded rings. In order to correct the flaw in the proof of [BIKS, Theorem 10], we actually consider the subring generated by a suitable transcendence basis of the graded ring over a generic fiber and compare it with that on the special fiber.

2. Framed G-connections

Let X be a compact connected Riemann surface, and let $x_1, \ldots, x_n$ be finitely many distinct points on X. Let

$$D = x_1 + \cdots + x_n \tag{2.1}$$

be the reduced effective divisor on X. For notational convenience, the subset $\{x_1, \ldots, x_n\} \subset X$ will also be denoted by D. Denote by K_X the holomorphic cotangent bundle of X.

2.1 Framed principal G-bundles

Let G be a connected complex reductive affine algebraic group. The Lie algebra of G will be denoted by $\mathfrak{g}$. Let

$$p : E_G \longrightarrow X \tag{2.2}$$

be a holomorphic principal G-bundle over X. For any point $x \in X$, the fiber $p^{-1}(x) \subset E_G$ will be denoted by $(E_G)_x$.

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Definition 2.1 (See [BLP2,p. 5]).For each point x of the subset D in (2.1), fix a closed complex Lie proper subgroup

$$H_x \subsetneq G\,.$$

A framing of E_G over the divisor D is a map

$$\phi:\ D \longrightarrow \bigcup_{x\in D} (E_G)_x/H_x$$

such that $\phi(x) \in (E_G)_x/H_x$ for every $x \in D$. A framed principal G-bundle on X is a holomorphic principal G-bundle E_G on X equipped with a framing over D.

A framing ϕ of E_G produces a reduction of structure group

$$\mathbf{H}_x \ :=\ q_x^{-1}(\phi(x)) \subset (E_G)_x \tag{2.3}$$

to H_x at each point $x \in D$, where $q_x : (E_G)_x \longrightarrow (E_G)_x/H_x$ is the quotient map.

2.2 Adjoint bundle for framed principal G-bundles

Let $T_{E_G/X} \longrightarrow E_G$ be the relative tangent bundle for the projection p in (2.2). Using the action of the group G on E_G, this relative tangent bundle $T_{E_G/X}$ is identified with the trivial vector bundle $E_G \times \mathfrak{g} \longrightarrow E_G$ with fiber $\mathfrak{g} = \mathrm{Lie}(G)$. The quotient $(T_{E_G/X})/G$ is a vector bundle over X. The above identification of $T_{E_G/X}$ with $E_G \times \mathfrak{g}$ produces an identification of $(T_{E_G/X})/G$ with the vector bundle on X associated to the principal G-bundle E_G for the adjoint action of G on $\mathfrak{g}$. This associated vector bundle, which is denoted by $\mathrm{ad}(E_G)$, is called the *adjoint bundle* for E_G. The fiber over any $x \in X$ for the natural projection $\mathrm{ad}(E_G) \longrightarrow X$ will be denoted by $\mathrm{ad}(E_G)_x$; it is a Lie algebra isomorphic to $\mathfrak{g}$.

Since the group G is reductive, its Lie algebra $\mathfrak{g}$ admits G-invariant nondegenerate symmetric bilinear forms. Fix a G-invariant nondegenerate symmetric bilinear form

$$\sigma\ :\ \mathrm{Sym}^2(\mathfrak{g}) \longrightarrow \mathbb{C} \tag{2.4}$$

on $\mathfrak{g}$. From the above construction of $\mathrm{ad}(E_G)$ it follows that given any point $z \in (E_G)_y$ there is a corresponding isomorphism of Lie algebras $I_z : \mathfrak{g} \longrightarrow \mathrm{ad}(E_G)_y$. Using I_z, the form σ in (2.4) produces a symmetric nondegenerate bilinear form on the fiber $\mathrm{ad}(E_G)_y$; this bilinear form on $\mathrm{ad}(E_G)_y$ constructed using σ is actually independent of the choice of the point z because σ is G-invariant. Let

$$\widehat{\sigma}\ :\ \mathrm{Sym}^2(\mathrm{ad}(E_G)) \longrightarrow \mathcal{O}_X \tag{2.5}$$

be the bilinear form constructed as above using σ.

Let ϕ be a framing of E_G over D. For every $x \in D$, define the Lie subalgebra

$$\mathcal{H}_x \ :=\ \mathrm{ad}(\mathbf{H}_x) \subset \mathrm{ad}(E_G)_x \tag{2.6}$$

(see (2.3)).

2.3 Framing of G-connections

Take a holomorphic principal G-bundle E_G over X. Let TE_G be the holomorphic tangent bundle of E_G. Consider the action of G on TE_G given by the tautological action of G on E_G. The quotient

$$\mathrm{At}(E_G)\ :=\ (TE_G)/G$$

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is a holomorphic vector bundle over X; it is called the *Atiyah algebra* for E_G. The Lie bracket operation of the vector fields on E_G produces a Lie algebra structure on the coherent sheaf associated to $\mathrm{At}(E_G)$. There is a short exact sequence of holomorphic vector bundles on X,

$$0 \longrightarrow \mathrm{ad}(E_G) \longrightarrow \mathrm{At}(E_G) \xrightarrow{p_{\mathrm{At}}} TX \longrightarrow 0\,, \tag{2.7}$$

where the projection p_{At} is given by the differential dp of the map p in (2.2) [At1]. All the homomorphisms in (2.7) are compatible with the Lie algebra structures. Define a holomorphic vector bundle $\mathrm{At}_D(E_G)$ over X as

$$\mathrm{At}_D(E_G) \;:=\; p_{\mathrm{At}}^{-1}(TX \otimes \mathcal{O}_X(-D)) \subset \mathrm{At}(E_G)\,.$$

Then (2.7) gives the following short exact sequence of holomorphic vector bundles on X:

$$0 \longrightarrow \mathrm{ad}(E_G) \longrightarrow \mathrm{At}_D(E_G) \xrightarrow{p_{\mathrm{At}_D}} TX(-D) \;:=\; TX \otimes \mathcal{O}_X(-D) \longrightarrow 0\,, \tag{2.8}$$

where p_{At_D} is the restriction, to $\mathrm{At}_D(E_G) \subset \mathrm{At}(E_G)$, of the homomorphism p_{At} in (2.7).

Definition 2.2 [At1].A holomorphic connection on E_G is a holomorphic homomorphism of vector bundles

$$\nabla \,:\, TX \longrightarrow \mathrm{At}(E_G)$$

such that $p_{\mathrm{At}} \circ \nabla = \mathrm{Id}_{TX}$, where p_{At} is the projection in (2.7). A D-twisted holomorphic connection on E_G (also called a logarithmic connection on E_G with polar part on D) is a holomorphic homomorphism of vector bundles

$$\nabla \,:\, TX(-D) \longrightarrow \mathrm{At}_D(E_G)$$

such that $p_{\mathrm{At}_D} \circ \nabla = \mathrm{Id}_{TX(-D)}$, where p_{At_D} is the homomorphism in (2.8).

For a D-twisted holomorphic connection ∇ on E_G, consider the commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathrm{ad}(E_G) & \longrightarrow & \mathrm{At}_D(E_G) & \xleftarrow{\nabla} & TX(-D) & \longrightarrow & 0 \\
 & & \| & & \big\downarrow \iota'' & & \big\downarrow \iota' & & \\
0 & \longrightarrow & \mathrm{ad}(E_G) & \longrightarrow & \mathrm{At}(E_G) & \xrightarrow{p_{\mathrm{At}}} & TX & \longrightarrow & 0
\end{array}$$

where ι' and ι'' are the natural inclusion homomorphisms. For any point $x \in D$, the homomorphism of fibers

$$\iota'(x) \,:\, TX(-D)_x \longrightarrow T_xX$$

vanishes, and hence $(p_{\mathrm{At}} \circ \iota'' \circ \nabla)(TX(-D)_x) = 0$ by the commutativity of the above diagram. Consequently, we have

$$(\iota'' \circ \nabla)(TX(-D)_x) \subset \mathrm{ad}(E_G)_x\,.$$

Note that for any point $x \in D$, using the Poincaré adjunction formula it follows that

$$a_x \,:\, TX(-D)_x \xrightarrow{\sim} \mathbb{C}. \tag{2.9}$$

The element

$$\mathrm{res}_x(\nabla) \;:=\; (\iota'' \circ \nabla)(1) \in \mathrm{ad}(E_G)_x$$

is called the residue of the logarithmic connection ∇ at x. To describe this residue explicitly, first recall that a holomorphic connection on E_G furnishes lift of holomorphic vector fields on any open subset U of X to G–invariant holomorphic vector fields on $E_G|_{p^{-1}(U)}$. Similarly, a D-twisted holomorphic connection ∇ furnishes lift of holomorphic vector fields on any open subset $U \subset X$, vanishing on $D \cap U$, to the G–invariant holomorphic vector fields on $E_G|_{p^{-1}(U)}$. In other words, these lifts are locally defined G–invariant holomorphic sections of $TE_G(-\log p^{-1}(D))$. Therefore, given a vector field v defined on a neighborhood of $x_i \in D$ of X, such that $v(x_i) = 0$ and $a_{x_i}(v(x_i)) \neq 0$ (see (2.9)), its lift $\widetilde{v}$ to E_G for ∇ may be nonzero on $p^{-1}(x_i)$ because $\widetilde{v}$ may be a nonzero vertical vector field on $p^{-1}(x_i)$. The residue of ∇ at x_i is $\widetilde{v}(p^{-1}(x_i))/a_{x_i}(v(x_i)) \in \mathrm{ad}(E_G)_{x_i}$ (see (2.9)).

For any $x \in D$, let

$$\mathcal{H}_x^{\perp} \subset \mathrm{ad}(E_G)_x \tag{2.10}$$

be the annihilator of $\mathcal{H}_x \subset \mathrm{ad}(E_G)_x$, defined in (2.6), with respect to the bilinear form $\widehat{\sigma}(x)$ in (2.5).

Definition 2.3. A framed G-connection is a triple of the form $(E_G,\, \nabla,\, \phi)$, where $(E_G,\, \phi)$ is a framed principal G-bundle and $\nabla : TX(-D) \longrightarrow \mathrm{At}_D(E_G)$ is a D-twisted connection such that $\mathrm{res}_x(\nabla) \in \mathcal{H}_x^{\perp} \subset \mathrm{ad}(E_G)_x$ for every $x \in D$, where $\mathcal{H}_x^{\perp}$ is constructed in (2.10).

2.4 Infinitesimal deformations

Consider the following 2-term complex of sheaves on X:

$$\mathcal{C}_\bullet : \mathrm{ad}(E_G)(-D) := \mathrm{ad}(E_G) \otimes \mathcal{O}_X(-D) \xrightarrow{\nabla} \mathrm{ad}(E_G) \otimes K_X(D) := \mathrm{ad}(E_G) \otimes K_X \otimes \mathcal{O}_X(D). \tag{2.11}$$

Lemma 2.4 (See [BLP2, Lemma 3.5] and [Ch, Proposition 4.4]). *Assume that $H_x = \{e\}$ for every $x \in D$. The infinitesimal deformations of the framed G-connection $(E_G,\, \nabla,\, \phi)$ are parametrized by the elements of the first hypercohomology $\mathbb{H}^1(\mathcal{C}_\bullet)$ of the complex in (2.11).*

Let

$$(E_G,\, \widehat{\nabla},\, \phi) \tag{2.12}$$

be a framed G-connection (see Definition 2.3). Consider the subspace $\mathcal{H}_x \subset \mathrm{ad}(E_G)_x$ in (2.6). Let $\mathrm{ad}_\phi(E_G)$ and $\mathrm{ad}_\phi^n(E_G)$ be the holomorphic vector bundles on X defined by the following short exact sequences of coherent analytic sheaves on X:

$$0 \longrightarrow \mathrm{ad}_\phi(E_G) \longrightarrow \mathrm{ad}(E_G) \longrightarrow \bigoplus_{x \in D} \mathrm{ad}(E_G)_x/\mathcal{H}_x \longrightarrow 0 \tag{2.13}$$

and

$$0 \longrightarrow \mathrm{ad}_\phi^n(E_G) \longrightarrow \mathrm{ad}(E_G) \longrightarrow \bigoplus_{x \in D} \mathrm{ad}(E_G)_x/\mathcal{H}_x^{\perp} \longrightarrow 0, \tag{2.14}$$

respectively.

Lemma 2.5. *The D-twisted connection $\widehat{\nabla}$ in (2.12) gives a holomorphic differential operator*

$$\nabla : \mathrm{ad}(E_G) \longrightarrow \mathrm{ad}(E_G) \otimes K_X(D) = \mathrm{ad}(E_G) \otimes K_X \otimes \mathcal{O}_X(D)\,.$$

If $\widehat{\nabla}$ is a framed G-connection, then ∇ sends the subsheaf $\mathrm{ad}_\phi(E_G)$ in (2.13) to $\mathrm{ad}_\phi^n(E_G) \otimes K_X(D)$, where $\mathrm{ad}_\phi^n(E_G)$ is constructed in (2.14).

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Proof. Let s be a holomorphic section of $\text{ad}(E_G)$ defined over an open subset $U \subset X$. Then s defines a G–invariant holomorphic vector field on $p^{-1}(U) \subset E_G$ which is vertical for the projection p in (2.2); this vertical vector field on $p^{-1}(U)$ will be denoted by $\widetilde{s}$. Take any $t \in H^0(U,\, TX(-D))$. Let

$$\widetilde{t} \;:=\; \widehat{\nabla}(t) \in H^0(p^{-1}(U),\, TE_G(-\log p^{-1}(D)))^G$$

be the horizontal lift of t for the D-twisted connection $\widehat{\nabla}$ in (2.12). Now consider the Lie bracket of vector fields

$$[\widetilde{t},\, \widetilde{s}] \in H^0(p^{-1}(U),\, TE_G).$$

Note that $[\widetilde{t},\, \widetilde{s}]$ is G–invariant because both $\widetilde{s}$ and $\widetilde{t}$ are. Furthermore, $[\widetilde{t},\, \widetilde{s}]$ is vertical for the projection p, because $\widetilde{s}$ is vertical and $\widetilde{t}$ is G–invariant. Indeed, for any holomorphic function f on U, evidently $\widetilde{s}(f \circ p) = 0$ (recall that $\widetilde{s}$ is vertical), and we also have that $\widetilde{t}(f \circ p)$ is G-invariant, so $\widetilde{s}(\widetilde{t}(f \circ p)) = 0$. Consequently, $[\widetilde{t},\, \widetilde{s}]$ produces a holomorphic section of $\text{ad}(E_G)$ over U; this section of $\text{ad}(E_G)$ over U will be denoted by $[t,\, s]'$. Next, note that a holomorphic function f on U satisfies

$$[\widetilde{f \cdot t},\, \widetilde{s}] = (f \circ p) \cdot [\widetilde{t},\, \widetilde{s}] - \widetilde{s}(f \circ p) \cdot \widetilde{t} = (f \circ p) \cdot [\widetilde{t},\, \widetilde{s}]$$

because $\widetilde{s}(f \circ p) = 0$ (recall that $\widetilde{s}$ is a vertical vector field). Consequently, there is a homomorphism

$$\nabla \;:\; \text{ad}(E_G) \longrightarrow \text{ad}(E_G) \otimes K_X(D)$$

uniquely defined by the equation

$$\langle \nabla(s),\, t \rangle = [t,\, s]',$$

where s and t are locally defined holomorphic sections of $\text{ad}(E_G)$ and $TX(-D)$ respectively, while $\langle -,\, - \rangle$ is the natural pairing $TX(-D) \otimes K_X(D) \longrightarrow \mathcal{O}_X$.

Recall from Definition 2.3 that $\text{res}_x(\nabla) \in \mathcal{H}_x^{\perp}$. Therefore, from the property of residues mentioned earlier, it follows immediately that $\widetilde{t}(x) \in \mathcal{H}_x^{\perp}$ for every $x \in D$. Now if s is a locally defined holomorphic section of $\text{ad}_\phi(E_G)$, then $\widetilde{s}(x) \in \mathcal{H}_x$. Next, note that

$$[\mathcal{H}_x^{\perp},\, \mathcal{H}_x] \subset \mathcal{H}_x^{\perp}, \tag{2.15}$$

because

$$\widehat{\sigma}(x)([a,\, b] \otimes c) = \widehat{\sigma}(x)(a \otimes [b,\, c])$$

for all $a,\, b,\, c \in \text{ad}(E_G)_x$ (this is derived using the given condition on σ that it is G-invariant). As a consequence of (2.15), the homomorphism ∇ maps the subsheaf $\text{ad}_\phi(E_G)$ to $\text{ad}^n_\phi(E_G) \otimes K_X(D)$. □

In view of Lemma 2.5, the following 2-term complex of sheaves on X is obtained:

$$\mathcal{D}_\bullet \;:\; \text{ad}_\phi(E_G) \xrightarrow{\;\nabla\;} \text{ad}^n_\phi(E_G) \otimes K_X(D)\,. \tag{2.16}$$

The next lemma is straightforward to prove.

Lemma 2.6. *The infinitesimal deformations of the framed G-connection $(E_G,\, \widehat{\nabla},\, \phi)$ in (2.12) are parametrized by the elements of the first hypercohomology $\mathbb{H}^1(\mathcal{D}_\bullet)$ of the complex in (2.16).*

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3. Construction of the moduli space

We now assume that $G = \mathrm{GL}(r, \mathbb{C})$. Fix a closed complex algebraic proper subgroup $H_x \subsetneq G$ for each $x \in D$, and set $H = \{H_x\}_{x \in D}$ to be the collection of subgroups indexed by the points of D. For a framed vector bundle (E, ϕ), if E_G is the principal $\mathrm{GL}(r, \mathbb{C})$-bundle associated to the vector bundle E, then $\mathrm{ad}(E_G) = \mathrm{End}(E)$. Define

$$\mathrm{End}_\phi(E) := \mathrm{ad}_\phi(E_G) \quad \text{and} \quad \mathrm{End}^n_\phi(E) := \mathrm{ad}^n_\phi(E_G)$$

(see Lemma 2.5).

3.1 Definition of the moduli functors

A framed $\mathrm{GL}(r, \mathbb{C})$-connection (E, ϕ, ∇) on X will be called *simple* if

$$\ker\Big(H^0(X, \mathcal{E}nd_\phi(E)) \xrightarrow{\nabla} H^0(X, \mathcal{E}nd^n_\phi(E) \otimes K_X(D)) \Big) = 0 .$$

Definition 3.1. Define a stack $\mathcal{M}^H_{\mathrm{FC}}(d)$ of simple framed $\mathrm{GL}(r, \mathbb{C})$-connections, for H, by breaking it into the following two cases, A and B.

(A) *If* $\mathbb{C}^* \cdot \mathrm{Id} \not\subset H_x$ *for some* $x \in D$, *then define a stack* $\mathcal{M}^H_{\mathrm{FC}}(d)$ *over the category of locally Noetherian schemes over* $\mathrm{Spec}\mathbb{C}$ *whose objects are quadruples* $(S, E, \phi = \{\phi_{x\times S}\}_{x\in D}, \nabla)$ *of the following type.*

(1) S *is a locally Noetherian scheme over* $\mathrm{Spec}\mathbb{C}$, *and* $E \longrightarrow X \times S$ *is a vector bundle of rank* r with $\deg(E|_{X\times s}) = d$ *for any geometric point* s *of* S.

(2) $\phi_{x\times S}$ *is a section of the structure map*

$$\mathrm{Isom}_S(\mathcal{O}^{\oplus r}_{x\times S},\ E|_{x\times S})/(H_x \times S) \longrightarrow x \times S.$$

Here the action of the group scheme $H_x \times S$ *over* S *on* $\mathrm{Isom}_S(\mathcal{O}^{\oplus r}_{x\times S},\ E|_{x\times S})$ *is the restriction of the natural transitive action of the group scheme* $\mathrm{GL}(r, \mathbb{C}) \times S$ *over* S *on* $\mathrm{Isom}_S(\mathcal{O}^{\oplus r}_{x\times S},\ E|_{x\times S})$ *given by the standard action of* $\mathrm{GL}(r, \mathbb{C})$ *on* $\mathcal{O}^{\oplus r}_{x\times S}$. *Define a* S*-scheme* $\widetilde{S}$ *and a map* $\widetilde{S} \longrightarrow \mathrm{Isom}_S(\mathcal{O}^{\oplus r}_{x\times S},\ E|_{x\times S})$ *such that the diagram*

$$\begin{array}{ccc} x \times \widetilde{S} & \longrightarrow & \mathrm{Isom}_S(\mathcal{O}^{\oplus r}_{x\times S},\ E|_{x\times S}) \\ \downarrow & & \downarrow \\ x \times S & \xrightarrow{\phi_{x\times S}} & \mathrm{Isom}_S(\mathcal{O}^{\oplus r}_{x\times S},\ E|_{x\times S})/(H_x \times S) \end{array}$$

is Cartesian. Let

$$\widetilde{\phi}_{x\times\widetilde{S}} : \mathcal{O}^{\oplus r}_{x\times\widetilde{S}} \xrightarrow{\sim} E_{\widetilde{S}}|_{x\times\widetilde{S}}$$

be the isomorphism given by the map $\widetilde{S} \longrightarrow \mathrm{Isom}_S(\mathcal{O}^{\oplus r}_{x\times S},\ E|_{x\times S})$.

(3) $\nabla : E \longrightarrow E \otimes K_X(D)$ *is a relative connection, relative to* S.

(4) *Let* $\mathrm{res}_{x\times\widetilde{S}}(\nabla_{\widetilde{S}}) \in \mathrm{End}(E_{\widetilde{S}})|_{x\times\widetilde{S}}$ *be the residue of the induced connection* $\nabla_{\widetilde{S}} : E_{\widetilde{S}} \longrightarrow E_{\widetilde{S}} \otimes K_X(D)$. *Then* $\widetilde{\phi}^{-1}_{x\times\widetilde{S}} \circ \mathrm{res}_{x\times\widetilde{S}}(\nabla_{\widetilde{S}}) \circ \widetilde{\phi}_{x\times\widetilde{S}} \in \mathfrak{h}^\perp \otimes \mathcal{O}_{\widetilde{S}}$.

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(5) *For each point* $s \in S$*, the framed* $\mathrm{GL}(r,\mathbb{C})$*-connection* $(E_s,\ \phi_s,\ \nabla_s)$ *is simple. Recall that* $(E_s,\ \phi_s,\ \nabla_s)$ *is simple if*

$$\ker\left(H^0(X,\ \mathcal{E}nd_{\phi_s}(E_s)) \xrightarrow{\nabla_s} H^0(X,\ \mathcal{E}nd^n_{\phi_s}(E_s)\otimes K_X(D))\right) = 0\,.$$

A morphism

$$(S,\ E,\ \phi,\ \nabla) \longrightarrow (S',\ E',\ \phi',\ \nabla')$$

in $\mathcal{M}^H_{\mathrm{FC}}$ *is a Cartesian square*

$$\begin{array}{ccc} E & \xrightarrow{\sigma} & E' \\ \downarrow & & \downarrow \\ S & \xrightarrow{\widetilde{\sigma}} & S' \end{array}$$

such that the diagram

$$\begin{array}{ccc} E & \xrightarrow{\nabla} & E\otimes K_X(D) \\ \cong\downarrow\sigma & & \cong\downarrow\sigma\otimes\mathrm{Id} \\ E'\times_{S'} S & \xrightarrow{\nabla'} & (E'\times_{S'} S)\otimes K_X(D) \end{array}$$

is commutative and $(\widetilde{\phi}'_{x\times\widetilde{S}})^{-1}\circ\sigma_{\widetilde{S}}\circ\widetilde{\phi}_{x\times\widetilde{S}} \in H_x\times\widetilde{S}$ *for each* $x \in D$*.*

(B) *If* $\mathbb{C}^*e \subset H_x$ *for all* $x \in D$*, then define* $\mathcal{M}^H_{\mathrm{FC}}(d)$ *to be the stackification of* $\mathrm{pre}{-}\mathcal{M}^H_{\mathrm{FC}}(d)$ *(see [Ols, Theorem 4.6.5]). Here,* $\mathrm{pre}{-}\mathcal{M}^H_{\mathrm{FC}}(d)$ *is the fibered category over the category of locally Noetherian schemes over* $\operatorname{Spec}\mathbb{C}$ *whose objects are quadruples* $(S,\ E,\ \phi = \{\phi_{x\times S}\}_{x\in D},\ \nabla)$ *that satisfy (1), (3) and (4) as above as well as the following (2') and (5').*

(2)′: $\phi_{x\times S}$ *is a section of the structure map*

$$\mathrm{Isom}_S(\mathcal{O}^{\oplus r}_{x\times S},\ E|_{x\times S})/(H_x\times S) \longrightarrow x\times S.$$

Here the action of the group scheme $H_x\times S$*, over* S*, on* $\mathrm{Isom}_S(\mathcal{O}^{\oplus r}_{x\times S},\ E|_{x\times S})$ *is the restriction of the natural transitive group action of the group scheme* $\mathrm{GL}(r,\mathbb{C})\times S$ *over* S *on* $\mathrm{Isom}_S(\mathcal{O}^{\oplus r}_{x\times S},\ E|_{x\times S})$ *given by the standard action of* $\mathrm{GL}(r,\mathbb{C})$ *on* $\mathcal{O}^{\oplus r}_{x\times S}$*. Define a* S*-scheme* $\widehat{S}$ *and a map* $\widehat{S} \longrightarrow \mathrm{Isom}_S(\mathcal{O}^{\oplus r}_{x\times S},\ E|_{x\times S})/(\mathbb{C}^*e\times S)$ *such that the diagram*

$$\begin{array}{ccc} x\times\widehat{S} & \longrightarrow & \mathrm{Isom}_S(\mathcal{O}^{\oplus r}_{x\times S},\ E|_{x\times S})/(\mathbb{C}^*e\times S) \\ \downarrow & & \downarrow \\ x\times S & \xrightarrow{\phi_{x\times S}} & \mathrm{Isom}_S(\mathcal{O}^{\oplus r}_{x\times S},\ E|_{x\times S})/(H_x\times S) \end{array}$$

is Cartesian. Denote by $\widehat{\phi}_{x\times\widehat{S}} :\ \mathbb{P}(\mathcal{O}^{\oplus r}_{x\times\widehat{S}}) \xrightarrow{\sim} \mathbb{P}(E_{\widehat{S}}|_{x\times\widehat{S}})$ *the isomorphism given by the map* $\widehat{S} \longrightarrow \mathrm{Isom}_S(\mathcal{O}^{\oplus r}_{x\times S},\ E|_{x\times S})/(\mathbb{C}^*e\times S)$*.*

(5)′: $(E_s,\ \phi_s,\ \nabla_s)$ *is simple for each point* $s \in S$*, that is,*

$$\ker(H^0(X,\ \mathcal{E}nd_{\phi_s}(E_s)) \xrightarrow{\nabla_s} H^0(X,\ \mathcal{E}nd^n_{\phi_s}(E_s)\otimes K_X(D))) = \mathbb{C}.$$

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A morphism

$$(S,\, E,\, \phi,\, \nabla) \longrightarrow (S',\, E',\, \phi',\, \nabla')$$

in pre$-\mathcal{M}_{\mathrm{FC}}^{H}(d)$ *is a triple* $(\mathcal{L},\, \sigma,\, \widetilde{\sigma})$, *where* $\mathcal{L}$ *is a line bundle on* S' *and* $\sigma,\, \widetilde{\sigma}$ *are maps that fit in a Cartesian square*

$$\begin{array}{ccc} E & \xrightarrow{\sigma} & E' \otimes \mathcal{L} \\ \downarrow & & \downarrow \\ S & \xrightarrow{\widetilde{\sigma}} & S' \end{array}$$

such that the diagram

$$\begin{array}{ccc} E & \xrightarrow{\quad\nabla\quad} & E \otimes K_X(D) \\ \cong \downarrow \sigma & & \cong \downarrow \sigma\otimes\mathrm{Id} \\ (E' \otimes \mathcal{L}) \times_{S'} S & \xrightarrow{\nabla'\otimes\mathcal{L}} & ((E' \otimes \mathcal{L}) \times_{S'} S) \otimes K_X(D) \end{array}$$

is commutative and

$$(\widehat{\phi}'_{x\times\widehat{S}})^{-1} \circ \overline{\sigma}_{x\times\widehat{S}} \circ \widehat{\phi}_{x\times\widehat{S}} \in (H_x/\mathbb{C}^* e) \times \widehat{S}$$

for each $x \in D$, *where* $\overline{\sigma}_{x\times\widehat{S}} :\, \mathbb{P}(E_{\widehat{S}}|_{x\times\widehat{S}}) \longrightarrow \mathbb{P}(E'_{\widehat{S}}|_{x\times\widehat{S}})$ *is induced by* σ.

We say that σ is an *automorphism* of a framed G-connection $(E,\, \phi,\, \nabla)$ if σ is a holomorphic automorphism of the vector bundle E on X such that the diagram

$$\begin{array}{ccc} E & \xrightarrow{\nabla} & E \otimes K_X(D) \\ \downarrow \sigma & & \downarrow \sigma\otimes\mathrm{Id} \\ E & \xrightarrow{\nabla} & E \otimes K_X(D) \end{array}$$

is commutative and $\sigma|_x \circ \phi_x$ coincides with ϕ_x in the quotient $\mathrm{Isom}(\mathcal{O}_x^{\oplus r},\, E|_x)/H_x$ for each $x \in D$. Denote by $\mathrm{Aut}(E,\, \phi,\, \nabla)$ the space of all automorphisms of a framed G-connection $(E,\, \phi,\, \nabla)$.

Proposition 3.2. *Assume that* $\mathbb{C}^* \cdot \mathrm{Id} \not\subset H_x$ *for some* $x \in D$. *Let* $(E,\, \phi,\, \nabla)$ *be a simple framed* G*-connection over* X *(see Definition 3.1(5)). Then* $\mathrm{Aut}(E, \phi, \nabla)$ *is a finite group.*

Proof. The space $\mathrm{Aut}(E,\, \phi,\, \nabla)$ has the structure of a group scheme of finite type over $\mathbb{C}$. We can see that the tangent space of $\mathrm{Aut}(E,\, \phi,\, \nabla)$ at the identity element is isomorphic to

$$\ker(H^0(X,\, \mathcal{E}nd_\phi(E)) \xrightarrow{\nabla} H^0(X,\, \mathcal{E}nd_\phi^m(E) \otimes K_X(D))),$$

which is zero because $(E,\, \phi,\, \nabla)$ is simple. Consequently, $\mathrm{Aut}(E,\, \phi,\, \nabla)$ is a finite group. □

Proposition 3.3. *Assume that* $H_x = \{e\}$ *for all* $x \in D$. *Let* $(E,\, \phi,\, \nabla)$ *be a simple framed* G*-connection over* X *associated to* $\{H_x\}_{x\in D}$. *Then* $\mathrm{Aut}(E,\, \phi,\, \nabla) = \{\mathrm{Id}_E\}$.

Proof. An automorphism $\sigma \in \mathrm{Aut}(E,\, \phi,\, \nabla)$ is an automorphism of the vector bundle E such that $\nabla \circ \sigma = \sigma \circ \nabla$ and $\phi_x \circ \sigma|_x \circ \phi_x^{-1} \in H_x$ for all $x \in D$. Since $H_x = \{e\}$ by the assumption,

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it follows that $\sigma|_x = \mathrm{Id}|_{E|_x}$ for all $x \in D$. Now set

$$\widetilde{\sigma} := \mathrm{Id}_E - \sigma : E \longrightarrow E .$$

Then $\widetilde{\sigma}|_x = 0$ for all $x \in D$, and it is straightforward to check that $\nabla \circ \widetilde{\sigma} - \widetilde{\sigma} \circ \nabla = 0$, that is,

$$\widetilde{\sigma} \in \ker(H^0(X, \mathcal{E}nd_\phi(E)) \xrightarrow{\nabla} H^0(X, \mathcal{E}nd^n_\phi(E) \otimes K_X(D))) .$$

Since (E, ϕ, ∇) is simple, it follows that $\widetilde{\sigma} = 0$, and hence $\sigma = \mathrm{Id}_E$. □

Proposition 3.4. *Assume that* $\mathbb{C}^* \cdot \mathrm{Id} \subset H_x$ *for all* $x \in D$*. If* (E, ϕ, ∇) *is a simple framed* G*-connection over* X*, then the quotient* $\mathrm{Aut}(E, \phi, \nabla)/(\mathbb{C}^* \cdot \mathrm{Id})$ *is a finite group.*

Proof. The tangent space of $\mathrm{Aut}(E, \phi, \nabla)/(\mathbb{C}^* \cdot \mathrm{Id})$ is zero, because we have that (E, ϕ, ∇) is simple. Consequently, $\mathrm{Aut}(E, \phi, \nabla)/(\mathbb{C}^* \cdot \mathrm{Id})$ is a finite group. □

3.2 Representation of moduli functors as Deligne–Mumford stacks

Proposition 3.5. *The stack* $\mathcal{M}^H_{\mathrm{FC}}(d)$ *in Definition 3.1 is a Deligne–Mumford stack.*

Proof. Fix a very ample line bundle $\mathcal{O}_X(1)$ on the curve X. Define a polynomial $\theta_d(m)$ in m to be

$$\theta_d(m) = r d_X m + d + r(1 - g) ,$$

where $d_X := \deg \mathcal{O}_X(1)$ and g is the genus of X. Let

$$\Sigma^d_{m_0} \tag{3.1}$$

denote the fibered category whose objects are simple framed $\mathrm{GL}(r, \mathbb{C})$-connections (E, ϕ, ∇) on $X \times S$ such that:

(1) $H^1(X, E_s(m_0 - 1)) = 0$ for each $s \in S$;

(2) $\chi(E_s(m)) = \theta_d(m)$ for each $s \in S$ and all $m \in \mathbb{Z}$.

The fibered categories $\Sigma^d_{m_0}$ in (3.1) form an open covering of $\mathcal{M}^H_{\mathrm{FC}}(d)$. So we only have to prove that each $\Sigma^d_{m_0}$ is a Deligne–Mumford stack.

Let

$$\mathcal{O}_{X \times Q^d_{m_0}}(-m_0)^{\oplus \theta_d(m_0)} \longrightarrow \mathcal{E}$$

be the universal quotient sheaf of the Quot-scheme $\mathrm{Quot}^{\theta_d}_{(\mathcal{O}_X(-m_0)^{\oplus \theta_d(m_0)}/X)}$. Define the open subset $Q^d_{m_0}$ of $\mathrm{Quot}^{\theta_d}_{(\mathcal{O}_X(-m_0)^{\oplus \theta_d(m_0)}/X)}$ by

$$Q^d_{m_0} := \left\{ s \in \mathrm{Quot}^{\theta_d}_{(\mathcal{O}_X(-m_0)^{\oplus \theta_d(m_0)}/X)} \;\middle|\; \begin{array}{l} \text{(i) } \mathrm{h}^0(\mathrm{X}, \mathcal{E}_\mathrm{s}(\mathrm{m}_0)) = \theta_{\mathrm{d}}(\mathrm{m}_0); \\ \text{(ii) } \mathrm{h}^\mathrm{i}(\mathrm{X}, \mathcal{E}_\mathrm{s}(\mathrm{m}_0 - \mathrm{i})) = 0 \text{ for all } \mathrm{i} > 0; \text{ and} \\ \text{(iii) } \mathcal{E}_\mathrm{s} \text{ is locally free.} \end{array} \right\} .$$

There is a locally free $\mathcal{O}_{Q^d_{m_0}}$–module $\mathscr{H}_D$ such that $\boldsymbol{V}^*(\mathscr{H}_D) := \mathrm{Spec}(\mathrm{Sym}^* \mathscr{H}_D)$ represents the functor

$$S \longmapsto \bigoplus_{x \in D} \mathrm{Hom}_{x \times S}(\mathcal{O}^{\oplus r}_{x \times S}, \mathcal{E}_{X \times S}|_{x \times S}) \in (\mathit{Sets})$$

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for any Noetherian schemes S over $Q^d_{m_0}$. There is a universal family

$$\varphi^x : \mathcal{O}^{\oplus r}_{x\times \boldsymbol{V}^*(\mathscr{H}_D)} \longrightarrow \mathcal{E}_{X\times \boldsymbol{V}^*(\mathscr{H}_D)}|_{x\times \boldsymbol{V}^*(\mathscr{H}_D)} \tag{3.2}$$

for every $x \in D$. Define $\widetilde{Q}^d_{m_0}$ as follows:

$$\widetilde{Q}^d_{m_0} := \{ s \in \boldsymbol{V}^*(\mathscr{H}_D) \mid \text{coker}\,(\varphi^{\,x}_{\,s}) = 0 \}.$$

Consider the map $\widetilde{Q}^d_{m_0} \longrightarrow Q^d_{m_0}$. For each Noetherian scheme S over $Q^d_{m_0}$, the natural transitive group action of $G \times S$ on

$$\text{Isom}_{x\times S}(\mathcal{O}^{\oplus r}_{x\times S},\ \mathcal{E}_{X\times S}|_{x\times S})$$

induces an action on $\widetilde{Q}^d_{m_0}$ of the group scheme $\left(\prod_{x\in D} G\right) \times Q^d_{m_0}$ over $Q^d_{m_0}$. The group scheme $\left(\prod_{x\in D} H_x\right) \times Q^d_{m_0}$ acts on $\widetilde{Q}^d_{m_0}$ by restricting this action of $(\prod_{x\in D} G) \times Q^d_{m_0}$ on $\widetilde{Q}^d_{m_0}$. Set

$$\widetilde{Q}^{d,H}_{m_0} := \widetilde{Q}^d_{m_0}\Big/\Big((\prod_{x\in D} H_x) \times Q^d_{m_0}\Big).$$

Let $\widetilde{\mathcal{E}}$ be the pull-back of the family $\mathcal{E}$ under the map $X \times \widetilde{Q}^{d,H}_{m_0} \longrightarrow X \times Q^d_{m_0}$. We have a family $\widetilde{\phi}_x$ of sections of

$$\text{Isom}_{\widetilde{Q}^{d,H}_{m_0}}(\mathcal{O}^{\oplus r}_{\widetilde{Q}^{d,H}_{m_0}},\ \widetilde{\mathcal{E}}|_{x\times \widetilde{Q}^{d,H}_{m_0}})/(H_x \times \widetilde{Q}^{d,H}_{m_0}) \longrightarrow \widetilde{Q}^{d,H}_{m_0}$$

induced by φ^x. Put $\widetilde{\phi} := \{\widetilde{\phi}_x\}_{x\in D}$.

Let

$$\pi : X \times \widetilde{Q}^{d,H}_{m_0} \longrightarrow \widetilde{Q}^{d,H}_{m_0} \tag{3.3}$$

be the natural projection map. Let $\text{At}(\widetilde{\mathcal{E}})$ be the Atiyah bundle for $\widetilde{\mathcal{E}}$. Then there is a short exact sequence

$$0 \longrightarrow \mathcal{E}nd(\widetilde{\mathcal{E}}) \longrightarrow \text{At}(\widetilde{\mathcal{E}}) \xrightarrow{\text{symb}_1} T_{X\times \widetilde{Q}^{d,H}_{m_0}/\widetilde{Q}^{d,H}_{m_0}} \longrightarrow 0.$$

Set $\text{At}_D(\widetilde{\mathcal{E}}) := \text{symb}_1^{-1}\left(T_{X\times \widetilde{Q}^{d,H}_{m_0}/\widetilde{Q}^{d,H}_{m_0}}(-D \times \widetilde{Q}^{d,H}_{m_0})\right)$. The natural short exact sequences of Atiyah bundles induces an exact sequence as follows.

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathcal{E}nd(\widetilde{\mathcal{E}}) \otimes K_X & \longrightarrow & \text{At}(\widetilde{\mathcal{E}}) \otimes K_X & \xrightarrow{\text{symb}_1} & \mathcal{O}_{X\times \widetilde{Q}^{d,H}_{m_0}} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{E}nd(\widetilde{\mathcal{E}}) \otimes K_X(D) & \longrightarrow & \text{At}_D(\widetilde{\mathcal{E}}) \otimes K_X(D) & \xrightarrow{\text{symb}_1^D} & \mathcal{O}_{X\times \widetilde{Q}^{d,H}_{m_0}} & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle p'} & & \downarrow{\scriptstyle p} & & & & \\
 & & \mathcal{E}nd(\widetilde{\mathcal{E}}) \otimes K_X(D)|_{D\times \widetilde{Q}^{d,H}_{m_0}} & \underset{\cong}{\xrightarrow{q}} & \left(\text{At}_D(\widetilde{\mathcal{E}}) \otimes K_X(D)\right) \Big/ \left(\text{At}(\widetilde{\mathcal{E}}) \otimes K_X\right) & & & &
\end{array}$$

In particular, there are two compositions of maps

$$\begin{aligned}
&\text{At}_D(\widetilde{\mathcal{E}}) \otimes K_X(D) \xrightarrow{p} \left(\text{At}_D(\widetilde{\mathcal{E}}) \otimes K_X(D)\right) \Big/ \left(\text{At}(\widetilde{\mathcal{E}}) \otimes K_X\right) \\
&\xrightarrow{q^{-1}} \mathcal{E}nd(\widetilde{\mathcal{E}}) \otimes K_X(D)|_{D\times \widetilde{Q}^{d,H}_{m_0}} \xrightarrow{\text{res}_{D\times \widetilde{Q}^{d,H}_{m_0}}} \text{End}(\widetilde{\mathcal{E}})|_{D\times \widetilde{Q}^{d,H}_{m_0}}
\end{aligned} \tag{3.4}$$

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and

$$\mathcal{E}nd(\widetilde{\mathcal{E}})\otimes K_X(D) \xrightarrow{p'} \mathcal{E}nd(\widetilde{\mathcal{E}})\otimes K_X(D)|_{D\times \widetilde{Q}^{d,H}_{m_0}} \xrightarrow{\mathrm{res}_{D\times\widetilde{Q}^{d,H}_{m_0}}} \mathrm{End}(\widetilde{\mathcal{E}})|_{D\times\widetilde{Q}^{d,H}_{m_0}}. \tag{3.5}$$

Here, $\mathrm{res}_{D\times\widetilde{Q}^{d,H}_{m_0}}$ is the residue map

$$\mathrm{res}_{D\times\widetilde{Q}^{d,H}_{m_0}} \ :\ \mathrm{End}(\widetilde{\mathcal{E}})\otimes K_X(D)|_{D\times\widetilde{Q}^{d,H}_{m_0}} \longrightarrow \mathrm{End}(\widetilde{\mathcal{E}})|_{D\times\widetilde{Q}^{d,H}_{m_0}}.$$

Using the family $\widetilde{\phi}$ of framings and the Lie subgroups H_x for each $x\in D$, we may define a subsheaf $\mathcal{H}^{\perp}_{x\times\widetilde{Q}^{d,H}_{m_0}} \subset \mathrm{End}(\widetilde{\mathcal{E}})|_{D\times\widetilde{Q}^{d,H}_{m_0}}$ as in (2.10). Define subsheaves $\mathrm{At}^{\widetilde{\phi}}_D(\widetilde{\mathcal{E}})\subset \mathrm{At}_D(\widetilde{\mathcal{E}})$ and $\mathcal{E}nd^n_{\widetilde{\phi}}(\widetilde{\mathcal{E}})\subset \mathcal{E}nd(\widetilde{\mathcal{E}})$ such that $\mathrm{At}^{\widetilde{\phi}}_D(\widetilde{\mathcal{E}})\otimes K_X(D)$ is the inverse image of $\mathcal{H}^{\perp}_{x\times\widetilde{Q}^{d,H}_{m_0}}$ under the composition of maps in (3.4) and $\mathcal{E}nd^n_{\widetilde{\phi}}(\widetilde{\mathcal{E}})\otimes K_X(D)$ is the inverse image of $\mathcal{H}^{\perp}_{x\times\widetilde{Q}^{d,H}_{m_0}}$ under the composition of maps in (3.5).

By [Grot1, Theorem 7.7.6], there is a unique coherent sheaf $\mathscr{H}$ on $\widetilde{Q}^{d,H}_{m_0}$ (up to isomorphism) such that

$$(\pi_{Q'})_*\left(\left(\mathrm{At}^{\widetilde{\phi}}_D(\widetilde{\mathcal{E}})\otimes K_X(D)\right)\otimes_{\mathcal{O}_{Q'}} M\right) \ \cong\ \mathcal{H}\,\mathrm{om}_{\mathcal{O}_{Q'}}(\mathscr{H}_{Q'}, M) \tag{3.6}$$

for any $\widetilde{Q}^{d,H}_{m_0}$-scheme Q' and any quasi-coherent sheaf M on Q'. Here, $\pi_{Q'}$ is the natural projection $X\times Q'\longrightarrow Q'$. Set

$$\boldsymbol{V}(\mathscr{H}) \ :=\ \mathrm{Spec}(\mathrm{Sym}^*(\mathscr{H}))\,.$$

There is a natural morphism $\varphi\in\mathcal{H}\mathrm{om}_{\mathcal{O}_{\boldsymbol{V}(\mathscr{H})}}(\mathscr{H}_{\boldsymbol{V}(\mathscr{H})},\, \mathcal{O}_{\boldsymbol{V}(\mathscr{H})})$ by the definition of $\boldsymbol{V}(\mathscr{H})$. In view of the isomorphism in (3.6), there is an element $\varphi'\in\pi_*((\mathrm{At}^{\widetilde{\phi}}_D(\widetilde{\mathcal{E}})\otimes K_X(D))_{\boldsymbol{V}(\mathscr{H})})$ corresponding to φ. The morphism symb^D_1 induces a morphism

$$(\pi_{\boldsymbol{V}(\mathscr{H})})_*\left(\left(\mathrm{At}^{\widetilde{\phi}}_D(\widetilde{\mathcal{E}})\otimes K_X(D)\right)_{\boldsymbol{V}(\mathscr{H})}\right) \longrightarrow (\pi_{\boldsymbol{V}(\mathscr{H})})_*(\mathcal{O}_{X\times\boldsymbol{V}(\mathscr{H})})\cong\mathcal{O}_{\boldsymbol{V}(\mathscr{H})}.$$

Using this morphism, there is a function $f_{\mathrm{symb}^D_1}$ on $\boldsymbol{V}(\mathscr{H})$ corresponding to φ'. Denote by $I_{\mathrm{symb}^D_1}$ the ideal sheaf of $\mathcal{O}_{\boldsymbol{V}(\mathscr{H})}$ generated by $f_{\mathrm{symb}^D_1}-1$. Put

$$Z^{d,H}_{m_0} \ :=\ \mathrm{Spec}(\mathcal{O}_{\boldsymbol{V}(\mathscr{H})}\,/\,I_{\mathrm{symb}^D_1})\,.$$

Also, denote by $\widetilde{\mathcal{E}}$ the pull-back of $\widetilde{\mathcal{E}}$ under the natural morphism $X\times Z^{d,H}_{m_0}\longrightarrow X\times\widetilde{Q}^{d,H}_{m_0,\mathrm{spl}}$, and let

$$\widetilde{\nabla}\ :\ \widetilde{\mathcal{E}}\longrightarrow\widetilde{\mathcal{E}}\otimes_{\mathcal{O}_{X\times\boldsymbol{V}^*(\mathscr{H})_\epsilon}} K_X(D)$$

be a universal relative connection on $\widetilde{\mathcal{E}}$, which is determined by φ'. Define the open subset $(Z^{d,H}_{m_0})'$ of $Z^{d,H}_{m_0}$ by

$$(Z^{d,H}_{m_0})' = \left\{s\in Z^{d,H}_{m_0}\ \middle|\ (\widetilde{\mathcal{E}},\widetilde{\phi},\widetilde{\nabla})|_{X\times\{s\}} \text{ is a simple framed connection}\right\}$$

and denote by $(\widetilde{\mathcal{E}},\, \widetilde{\phi}=\{[\widetilde{\phi}_{x\times(Z^{d,H}_{m_0})'}]\}_{x\in D},\, \widetilde{\nabla})$ a universal family of m_0-regular simple framed G-connections on $X\times(Z^{d,H}_{m_0})'$. Here, m_0-regular means $H^1(X,\,\widetilde{\mathcal{E}}_s(m_0-1))=0$ for each $s\in(Z^{d,H}_{m_0})'$.

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Now, consider the case where $\mathbb{C}^* \cdot \mathrm{Id} \not\subset H_x$ for some $x \in D$. There exists an action of $\mathrm{GL}(\theta_d(m_0), \mathbb{C})$ on $(Z^{d,H}_{m_0})'$ given by

$$\left(\mathcal{O}_{X\times S}^{\oplus \theta_d(m)} \xrightarrow{q} E,\, \phi,\, \nabla\right) \longmapsto \left(\mathcal{O}_{X\times S}^{\oplus \theta_d(m)} \xrightarrow{q\circ g} E,\, \phi,\, \nabla\right)$$

on S-points for $g \in \mathrm{GL}(\theta_d(m_0), \mathbb{C})_S$. Consider the map

$$\begin{aligned} (Z^{d,H}_{m_0})' &\longrightarrow \Sigma^d_{m_0} \\ (\mathcal{O}_{X\times S}^{\oplus \theta_d(m)} \xrightarrow{q} E,\, \phi,\, \nabla) &\longmapsto (S,\, E,\, \phi,\, \nabla). \end{aligned}$$

This map gives an isomorphism

$$\Sigma^d_{m_0} \;\cong\; [\,(Z^{d,H}_{m_0})' \,/\, \mathrm{GL}(\theta_d(m_0), \mathbb{C})\,]\,.$$

Here, $[\,(Z^{d,H}_{m_0})' \,/\, \mathrm{GL}(\theta_d(m_0), \mathbb{C})\,]$ is a quotient stack. Consequently, $\Sigma^d_{m_0}$ is an algebraic stack. Using Proposition 3.2, it follows that $\Sigma^d_{m_0}$ is in fact a Deligne–Mumford stack (see [Ols, Corollary 8.4.2]).

Next, we consider the case where $\mathbb{C}^* \cdot \mathrm{Id} \subset H_x$ for every $x \in D$. The $\mathbb{C}^*$-action on $(Z^{d,H}_{m_0})'$ is trivial, because $\mathbb{C}^* \cdot \mathrm{Id} \subset H_x$ for all $x \in D$. There exists a natural action of $\mathrm{PGL}(\theta_d(m_0),\, \mathbb{C})$ on $(Z^{d,H}_{m_0})'$. Define a map

$$\begin{aligned} (Z^{d,H}_{m_0})' &\longrightarrow \Sigma^d_{m_0} \\ (\mathcal{O}_{X\times S}^{\oplus \theta_d(m)} \xrightarrow{q} E,\, \phi,\, \nabla) &\longmapsto (S,\, E,\, \phi,\, \nabla). \end{aligned}$$

It is straightforward to check that this map gives an isomorphism

$$\Sigma^d_{m_0} \;\cong\; [\,(Z^{d,H}_{m_0})' \,/\, \mathrm{PGL}(\theta_d(m_0), \mathbb{C})\,].$$

Then $\Sigma^d_{m_0}$ is an algebraic stack. By Proposition 3.4, it follows that $\Sigma^d_{m_0}$ is in fact a Deligne–Mumford stack (see [Ols, Corollary 8.4.2]). □

Remark 3.6. If $H_x = e$ for all x *in* D, then $\mathcal{M}^H_{\mathrm{FC}}(d)$ is an algebraic space by Proposition 3.3.

Remark 3.7. In the proof of Proposition 3.5, we introduced the coherent sheaf $\mathscr{H}$ which is characterized by the property (3.6). Since $\mathscr{H}$ is not necessarily locally free, we cannot see the irreducibility of the moduli space $\mathcal{M}^H_{\mathrm{FC}}(d)$ immediately from its construction.

Define an open substack of $\mathcal{M}^H_{\mathrm{FC}}(d)$ as follows:

$$\mathcal{M}^H_{\mathrm{FC}}(d)^\circ \;:=\; \{(S,\, E,\, \phi,\, \nabla) \in \mathcal{M}^H_{\mathrm{FC}}(d) \mid (E_s,\, \phi_s) \text{ is simple for each s} \in \mathrm{S}\}. \tag{3.7}$$

Here we say that $(E_s,\, \phi_s)$ is simple if

$$\begin{cases} H^0(X,\, \mathcal{E}nd_{\phi_s}(E_s)) = 0 \text{when } \mathbb{C}^* \cdot \mathrm{Id} \not\subset \mathrm{H_x} \text{ for some x} \in \mathrm{D}, \\ H^0(X,\, \mathcal{E}nd_{\phi_s}(E_s)) = \mathbb{C} \text{when } \mathbb{C}^* \cdot \mathrm{Id} \subset \mathrm{H_x} \text{ for all x} \in \mathrm{D}. \end{cases}$$

We adopt the above definition of simple framed bundle in order that the loci $\mathcal{M}^H_{\mathrm{FC}}(d)^\circ$ becomes open in $\mathcal{M}^H_{\mathrm{FC}}(d)$.

Proposition 3.8. *The open substack $\mathcal{M}^H_{\mathrm{FC}}(d)^\circ$ in (3.7) is irreducible.*

Proof. Fix a very ample line bundle $\mathcal{O}_X(1)$ on the curve X. Let $\theta_d(m)$ be a polynomial in m defined as in Proposition 3.5. Let $(\Sigma^d_{m_0})^\circ$ denote the substack of $\mathcal{M}^H_{\mathrm{FC}}(d)^\circ$ whose objects are framed $\mathrm{GL}(r, \mathbb{C})$-connections $(E,\, \phi,\, \nabla)$ on X such that:

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(1) $(E,\, \phi)$ is simple;
(2) $H^1(X,\, E(m_0-1)) = 0$;
(3) $\chi(E(m)) = \theta_d(m)$ for all $m \in \mathbb{Z}$.

To prove the proposition it suffices to show that $(\Sigma^d_{m_0})^\circ$ is irreducible.

Let V be a $\theta_d(m_0)$-dimensional vector space so that the underlying vector bundle E of any object of $(\Sigma^d_{m_0})^\circ$ is described as the following quotient:

$$V \otimes \mathcal{O}_X(-m_0) \longrightarrow E.$$

Take a subspace $V_r \subset V$ such that $\dim(V_r) = r$. Taking the dual of the above quotient, and tensoring with $\mathcal{O}_X(-m_0)$, we have the following short exact sequence

$$0 \longrightarrow E^\vee(-m_0) \longrightarrow V_r^\vee \otimes \mathcal{O}_X \longrightarrow F \longrightarrow 0,$$

where F is the quotient for the injective map $E^\vee(-m_0) \longrightarrow V_r^\vee \otimes \mathcal{O}_X$. So for each object of $(\Sigma^d_{m_0})^\circ$, there is a point of $\mathrm{Quot}^N_{(V_r^\vee \otimes \mathcal{O}_X)/X}$ which determines the underlying vector bundle of the object. Here N is the length of F. Note that N remains constant for the underlying vector bundles. We will show that $\mathrm{Quot}^N_{(V_r^\vee \otimes \mathcal{O}_X)/X}$ is irreducible.

The Quot-scheme $\mathrm{Quot}^N_{(V_r^\vee \otimes \mathcal{O}_X)/X}$ is smooth, because the obstructions to deformations of

$$[\, q :\; V_r^\vee \otimes \mathcal{O}_X \longrightarrow F\,] \in \mathrm{Quot}^N_{(V_r^\vee \otimes \mathcal{O}_X)/X}$$

lie in

$$\mathrm{Ext}^1(\mathrm{Ker}\, q,\; F) \cong H^1((\mathrm{Ker}\, q)^\vee \otimes F) = 0\,.$$

Define the map

$$\begin{aligned} f_N \;:\; \mathrm{Quot}^N_{(V_r^\vee \otimes \mathcal{O}_X)/X} &\longrightarrow \mathrm{Hilb}^N_X \\ [\, q\colon V_r^\vee \otimes \mathcal{O}_X \longrightarrow F\,] &\longmapsto \mathrm{Divisor}(\det(\mathrm{Ker}\, q \longrightarrow V_r^\vee \otimes \mathcal{O}_X)). \end{aligned}$$

Let H' be the Zariski open subset of Hilb^N_X which consists of distinct points on X, or in other words, H' parametrizes the reduced subschemes. This Zariski open subset H' is in fact irreducible. The map

$$f_N^{-1}(H') \longrightarrow H'$$

is a $(\mathbb{P}^{r-1} \times \cdots \times \mathbb{P}^{r-1})$-bundle; here $\mathbb{P}^{r-1} \times \cdots \times \mathbb{P}^{r-1}$ is the product of N-copies of $\mathbb{P}^{r-1}$. Hence $f_N^{-1}(H')$ is irreducible. Take a point $x = N_1 z_1 + \cdots + N_l z_l$ on Hilb^N_X, where $\sum_{i=1}^l N_i = N$ and $z_1, \cdots, z_l$ are distinct points on X. A point on $f_N^{-1}(x)$ can be described as a collection $(\, q_i :\; V_r^\vee \otimes \mathcal{O}_{z_i,X} \longrightarrow F_i\,)_{i=1}^l$ for which $\mathrm{length}(F_i) = N_i$. Consider the map $(\mathrm{Ker}\, q_i)_{z_i} \longrightarrow V_r^\vee \otimes \mathcal{O}_{z_i,X}$ corresponding to a point on $f_N^{-1}(x)$. Note that $(\mathrm{Ker}\, q_i)_{z_i} \cong \mathcal{O}^{\oplus r}_{z_i,X}$. We have a matrix representation of $(\mathrm{Ker}\, q_i)_{z_i} \longrightarrow V_r^\vee \otimes \mathcal{O}_{z_i,X}$ as follows:

$$\begin{pmatrix} 1 & & & & & \\ & \ddots & & & & \\ & & 1 & & & \\ & & & z^{l_{i,1}} & & \\ & & & & \ddots & \\ & & & & & z^{l_{i,s_i}} \end{pmatrix},$$

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where the maximal ideal $\mathfrak{m}_{z_i}$ is $\{z=0\}$, and $1,\ z^{l_{i,1}},\ \ldots,\ z^{l_{i,s_i}}$ are invariant factors of

$$(\operatorname{Ker} q_i)_{z_i} \longrightarrow V_r^{\vee} \otimes \mathcal{O}_{z_i,X}\,.$$

For any tuple of complex numbers $a_1^{(i)},\ \ldots,\ a_{s_i}^{(i)}$, there is a deformation of $(\operatorname{Ker} q_i)_{z_i} \longrightarrow V_r^{\vee} \otimes \mathcal{O}_{z_i,X}$

$$\begin{pmatrix} 1 & & & & & \\ & \ddots & & & & \\ & & 1 & & & \\ & & & z^{l_{i,1}}+ta_1^{(i)} & & \\ & & & & \ddots & \\ & & & & & z^{l_{i,s_i}}+ta_{s_i}^{(i)} \end{pmatrix},$$

over $X \times \operatorname{Spec} \mathbb{C}[t]$. When the complex numbers $a_1^{(i)},\ \cdots,\ a_{s_i}^{(i)}$ are generic, we have a deformation moving from a point on $f_N^{-1}(x)$, where $x = N_1 z_1 + \cdots + N_l z_l$, to a point on $f_N^{-1}(H')$. Therefore, $\operatorname{Quot}^N_{(V_r^{\vee}\otimes\mathcal{O}_X)/X}$ is irreducible.

Consider the open subset

$$Q' \ := \ \left\{ [\,q\,] \in \operatorname{Quot}^N_{(V_r^{\vee}\otimes\mathcal{O}_X)/X} \ \middle|\ \mathrm{E_q \text{ satisfies } H^1(E_q(m_0-1)) = 0} \right\} \subset \operatorname{Quot}^N_{(V_r^{\vee}\otimes\mathcal{O}_X)/X}.$$

Here, denote $E_q \ := \ (\ker q)^{\vee}(-m_0)$ for a quotient q. By definition, E_q is locally free and satisfies the condition $\chi(E_q(m)) = \theta_d(m)$ for all $m \in \mathbb{Z}$. Let $\widetilde{Q}'$ be the scheme over Q' which parametrizes quotients q in Q' and framings of E_q, which is constructed as in the proof of Proposition 3.5. Now define $(\widetilde{Q}')^{\circ}$ as follows:

$$(\widetilde{Q}')^{\circ} \ := \ \left\{ s\ in\ \widetilde{Q}' \middle|\ (\widetilde{\mathcal{E}}_{\mathrm{s}},\ \widetilde{\phi}_{\mathrm{s}})\text{is simple} \right\}.$$

Here, $(\widetilde{\mathcal{E}},\ \widetilde{\phi})$ is the family of vector bundles E_q and framings of E_q in $\widetilde{Q}'$ induced by the universal family of $\widetilde{Q}'$. Since $\operatorname{Quot}^N_{(V_r^{\vee}\otimes\mathcal{O}_X)/X}$ is irreducible, $(\widetilde{Q}')^{\circ}$ is also irreducible. Let $(Z')^{\circ}$ be the scheme over $(\widetilde{Q}')^{\circ}$ which parametrizes quotients q with framings of E_q in $(\widetilde{Q}')^{\circ}$ and connections on E_q that are compatible with the framings. The scheme $(Z')^{\circ}$ is also constructed as in the proof of Proposition 3.5. It is straightforward to check that $(Z')^{\circ}$ is smooth and each fiber of $(Z')^{\circ} \longrightarrow (\widetilde{Q}')^{\circ}$ is an affine space which is isomorphic to $H^0(X,\ \operatorname{End}^n_{\phi}(E) \otimes K_X(D))$. So $(Z')^{\circ}$ is irreducible. Since a natural map from $(Z')^{\circ}$ to $(\Sigma^d_{m_0})^{\circ}$ is induced and this map is surjective, we conclude that $(\Sigma^d_{m_0})^{\circ}$ is irreducible. This completes the proof of the proposition. □

4. Symplectic structures of the moduli spaces

Throughout this section it is assumed that $G = \mathrm{GL}(r, \mathbb{C})$.

4.1 Cotangent bundle of the moduli space of simple framed bundles

In this subsection, we assume that $H_x = \{e\} \subset \mathrm{GL}(r, \mathbb{C})$ for all $x \in D$. Let $\mathcal{N}^e(d)$ be the following moduli space:

$$\mathcal{N}^e(d) = \left\{ (E,\ \phi = \{\phi_x\}_{x\in D}) \ \middle|\ \begin{array}{l} \text{E is a vector bundle of degree d and} \\ \text{(E, } \phi \text{) is a simple framed principal} \\ \text{G} - \text{bundle, where } \mathrm{H_x} = \{\mathrm{e}\} \text{ for all x} \in \text{D.} \end{array} \right\} \Big/ \sim_e. \tag{4.1}$$

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Here, $(E,\ \phi) \sim_e (E',\ \phi')$ if there exists an isomorphism $\sigma :\ E \longrightarrow E'$ of vector bundles such that the composition of homomorphisms $(\phi'_x)^{-1} \circ \sigma|_x \circ \phi_x$ is the identity map of $\mathbb{C}^r$ for each $x \in D$. Since the tangent space of $\mathcal{N}^e(d)$ at $(E,\ \phi)$ is $H^1(X,\ \mathcal{E}nd(E)(-D))$ [BLP1, Lemma 2.5], using the Serre duality it follows that the cotangent space of $\mathcal{N}^e(d)$ at $(E,\ \phi)$ is $H^0(X,\ \mathcal{E}nd(E) \otimes K_X(D))$. Let $T^*\mathcal{N}^e(d)$ be the cotangent bundle of $\mathcal{N}^e(d)$. For $\theta \in H^0(X,\ \mathcal{E}nd(E) \otimes K_X(D))$, define the following 2-term complex:

$$\mathcal{C}^{\mathrm{Higgs}}_\bullet :\ \mathcal{C}^{\mathrm{Higgs}}_0 := \mathcal{E}nd(E)(-D) \xrightarrow{[\theta,\,\cdot]} \mathcal{C}^{\mathrm{Higgs}}_1 := \mathcal{E}nd(E) \otimes K_X(D)\,.$$

The tangent space $T_{(E,\phi,\theta)}T^*\mathcal{N}^e(d)$ at $(E,\ \phi,\ \theta)$ is $\mathbb{H}^1(\mathcal{C}^{\mathrm{Higgs}}_\bullet)$ [BLP1, Lemma 2.7]. Given an affine open covering $\{U_\alpha\}$ of X, the hypercohomology $\mathbb{H}^1(\mathcal{C}^{\mathrm{Higgs}}_\bullet)$ admits a description in terms of Čech cohomology. In this description, the 1-cocycles are pairs $(\{u_{\alpha\beta}\},\ \{v_\alpha\})$, where

$$u_{\alpha\beta} \in \mathcal{E}nd(E)(-D)(U_\alpha \cap U_\beta) \quad \text{and} \quad \mathrm{v}_\alpha \in \mathcal{E}\mathrm{nd}(\mathrm{E}) \otimes \mathrm{K}_\mathrm{X}(\mathrm{D})(\mathrm{U}_\alpha)$$

such that $u_{\beta\gamma} - u_{\alpha\gamma} + u_{\alpha\beta} = 0$ and $v_\beta - v_\alpha = [\theta,\ u_{\alpha\beta}]$. The 1-coboundaries are of the form $(\{w_\alpha - w_\beta\},\ \{[w_\alpha,\ \theta]\})$, where $w_\alpha \in \mathcal{E}nd(E)(-D)(U_\alpha)$.

We define a canonical 1-form $\phi_{\mathcal{N}^e(d)}$ on the cotangent bundle $T^*\mathcal{N}^e(d)$ by

$$\begin{aligned} \phi_{\mathcal{N}^e(d)} :\ \mathbb{H}^1(\mathcal{C}^{\mathrm{Higgs}}_\bullet) &\longrightarrow H^1(K_X) \\ [(\{u_{\alpha\beta}\},\ \{v_\alpha\})] &\longmapsto [\{\mathrm{Tr}(\theta|_{U_\alpha} u_{\alpha\beta})\}]. \end{aligned} \tag{4.2}$$

Lemma 4.1. *Let $\Phi_{T^*\mathcal{N}^e(d)}$ be the Liouville 2-form on the cotangent bundle $T^*\mathcal{N}^e(d)$, that is, $\Phi_{T^*\mathcal{N}^e(d)}$ is the exterior derivative of the canonical 1-form $\phi_{\mathcal{N}^e(d)}$ in (4.2). The Liouville 2-form $\Phi_{T^*\mathcal{N}^e(d)}$ coincides with the bilinear form*

$$\begin{aligned} \mathbb{H}^1(\mathcal{C}^{\mathrm{Higgs}}_\bullet) \otimes \mathbb{H}^1(\mathcal{C}^{\mathrm{Higgs}}_\bullet) &\longrightarrow H^1(K_X) \\ [(\{u_{\alpha\beta}\},\ \{v_\alpha\})] \otimes [(\{u'_{\alpha\beta}\},\ \{v'_\alpha\})] &\longmapsto [\{\mathrm{Tr}(v_\alpha u'_{\alpha\beta}) - \mathrm{Tr}(u_{\alpha\beta} v'_\beta)\}] \end{aligned}$$

on the Čech cohomology.

Proof. Let v and v' be tangent vectors of $T^*\mathcal{N}^e(d)$ at $(E,\ \phi,\ \theta) \in T^*\mathcal{N}^e(d)$. Let

$$D_v :\ \mathcal{O}_{T^*\mathcal{N}^e(d)} \longrightarrow \mathcal{O}_{T^*\mathcal{N}^e(d)}$$

be the derivative corresponding to v. Take an affine open subset $U \subset T^*\mathcal{N}^e(d)$ such that $(E,\ \phi,\ \theta) \in U$, and also take an affine open covering $\{U_\alpha\}$ of $X \times U$ such that there is a trivialization

$$g_\alpha :\ E|_{U_\alpha} \xrightarrow{\sim} \mathcal{O}_{U_\alpha}^{\oplus r}$$

for each U_α. Set $g_{\alpha\beta} := g_\alpha \circ g_\beta^{-1}$ and $\theta_\alpha := g_\alpha \circ \theta|_{U_\alpha} \circ g_\alpha^{-1}$. We may describe the tangent vector v as

$$v = [(\{u_{\alpha\beta}\},\ \{v_\alpha\})]\,,$$

where $u_{\alpha\beta} := g_\alpha^{-1} \circ (D_v(g_{\alpha\beta})g_{\alpha\beta}^{-1}) \circ g_\alpha$ and $v_\alpha := g_\alpha^{-1} \circ (D_v\theta_\alpha) \circ g_\alpha$. The exterior derivative of $\phi_{\mathcal{N}^e(d)}$ is computed as follows:

$$\begin{aligned} D_v\phi_{\mathcal{N}^e(d)}(v') - D_{v'}\phi_{\mathcal{N}^e(d)}(v) + \phi_{\mathcal{N}^e(d)}([v,\ v']) &= D_v\left(\mathrm{Tr}(\theta_\alpha D_{v'}(g_{\alpha\beta})g_{\alpha\beta}^{-1})\right) \\ &\quad - D_{v'}\left(\mathrm{Tr}(\theta_\alpha D_v(g_{\alpha\beta})g_{\alpha\beta}^{-1})\right) + \left(\mathrm{Tr}(\theta_\alpha(D_{v'} \circ D_v - D_v \circ D_{v'})(g_{\alpha\beta}))g_{\alpha\beta}^{-1}\right) \end{aligned}$$

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$$
\begin{aligned}
&= \mathrm{Tr}\left(D_v(g_{\alpha\beta}^{-1}\theta_\alpha)D_{v'}(g_{\alpha\beta})\right) - \mathrm{Tr}\left(D_{v'}(g_{\alpha\beta}^{-1}\theta_\alpha)D_v(g_{\alpha\beta})\right) \\
&= \left(\mathrm{Tr}(D_v(\theta_\alpha)D_{v'}(g_{\alpha\beta})g_{\alpha\beta}^{-1}) - \mathrm{Tr}(D_{v'}(\theta_\alpha)D_v(g_{\alpha\beta})g_{\alpha\beta}^{-1})\right) \\
&\quad - \left(\mathrm{Tr}(D_v(g_{\alpha\beta})g_{\alpha\beta}^{-1}\theta_\alpha D_{v'}(g_{\alpha\beta})g_{\alpha\beta}^{-1})\right) + \left(\mathrm{Tr}(D_{v'}(g_{\alpha\beta})g_{\alpha\beta}^{-1}\theta_\alpha D_v(g_{\alpha\beta})g_{\alpha\beta}^{-1})\right) \\
&= \left(\mathrm{Tr}(v_\alpha u'_{\alpha\beta}) - \mathrm{Tr}(v'_\alpha u_{\alpha\beta})\right) + \mathrm{Tr}([u'_{\alpha\beta},\, \theta_\alpha]u_{\alpha\beta}) \\
&= \mathrm{Tr}(v_\alpha u'_{\alpha\beta}) - \mathrm{Tr}(u_{\alpha\beta}v'_\beta).
\end{aligned}
$$

This completes the proof of the lemma. □

4.2 2-form on $\mathcal{M}^e_{\mathrm{FC}}(d)$

As before, assume that $H_x = \{e\} \subset \mathrm{GL}(r, \mathbb{C})$ for every $x \in D$. Let $\mathbb{K}$ denote the following complex of coherent sheaves on X:

$$
\mathbb{K} \,:\, \mathcal{O}_X \xrightarrow{\ d\ } K_X \longrightarrow 0, \tag{4.3}
$$

where $\mathcal{O}_X$ and K_X are at the 0-th position and 1-position, respectively, and d is the de Rham differential.

Consider the complex $\mathcal{C}_\bullet$ in (2.11). Define a pairing

$$
\begin{aligned}
\Theta^e \,:\, \mathbb{H}^1(\mathcal{C}_\bullet) \otimes \mathbb{H}^1(\mathcal{C}_\bullet) &\longmapsto \mathbb{H}^2(\mathbb{K}) \cong \mathbb{C} \\
[(\{u_{\alpha\beta}\},\, \{v_\alpha\})] \otimes [(\{u'_{\alpha\beta}\},\, \{v'_\alpha\})] &\longmapsto [(\{\mathrm{Tr}(u_{\alpha\beta}u'_{\beta\gamma})\},\, -\{\mathrm{Tr}(u_{\alpha\beta}v'_\beta) - \mathrm{Tr}(v_\alpha u'_{\alpha\beta})\})]
\end{aligned} \tag{4.4}
$$

in terms of the Čech cohomology with respect to an affine open covering $\{U_\alpha\}$ of X.

Lemma 4.2. *The pairing Θ^e in (4.4) satisfies the identity*

$$
\Theta^e(v,\, v) = 0\,.
$$

Thus, Θ^e is skew-symmetric, and hence produces a 2-form on $\mathcal{M}^e_{\mathrm{FC}}(d)$ (see Lemma 2.4).

Proof. Let $v = [(\{u_{\alpha\beta}\},\, \{v_\alpha\})]$ be an element of $\mathbb{H}^1(\mathcal{C}_\bullet)$. We compute $\Theta^e(v,\, v)$ as follows:

$$
\begin{aligned}
\Theta^e(v,\, v) &= [(\{\mathrm{Tr}(u_{\alpha\beta}u_{\beta\gamma})\},\, -\{\mathrm{Tr}(u_{\alpha\beta}v_\beta) - \mathrm{Tr}(v_\alpha u_{\alpha\beta})\})] \\
&= [(\{\mathrm{Tr}(u_{\alpha\beta}u_{\beta\gamma})\},\, -\{\mathrm{Tr}(u_{\alpha\beta}(v_\beta - v_\alpha))\})] \\
&= [(\{\mathrm{Tr}(u_{\alpha\beta}u_{\beta\gamma})\},\, -\{\mathrm{Tr}(u_{\alpha\beta}(\nabla \circ u_{\alpha\beta} - u_{\alpha\beta} \circ \nabla))\})] \\
&= \left[\left(\{\mathrm{Tr}(u_{\alpha\beta}u_{\beta\gamma})\}\,,\, -\left\{d\left(\tfrac{1}{2}\mathrm{Tr}(u_{\alpha\beta}^2)\right)\right\}\right)\right].
\end{aligned}
$$

On the other hand,

$$
\begin{aligned}
\tfrac{1}{2}\mathrm{Tr}(u_{\alpha\beta}^2) - \tfrac{1}{2}\mathrm{Tr}(u_{\alpha\gamma}^2) + \tfrac{1}{2}\mathrm{Tr}(u_{\beta\gamma}^2) &= \tfrac{1}{2}\mathrm{Tr}((u_{\alpha\beta} - u_{\alpha\gamma})(u_{\alpha\beta} + u_{\alpha\gamma})) + \tfrac{1}{2}\mathrm{Tr}(u_{\beta\gamma}^2) \\
&= \tfrac{1}{2}\mathrm{Tr}((u_{\beta\gamma})(u_{\beta\gamma} - u_{\alpha\beta} - u_{\alpha\gamma})) \\
&= -\mathrm{Tr}(u_{\alpha\beta}u_{\beta\gamma}).
\end{aligned}
$$

Combining these, it follows that $\Theta^e(v,\, v) = 0$ in $\mathbb{H}^2(\mathbb{K})$. □

Remark 4.3. We have constructed a 2-form Θ^e on $\mathcal{M}^e_{\mathrm{FC}}(d)$ by (4.4). On the other hand, there exists another definition of this 2-form from a differential geometric perspective; this will be explained below. First, recall a description of $\mathbb{H}^1(\mathcal{C}_\bullet)$ as a Dolbeault cohomology. (See the proof of

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Theorem 3.2 of [Bi].) Let $\overline{\partial}'$ and $\overline{\partial}_1'$ be the Dolbeault operators for the holomorphic vector bundles $\mathcal{E}nd(E)(-D)$ and $\mathcal{E}nd(E)(D)$, respectively. Consider the Dolbeault resolution of the complex $\mathcal{C}_\bullet$:

$$\begin{array}{ccc} 0 & & 0 \\ \downarrow & & \downarrow \\ \mathcal{E}nd(E)(-D) & \xrightarrow{\nabla} & \mathcal{E}nd(E)\otimes K_X(D) \\ \downarrow & & \downarrow \\ \Omega_X^{0,0}(\mathcal{E}nd(E)(-D)) & \xrightarrow{\nabla} & \Omega_X^{1,0}(\mathcal{E}nd(E)(D)) \\ \downarrow \overline{\partial}' & & \downarrow \overline{\partial}_1' \\ \Omega_X^{0,1}(\mathcal{E}nd(E)(-D)) & \xrightarrow{\nabla'} & \Omega_X^{1,1}(\mathcal{E}nd(E)(D)) \\ \downarrow & & \downarrow \\ 0 & & 0 \end{array}$$

where ∇' is constructed using ∇ and the usual differential operator ∂ on $(0,\, 1)$-forms on X. Note that

$$\overline{\partial}_1' \circ \nabla + \nabla' \circ \overline{\partial}' = 0\,.$$

This produces the following complex of vector spaces

$$0 \longrightarrow C^\infty(X,\, \mathcal{E}nd(E)(-D)) \xrightarrow{\overline{\partial}'\oplus\nabla} C^\infty(X,\, \Omega_X^{0,1}(\mathcal{E}nd(E)(-D))) \oplus C^\infty(X,\, \mathcal{E}nd(E)\otimes K_X(D))$$
$$\xrightarrow{\nabla'+\overline{\partial}'} C^\infty(X,\, \Omega_X^{1,1}(\mathcal{E}nd(E)(D))) \longrightarrow 0.$$

Since the Dolbeault complex is a fine resolution of $\mathcal{C}_\bullet$, it follows immediately that

$$\mathbb{H}^1(\mathcal{C}_\bullet) \;=\; \frac{\mathrm{Ker}(\nabla\,' + \text{-}\partial\,')}{\mathrm{Im}(\overline{\partial}' \oplus \nabla)}.$$

Then the 2-form Θ^e in (4.4) can be described using the Dolbeault cohomology in the following way:

$$(\omega_1,\, \omega_2)\otimes(\omega_1',\, \omega_2') \longmapsto \int_X \mathrm{Tr}(\omega_1\wedge\omega_2') + \int_X \mathrm{Tr}(\omega_2\wedge\omega_1')\,.$$

4.3 Symplectic structure on $\mathcal{M}^e_{\mathrm{FC}}(d)^\circ$

Now take the restriction of Θ^e to $\mathcal{M}^e_{\mathrm{FC}}(d)^\circ$; here, $\mathcal{M}^e_{\mathrm{FC}}(d)^\circ$ is the open substack of $\mathcal{M}^e_{\mathrm{FC}}(d)$ defined in (3.7), or in other words, the underlying framed bundle $(E,\, \phi)$ of any $(E,\, \phi,\, \nabla) \in \mathcal{M}^e_{\mathrm{FC}}(d)^\circ$ satisfies the condition that it is simple. Denote this restriction of Θ^e to $\mathcal{M}^e_{\mathrm{FC}}(d)^\circ$ by $\Theta^e|_{\mathcal{M}^e_{\mathrm{FC}}(d)^\circ}$. It will be shown that this restriction of Θ^e is a symplectic form.

Let

$$p_1 :\; \mathcal{M}^e_{\mathrm{FC}}(d)^\circ \longrightarrow \mathcal{N}^e(d) \tag{4.5}$$

be the forgetful map that simply forgets the connection. Take an analytic open subset $U \subset \mathcal{N}^e(d)$, which is assumed to be small enough. Then there exist sections, over U, of the map p_1

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in (4.5). Let

$$s:\, U \longrightarrow p_1^{-1}(U)$$

be a holomorphic section. Using s, an isomorphism

$$\begin{aligned} P_1 :\, T^*U &\xrightarrow{\cong} p_1^{-1}(U) \\ (y,\, v) &\longmapsto s(y)+v \end{aligned} \tag{4.6}$$

is obtained. The restriction, to $p_1^{-1}(U)$, of the form $\Theta^e|_{\mathcal{M}^e_{\mathrm{FC}}(d)^\circ}$ is denoted by $\Theta^e|_{p_1^{-1}(U)}$.

Lemma 4.4. *Let Φ_U be the Liouville 2-form on the cotangent bundle T^*U. Then,*

$$\Theta^e|_{p_1^{-1}(U)} - (P_1^{-1})^*\Phi_U \,=\, p_1^*(s^*\Theta^e|_{p_1^{-1}(U)})\,,$$

where $\Theta^e|_{p_1^{-1}(U)}$ is the restriction of the form in (4.4) and p_1 is the projection in (4.5), while P_1 is the isomorphism in (4.6).

Proof. Take a point $z=(E,\,\phi,\,\nabla)$ of $p_1^{-1}(U)$. Let $\nabla(E,\,\phi)$ be the connection associated to the point $s\circ p_1(z)$. The image of z under the map P_1^{-1} in (4.6) is as follows:

$$P_1^{-1}(z) \,=\, P_1^{-1}(E,\,\phi,\,\nabla) \,=\, (E,\,\phi,\,\nabla-\nabla(E,\phi))\,.$$

Let $[(\{u_{\alpha\beta}\},\,\{v_\beta\})]$ be an element of $\mathbb{H}^1((\mathcal{C}_\bullet)_z)$, where $(\mathcal{C}_\bullet)_z$ is the complex in (2.11) associated to $z=(E,\,\phi,\,\nabla)$. Recall from Lemma 2.4 that $\mathbb{H}^1((\mathcal{C}_\bullet)_z)$ is the tangent space of $p_1^{-1}(U)$ at z. Note that $u_{\alpha\beta}$ and v_α satisfy the equality

$$v_\beta - v_\alpha \,=\, \nabla\circ u_{\alpha\beta} - u_{\alpha\beta}\circ\nabla.$$

Let $[(\{u_{\alpha\beta}\},\,\{v^s_\alpha\})]$ be the element of $\mathbb{H}^1((\mathcal{C}_\bullet)_{s\circ p_1(z)})$ such that

$$(s\circ p_1)_*([(\{u_{\alpha\beta}\},\,\{v_\beta\})]) \,=\, [(\{u_{\alpha\beta}\},\,\{v^s_\beta\})]\,.$$

Note that $u_{\alpha\beta}$ and v^s_α satisfy the equality

$$v^s_\beta - v^s_\alpha \,=\, \nabla(E,\phi)\circ u_{\alpha\beta} - u_{\alpha\beta}\circ\nabla(E,\phi)\,.$$

Since

$$(v_\beta - v^s_\beta) - (v_\alpha - v^s_\alpha) \,=\, [\nabla-\nabla(E,\phi),\, u_{\alpha\beta}]\,,$$

it follows that $[(\{u_{\alpha\beta}\},\,\{v_\alpha - v^s_\alpha\})]$ is an element of $\mathbb{H}^1(\mathcal{C}^{\mathrm{Higgs}}_\bullet)$; recall that $\mathbb{H}^1(\mathcal{C}^{\mathrm{Higgs}}_\bullet)$ is the tangent space of T^*U at $P_1^{-1}(z)$ (see (4.6)). There is a map

$$\begin{aligned} \mathbb{H}^1((\mathcal{C}_\bullet)_z) &\longrightarrow \mathbb{H}^1(\mathcal{C}^{\mathrm{Higgs}}_\bullet) \\ [(\{u_{\alpha\beta}\},\,\{v_\alpha\})] &\longmapsto [(\{u_{\alpha\beta}\},\,\{v_\alpha - v^s_\alpha\})]. \end{aligned}$$

This map coincides with $(P_1^{-1})_*:\, \mathbb{H}^1((\mathcal{C}_\bullet)_z) \longrightarrow \mathbb{H}^1(\mathcal{C}^{\mathrm{Higgs}}_\bullet)$.

Now we compute the map

$$(\Theta^e - (P_1^{-1})^*\Phi_U):\, \mathbb{H}^1((\mathcal{C}_\bullet)_z)\otimes\mathbb{H}^1((\mathcal{C}_\bullet)_z) \longrightarrow \mathbb{H}^2(\mathbb{K})$$

as follows:

$$\begin{aligned} (\Theta^e - (P_1^{-1})^*\Phi_U)(v,v') \,&=\, [(\{\mathrm{Tr}(u_{\alpha\beta}u'_{\beta\gamma})\},\, -\{\mathrm{Tr}(u_{\alpha\beta}v'_\beta) - \mathrm{Tr}(v_\alpha u'_{\alpha\beta})\})] \\ &\qquad - [\{0\},\, -\{\mathrm{Tr}(u_{\alpha\beta}(v'_\beta - (v^s_\beta)')) - \mathrm{Tr}((v_\alpha - v^s_\alpha)u'_{\alpha\beta})\}] \\ &=\, [(\{\mathrm{Tr}(u_{\alpha\beta}u'_{\beta\gamma})\},\, -\{\mathrm{Tr}(u_{\alpha\beta}(v^s_\beta)') - \mathrm{Tr}(v^s_\alpha u'_{\alpha\beta})\})] \in \mathbb{H}^2(\mathbb{K}). \end{aligned}$$

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On the other hand, we compute

$$p_1^*(s^*\Theta^e):\ \mathbb{H}^1((\mathcal{C}_\bullet)_z)\otimes\mathbb{H}^1((\mathcal{C}_\bullet)_z)\xrightarrow{(s\circ p_1)_*\otimes(s\circ p_1)_*}\mathbb{H}^1((\mathcal{C}_\bullet)_{s\circ p_1(z)})\otimes\mathbb{H}^1((\mathcal{C}_\bullet)_{s\circ p_1(z)})\xrightarrow{\Theta^e}\mathbb{H}^1(\mathbb{K})$$

as follows:

$$\begin{aligned}p_1^*(s^*\Theta^e)(v,\ v') &= p_1^*(s^*\Theta^e)([(\{u_{\alpha\beta}\},\ \{v^s_\alpha\})],[(\{u'_{\alpha\beta}\},\ \{(v^s_\alpha)'\})])\\ &= [(\{\mathrm{Tr}(u_{\alpha\beta}u'_{\beta\gamma})\},\ -\{\mathrm{Tr}(u_{\alpha\beta}(v^s_\beta)')-\mathrm{Tr}(v^s_\alpha u'_{\alpha\beta})\})]\in\mathbb{H}^1(\mathbb{K}).\end{aligned}$$

Therefore, we have the equality $\Theta^e|_{p_1^{-1}(U)}-(P_1^{-1})^*\Phi_U = p_1^*(s^*\Theta^e|_{p_1^{-1}(U)})$. □

It will now be shown that the restriction of Θ^e to $\mathcal{M}^e_{\mathrm{FC}}(d)^\circ$ is nondegenerate.

COROLLARY 4.5. *The 2-form $\Theta^e|_{\mathcal{M}^e_{\mathrm{FC}}(d)^\circ}$ is nondegenerate.*

Proof. For any point $(E,\ \phi,\ \nabla)\in\mathcal{M}^e_{\mathrm{FC}}(d)^\circ$ and any tangent vectors $v,\ w\in T_{(E,\phi,\nabla)}\mathcal{M}^e_{\mathrm{FC}}(d)^\circ$, we have

$$p_1^*s^*\Theta^e(E,\phi,\nabla)(v,\ w)\ =\ 0$$

when one of v and w is vertical for the projection p_1 in (4.5). So, if w is vertical, from Lemma 4.4 it follows that

$$\Theta^e(E,\ \phi,\ \nabla)(v,\ w)\ =\ (P_1^{-1})^*\Phi_U(E,\ \phi,\ \nabla)(v,\ w)\,. \tag{4.7}$$

Since Φ_U is a symplectic form, there is a tangent vector $v\in T_{(E,\phi,\nabla)}\mathcal{M}^e_{\mathrm{FC}}(d)^\circ$ such that

$$(P_1^{-1})^*\Phi_U(E,\ \phi,\ \nabla)(v,\ w)\neq 0.$$

Now from (4.7) it follows that $\Theta^e(E,\ \phi,\ \nabla)(v,\ w)\neq 0$.

Since the vertical tangent spaces for the projection $T^*U\longrightarrow U$ are Lagrangian for the Liouville 2-form Φ_U, given any non-vertical tangent vector

$$v\in T_{(E,\phi,\nabla)}\mathcal{M}^e_{\mathrm{FC}}(d)^\circ$$

for the projection $T^*U\longrightarrow U$, there is a vertical tangent vector

$$w\in T_{(E,\phi,\nabla)}\mathcal{M}^e_{\mathrm{FC}}(d)^\circ$$

for the projection $T^*U\longrightarrow U$ such that $(P_1^{-1})^*\Phi_U(E,\ \phi,\ \nabla)(v,\ w)\neq 0$. Now from (4.7) it follows that $\Theta^e(E,\ \phi,\ \nabla)(v,\ w)\neq 0$. Consequently, the form $\Theta^e|_{p^{-1}(U)}$ is nondegenerate. □

Remark 4.6. It was shown above that the restriction of Θ^e to $\mathcal{M}^e_{\mathrm{FC}}(d)^\circ$ is nondegenerate by using Lemma 4.4. We will show that the 2-form Θ^e on $\mathcal{M}^e_{\mathrm{FC}}(d)$ is nondegenerate by using the Serre duality (Proposition 4.18 below). So it can be shown that the restriction of Θ^e to $\mathcal{M}^e_{\mathrm{FC}}(d)^\circ$ is nondegenerate without using Lemma 4.4. Nevertheless, we have discussed nondegeneracy of the restriction of Θ^e by using this lemma because this argument highlights another important perspective. Lemma 4.4 will be used below in the proof of the d-closedness of the restriction of Θ^e. Moreover, the d-closedness of the restriction of Θ^e will be used below in the proof of the d-closedness of Θ^e on $\mathcal{M}^e_{\mathrm{FC}}(d)$.

PROPOSITION 4.7. *Assume that $g\geqslant 2$. Let $\mathcal{M}^e_{\mathrm{FC}}(d)^\circ$ be the open subspace of $\mathcal{M}^e_{\mathrm{FC}}(d)$ defined in (3.7) for $H=\{e\}$. Then the restriction $\Theta^e|_{\mathcal{M}^e_{\mathrm{FC}}(d)^\circ}$ of the nondegenerate 2-form Θ^e in (4.4) is d-closed.*

Proof. The moduli space $\mathcal{N}^e(d)$ in (4.1) has the open subset $\mathcal{N}^e(d)^\circ$ defined by

$$\mathcal{N}^e(d)^\circ\ :=\ \{(E,\ \phi)\in\mathcal{N}^e(d)\mid \text{E is a stable vector bundle}\}\,.$$

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Also, let $\mathcal{M}^e_{\rm FC}(d)^{\circ\circ} \subset \mathcal{M}^e_{\rm FC}(d)^{\circ}$ (see Definition 3.1) be the open subset

$$\mathcal{M}^e_{\rm FC}(d)^{\circ\circ} \,:=\, \{(E,\,\phi,\,\nabla) \in \mathcal{M}^e_{\rm FC}(d)^{\circ} \mid E \text{ is a stable vector bundle}\}\,. \tag{4.8}$$

The openness of both $\mathcal{N}^e(d)^{\circ}$ and $\mathcal{M}^e_{\rm FC}(d)^{\circ\circ}$ follows from [Maru, p. 635, Theorem 2.8(B)]. The moduli spaces $\mathcal{N}^e(d)^{\circ}$ and $\mathcal{M}^e_{\rm FC}(d)^{\circ\circ}$ are non-empty because $g \geqslant 2$.

To prove that the form $\Theta^e|_{\mathcal{M}^e_{\rm FC}(d)^{\circ}}$ on $\mathcal{M}^e_{\rm FC}(d)^{\circ}$ is closed, it suffices to show that the restriction of $\Theta^e|_{\mathcal{M}^e_{\rm FC}(d)^{\circ}}$ to $\mathcal{M}^e_{\rm FC}(d)^{\circ\circ}$ is closed.

Let $p_{1,0} : \mathcal{M}^e_{\rm FC}(d)^{\circ\circ} \longrightarrow \mathcal{N}^e(d)^{\circ}$ be the restriction of the forgetful map p_1 in (4.5). Take a sufficiently small analytic open subset $U \subset \mathcal{N}^e(d)^{\circ}$ such that there is a holomorphic section

$$s :\, U \,\longrightarrow\, p_{1,0}^{-1}(U)\,,$$

over U, of $p_{1,0}$. Now, Lemma 4.4 says that

$$\Theta^e - (P_1^{-1})^*\Phi_U \,=\, p_{1,0}^*(s^*\Theta^e)$$

on $p_{1,0}^{-1}(U)$. This implies that

$$d\Theta^e \,=\, p_{1,0}^* d(s^*\Theta^e) \tag{4.9}$$

on $p_{1,0}^{-1}(U)$, because the Liouville 2-form is d-closed.

In view of (4.9), to prove the theorem it suffices to show the existence of a local holomorphic section $s :\, U \longrightarrow p_{1,0}^{-1}(U)$ of the map $p_{1,0}^{-1}$ such that $d(s^*\Theta^e) = 0$.

We shall construct a holomorphic section $s :\, U \longrightarrow p_{1,0}^{-1}(U)$ such that

$$d(s^*\Theta^e) \,=\, 0\,.$$

For that, first define a moduli space

$$\mathcal{M}^e_{\rm FC}(d)^{\circ\circ}_0 \,:=\, \left\{ (E,\phi,\nabla) \,\middle|\, \begin{array}{l} E \textit{ is a stable vector bundle of degree } d, \textit{ and} \\ (E,\ \phi\ ,\nabla\ ) \textit{ is a framed } \mathrm{GL}(r,\mathbb{C}) - \textit{connection such that} \\ \mathrm{res}_{x_i}(\nabla\ ) = 0 \text{ for } i = 1,\ldots\ , n-1 \text{ and } \mathrm{res}_{x_n}(\ \nabla\ ) = -\frac{d}{r}e \end{array} \right\} \Big/\sim,$$

where e is the identity matrix. There is the natural inclusion map

$$\iota \,:\, \mathcal{M}^e_{\rm FC}(d)^{\circ\circ}_0 \,\hookrightarrow\, \mathcal{M}^e_{\rm FC}(d)^{\circ\circ}\,, \tag{4.10}$$

where $\mathcal{M}^e_{\rm FC}(d)^{\circ\circ}$ is defined in (4.8). Also, define two moduli spaces

$$\mathcal{M}(d)^{\circ\circ}_0 \,:=\, \left\{ (E,\nabla) \,\middle|\, \begin{array}{l} \mathrm{E \text{ is a stable vector bundle of rank r and degree d, and}} \\ \nabla\ :\ \ \mathrm{E} \ \longrightarrow\ \mathrm{E} \otimes \mathrm{K_X(D)} \text{ is a connection such that} \\ \mathrm{res}_{x_i}(\ \nabla\ ) = 0 \text{ for } i = 1,\ldots\ , n-1 \text{ and } \mathrm{res}_{x_n}(\nabla\ ) = -\frac{d}{r}e \end{array} \right\} \Big/\sim$$

and

$$\mathcal{N}(d)^{\circ} \,=\, \{E \,|\, \mathrm{E \text{ is a stable vector bundle of rank r and degree d}}\}/\sim.$$

There are the forgetful maps

$$q_1 :\, \mathcal{M}(d)^{\circ\circ}_0 \longrightarrow \mathcal{N}(d)^{\circ}, \quad q_2 :\, \mathcal{M}^e_{\rm FC}(d)^{\circ\circ}_0 \longrightarrow \mathcal{M}(d)^{\circ\circ}_0 \quad \text{and} \quad p_2 :\, \mathcal{N}^e(d)^{\circ} \longrightarrow \mathcal{N}(d)^{\circ}\,, \tag{4.11}$$

where q_2 and p_2 forget the framing while q_1 forgets the connection.

Take an analytic open subset $U_0 \subset \mathcal{N}(d)^{\circ}$. Assume that U_0 is small enough and that the image of U under the forgetful map $p_2 :\, \mathcal{N}^e(d)^{\circ} \longrightarrow \mathcal{N}(d)^{\circ}$ is contained in U_0, by shrinking

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sufficiently the analytic open subset U. Take a holomorphic section

$$\begin{aligned} s_0 :\ U_0 &\longrightarrow q_1^{-1}(U_0) \\ E &\longmapsto (E,\, \nabla(E)) \end{aligned}$$

of the forgetful map $q_1 : \mathcal{M}(d)_0^{\circ\circ} \longrightarrow \mathcal{N}(d)^{\circ}$. Since $H_x = \{e\}$ for all $x \in D$, we may define a section $\widetilde{s}$ on U

$$\begin{aligned} \widetilde{s} :\ U &\longrightarrow \mathcal{M}^e_{\mathrm{FC}}(d)_0^{\circ\circ} \\ (E,\, \phi) &\longmapsto (E,\, \phi,\, \nabla(E)) \end{aligned}$$

using the section s_0. Define the section s on U of $p_{1,0} : p_1^{-1}(U) \longrightarrow U$ by

$$s = \iota \circ \widetilde{s}\,.$$

Now we shall prove that

$$d(s^*\Theta^e) = 0 \tag{4.12}$$

for such a section.

To prove (4.12), first recall that the moduli space $\mathcal{M}(d)_0^{\circ\circ}$ is equipped with a natural symplectic structure. We briefly describe this symplectic structure on $\mathcal{M}(d)_0^{\circ\circ}$. The tangent space to $\mathcal{M}(d)_0^{\circ\circ}$ at any point $(E,\, \nabla)$ is isomorphic to the first hypercohomology $\mathbb{H}^1(\mathcal{C}^0_\bullet)$, where

$$\mathcal{C}^0_\bullet\ :\ \mathcal{C}^0_0 = \mathcal{E}nd(E) \xrightarrow{\ \nabla\ } \mathcal{C}^0_1 = \mathcal{E}nd(E) \otimes K_X\,. \tag{4.13}$$

Define a nondegenerate 2-form Θ_0 on $\mathcal{M}(d)_0^{\circ\circ}$

$$\Theta_0(E,\, \nabla) :\ \mathbb{H}^1(\mathcal{C}^0_\bullet) \otimes \mathbb{H}^1(\mathcal{C}^0_\bullet) \longrightarrow \mathbb{C}$$

exactly as done in (4.4). This 2-form Θ_0 is d-closed, which is proved in [Gol].

Secondly, we show that

$$\iota^*\Theta^e = q_2^*\Theta_0\,, \tag{4.14}$$

where q_2 and ι are the maps in (4.11) and (4.10) respectively. To prove (4.14), note that the tangent space of $\mathcal{M}^e_{\mathrm{FC}}(d)_0^{\circ\circ}$ at $(E,\, \phi,\, \nabla)$ is isomorphic to the first hypercohomology $\mathbb{H}^1(\mathcal{C}'_\bullet)$ of the complex

$$\mathcal{C}'_\bullet :\ \mathcal{C}'_0 = \mathcal{E}nd(E)(-D) \xrightarrow{\ \nabla\ } \mathcal{C}'_1 = \mathcal{E}nd(E) \otimes K_X\,. \tag{4.15}$$

For $[(\{u_{\alpha\beta}\}, \{v_\alpha\})] \in \mathbb{H}^1(\mathcal{C}'_\bullet)$, we have that

$$\iota_*[(\{u_{\alpha\beta}\},\, \{v_\alpha\})] = [(\{u_{\alpha\beta}\},\, \{v_\alpha\})] \in \mathbb{H}^1(\mathcal{C}_\bullet)$$

and

$$(q_2)_*[(\{u_{\alpha\beta}\},\, \{v_\alpha\})] = [(\{u_{\alpha\beta}\},\, \{v_\alpha\})] \in \mathbb{H}^1(\mathcal{C}^0_\bullet).$$

Therefore, $\iota^*\Theta^e$ and $q_2^*\Theta_0$ have the following identical description:

$$\begin{aligned} \mathbb{H}^1(\mathcal{C}'_\bullet) \otimes \mathbb{H}^1(\mathcal{C}'_\bullet) &\longrightarrow \mathbb{H}^2(\mathbb{K}) \cong \mathbb{C} \\ [(\{u_{\alpha\beta}\},\, \{v_\alpha\})] \otimes [(\{u'_{\alpha\beta}\},\, \{v'_\alpha\})] &\longmapsto [(\{\mathrm{Tr}(u_{\alpha\beta}u'_{\beta\gamma})\},\, -\{\mathrm{Tr}(u_{\alpha\beta}v'_\beta) - \mathrm{Tr}(v_\alpha u'_{\alpha\beta})\})]. \end{aligned}$$

This proves (4.14).

Thirdly, by the equality $\iota^*\Theta^e = q_2^*\Theta_0$ in (4.14), we have

$$s^*\Theta^e = \widetilde{s}^*(q_2^*\Theta_0)\,.$$

Since Θ_0 is d-closed, it follows that $d(s^*\Theta^e) = 0$, proving (4.12).

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Finally, from the combination of (4.12) and the equality $d\Theta^e = p_{1,0}^* d(s^*\Theta^e)$ (see (4.9)), it follows that $d\Theta^e = 0$ on $p_{1,0}^{-1}(U)$. This implies that the 2-form Θ^e is d-closed on $\mathcal{M}^e_{\mathrm{FC}}(d)^\circ$. As noted before, this proves the theorem. □

Next, we show the d-closedness of Θ^e in (4.4) when $g = 0$ and $g = 1$. For this purpose, we recall the definition of *parabolic connections*. Let

$$(X,\, \boldsymbol{x}) \;:=\; (X,\, (x_1,\, \dots,\, x_n))$$

be an n-pointed smooth projective curve of genus g over $\mathbb{C}$, where $x_1, \cdots, x_n$ are distinct points of X. Denote the reduced divisor $x_1 + \cdots + x_n$ on X by $D(\boldsymbol{x})$ or simply by D if there is no possibility of confusion. Take a positive integer r.

DEFINITION 4.8. A *$\boldsymbol{x}$-quasi-parabolic bundle* of rank r and degree d is a pair $(E, \boldsymbol{l} = \{l_*^{(i)}\}_{1\leqslant i\leqslant n})$, where:

(1) *E is an algebraic vector bundle on X of rank r and degree d;*

(2) *$l_*^{(i)}$ is a filtration of subspaces $E|_{x_i} = l_0^{(i)} \supset l_1^{(i)} \supset \cdots \supset l_r^{(i)} = 0$ for every $1 \leqslant i \leqslant n$ such that* $\dim(l_j^{(i)}/l_{j+1}^{(i)}) = 1$.

Let $\boldsymbol{\alpha}$ be a tuple $(\alpha_j^{(i)})^{1\leqslant i\leqslant n}_{1\leqslant j\leqslant r}$ of real numbers which satisfy the condition

$$0 < \alpha_1^{(i)} < \alpha_2^{(i)} < \cdots < \alpha_r^{(i)} < 1$$

for each $1 \leqslant i \leqslant n$ and $\alpha_j^{(i)} \neq \alpha_{j'}^{(i')}$ for all $(i,\, j) \neq (i',\, j')$. We call the tuple $\boldsymbol{\alpha}$ a parabolic weight. Take an element

$$\boldsymbol{\nu} = (\nu_j^{(i)})^{1\leqslant i\leqslant n}_{0\leqslant j\leqslant r-1} \in \mathbb{C}^{nr}$$

such that $\sum_{i,j} \nu_j^{(i)} = -d \in \mathbb{Z}$.

DEFINITION 4.9. A quadruple $(E,\, \nabla,\, \boldsymbol{l} = \{l_*^{(i)}\}_{1\leqslant i\leqslant n}, \boldsymbol{\alpha})$ is called an $(\boldsymbol{x},\, \boldsymbol{\nu})$-parabolic connection of rank r and degree d if:

(1) $(E,\, \boldsymbol{l} = \{l_*^{(i)}\}_{1\leqslant i\leqslant n})$ is an $\boldsymbol{x}$-quasi-parabolic bundle of rank r and degree d;

(2) $\nabla :\; E \longrightarrow E \otimes K_X(D)$ *is a logarithmic connection whose residue* $\mathrm{res}_{x_i}(\nabla) :\; E|_{x_i} \longrightarrow E|_{x_i}$ *at each point x_i for $1 \leqslant i \leqslant n$ satisfies the condition* $(\mathrm{res}_{x_i}(\nabla) - \nu_j^{(i)}\mathrm{Id}_{E|_{x_i}})(l_j^{(i)}) \subset l_{j+1}^{(i)}$ *for all* $j = 0,\, \dots,\, r-1$.

DEFINITION 4.10. An $(\boldsymbol{x},\; \boldsymbol{\nu})$-parabolic connection $(E,\; \nabla,\; \boldsymbol{l},\, \boldsymbol{\alpha})$ is said to be $\boldsymbol{\alpha}$-stable if the inequality

$$\frac{\deg F + \sum_{i=1}^n \sum_{j=1}^r \alpha_j^{(i)} \dim((F|_{x_i} \cap l_{j-1}^{(i)})/(F|_{x_i} \cap l_j^{(i)}))}{\mathrm{rank} F}$$

$$< \;\frac{\deg E + \sum_{i=1}^n \sum_{j=1}^r \alpha_j^{(i)} \dim(l_{j-1}^{(i)}/l_j^{(i)})}{\mathrm{rank} E}$$

holds for every subbundle $0 \neq F \subsetneq E$ for which $\nabla(F) \subset F \otimes \Omega_X^1(D)$. We say that $(E,\; \nabla,\; \boldsymbol{l},\; \boldsymbol{\alpha})$ is $\boldsymbol{\alpha}$-semistable if the weaker inequality "$\leqslant$' holds (instead of "$<$').

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Remark 4.11. In the non-abelian Hodge correspondence (see [Sim1]), the parabolic weight $\boldsymbol{\alpha}$ is an important datum needed to connect to the parabolic Higgs bundles. Since we focus on the algebraic moduli spaces, we omit the parabolic weight $\boldsymbol{\alpha}$ to denote the parabolic connection. So we denote a parabolic connection by $(E,\, \nabla,\, \boldsymbol{l})$, even though there is the parabolic weight $\boldsymbol{\alpha}$ in the background.

In the inequality for the stability condition in Definition 4.10, we may replace the parabolic weight with a tuple of rational numbers which is very close to $\boldsymbol{\alpha}$. We have the following.

Theorem 4.12 ([Ina, Theorem 2.2]). *The moduli space $\mathcal{M}^{\boldsymbol{\alpha}}_{\mathrm{PC}}(\boldsymbol{\nu})$ of $\boldsymbol{\alpha}$-stable $(\boldsymbol{x},\boldsymbol{\nu})$-parabolic connections exists as a quasi-projective scheme over* $\mathrm{Spec}\mathbb{C}$.

Let $(E,\, \boldsymbol{l})$ be a $\boldsymbol{x}$-quasi-parabolic bundle. Set

$$\mathrm{End}(E,\, \boldsymbol{l}) \ := \ \left\{ u \in \ \mathrm{Hom}_{\mathcal{O}_X}(E,\, E) \ \middle| \ \mathrm{u}|_{\mathrm{x_i}}(\mathrm{l}^{(\mathrm{i})}_{\mathrm{j}}) \subset \ \ \mathrm{l}^{(\mathrm{i})}_{\mathrm{j}} \text{ for any i, j} \right\}.$$

We denote the invertible elements of $\mathrm{End}(E,\, \boldsymbol{l})$ by $\mathrm{Aut}(E,\, \boldsymbol{l})$.

Definition 4.13. A $\boldsymbol{x}$-quasi-parabolic bundle $(E,\, \boldsymbol{l})$ is said to be simple if $\mathrm{End}(E,\, \boldsymbol{l}) = \mathbb{C}$, which is equivalent to the condition $\mathrm{Aut}(E,\, \boldsymbol{l}) = \mathbb{C}^*$.

Remark 4.14. For each x_i, let $\mathbf{H}_{x_i} \subset \mathrm{GL}_r(\mathbb{C})$ be the Borel subgroup consisting of the upper triangular matrices. Then a framed $\mathrm{GL}_r(\mathbb{C})$–bundle with respect to the structure subgroups $(\mathbf{H}_{x_i} \subset \mathrm{GL}_r(\mathbb{C}))_{1\leqslant i\leqslant n}$ is equivalent to an $\boldsymbol{x}$-quasi-parabolic bundle. The above definition of simple quasi-parabolic bundle is equivalent to that of a simple framed bundle with this structure subgroup in the sense of Definition 3.1. A framed $\mathrm{GL}_r(\mathbb{C})$–connection with respect to the structure subgroups $(\mathbf{H}_{x_i} \subset \mathrm{GL}_r(\mathbb{C}))_{1\leqslant i\leqslant n}$ is equivalent to an $(\boldsymbol{x},\ \mathbf{0})$-parabolic connection, where $\mathbf{0} \in \mathbb{C}^{nr}$ is defined by $\nu^{(i)}_j = 0$ for any $i,\ j$.

For an $(\boldsymbol{x},\ \boldsymbol{\nu})$-parabolic connection $(E,\, \nabla,\, \boldsymbol{l})$, set

$$\mathrm{End}(E,\, \nabla,\, \boldsymbol{l}) \ := \ \{u \in \ \mathrm{End}(E,\, \boldsymbol{l}) \ | \ \ \nabla \circ u = (u \otimes \mathrm{id}) \circ \nabla\}\,,$$

and denote by $\mathrm{Aut}(E,\, \nabla,\, \boldsymbol{l})$ the invertible elements in $\mathrm{End}(E,\, \nabla,\, \boldsymbol{l})$.

Definition 4.15. An $(\boldsymbol{x},\ \boldsymbol{\nu})$-parabolic connection $(E,\, \nabla,\, \boldsymbol{l})$ is said to be simple if $\mathrm{End}(E,\, \nabla,\, \boldsymbol{l}) = \mathbb{C}$, which is equivalent to the condition $\mathrm{Aut}(E,\, \nabla,\, \boldsymbol{l}) = \mathbb{C}^*$.

An argument similar to the one in Proposition 3.5 proves the following proposition.

Proposition 4.16. *The moduli space $\mathcal{M}_{\mathrm{PC}}(\boldsymbol{\nu})$ of simple $(\boldsymbol{x},\boldsymbol{\nu})$-parabolic connections exists as an algebraic space. The moduli space $\mathcal{M}^{\boldsymbol{\alpha}}_{\mathrm{PC}}(\boldsymbol{\nu})$ of $\boldsymbol{\alpha}$-stable $(\boldsymbol{x},\boldsymbol{\nu})$-parabolic connections is a Zariski open subspace of $\mathcal{M}_{\mathrm{PC}}(\boldsymbol{\nu})$.*

Proposition 4.17. *Assume that either $g = 0$ or $g = 1$ and:*

(1) *$nr - 2r - 2 > 0$ if $g = 0$;*

(2) *$n \geqslant 2$ if $g = 1$.*

Let $\mathcal{M}^e_{\mathrm{FC}}(d)^\circ$ be the open subspace of $\mathcal{M}^e_{\mathrm{FC}}(d)$ defined in (3.7) for $H = \{e\}$. Then the restriction $\Theta^e|_{\mathcal{M}^e_{\mathrm{FC}}(d)^\circ}$ of the nondegenerate 2-form Θ^e in (4.4) is d-closed.

Proof. Consider the forgetful map $p_1 : \mathcal{M}^e_{\mathrm{FC}}(d)^\circ \longrightarrow \mathcal{N}^e(d)$ in (4.5), and take a sufficiently small analytic open subset $U \subset \mathcal{N}^e(d)$. For a holomorphic section $s : U \longrightarrow p_1^{-1}(U)$ of p_1 such

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that

$$d(s^*\Theta^e|_{\mathcal{M}^e_{\rm FC}(d)^\circ}) = 0,$$

we have $d\Theta^e|_{\mathcal{M}^e_{\rm FC}(d)^\circ} = 0$ by the same argument as in the proof of Theorem 4.7. We will now construct such a section s.

Let $\mathcal{N}_{\rm par}(d)$ be the moduli space of simple $\boldsymbol{x}$-quasi-parabolic bundles of rank r and degree d. For each $x \in D$, set the complex Lie proper subgroup H_x to be the subgroup of $\mathrm{GL}(r,\mathbb{C})$ consisting of the upper triangular matrices. It may be mentioned that an $\boldsymbol{x}$-quasi-parabolic bundle is the same as a framed bundle with respect to $\{H_x\}_{x\in D}$. For a framed bundle $(E,\,\phi)$, we can associate a quasi-parabolic bundle $(E,\,\boldsymbol{l})$ whose filtration $l^{(i)}_*$ on $E|_{x_i}$ is induced by the framing ϕ_{x_i} of $E|_{x_i}$ for each $1 \leqslant i \leqslant n$. Setting

$$\mathcal{N}^e(d)^\circ \ := \ \{(E,\phi) \mid \text{The quasi}-\text{parabolic bundle } (\mathrm{E},\,\boldsymbol{l}) \text{ induced by the framing } \phi \ \text{ is simple}\},$$

there is a natural morphism

$$q_B \ : \ \mathcal{N}^e(d)^\circ \ \longrightarrow \ \mathcal{N}^{\rm par}(d)^\circ. \tag{4.16}$$

Take an element

$$\boldsymbol{\nu} = (\nu^{(i)}_j)^{1\leqslant i\leqslant n}_{0\leqslant j\leqslant r-1} \in \mathbb{C}^{nr}$$

such that $\sum_{i,j}\nu^{(i)}_j = -d$. Let $\mathcal{M}_{\rm PC}(\boldsymbol{\nu})^\circ$ be the moduli space defined by

$$\mathcal{M}_{\rm PC}(\boldsymbol{\nu})^\circ \ := \ \left\{(E,\,\nabla,\,\boldsymbol{l}) \in \mathcal{M}_{\rm PC}(\boldsymbol{\nu}) \,\middle|\, (\mathrm{E},\ \boldsymbol{l}) \text{ is a simple } \boldsymbol{x}-\text{quasi}-\text{parabolic bundle}\right\} \Big/\sim.$$

Define the locally closed subspace $\mathcal{M}^e_{\rm FC}(\boldsymbol{\nu})^\circ$ of $\mathcal{M}^e_{\rm FC}(d)^\circ$ by

$$\mathcal{M}^e_{\rm FC}(\boldsymbol{\nu})^\circ \ := $$

$$\left\{(E,\,\phi,\,\nabla) \in \mathcal{M}^e_{\rm FC}(d)^\circ \,\middle|\, \begin{array}{l} \text{the framed bundle } (\mathrm{E},\phi\ ) \text{ belongs to } \mathcal{N}^{\rm e}(\mathrm{d})^\circ \\ \text{and } (\mathrm{E},\,\boldsymbol{l}) := \mathrm{q_B}(\mathrm{E},\,\phi) \text{ satisfies} \\ (\mathrm{res_x}(\nabla) - \nu^{(\mathrm{i})}_{\mathrm{j}}\,\mathrm{Id}_{\mathrm{E}|_{\mathrm{x_i}}})\,(\mathrm{l}^{(\mathrm{i})}_{\mathrm{j}}) \subset \mathrm{l}^{(\mathrm{i})}_{\mathrm{j}+1} \text{ for } 0 \leqslant \mathrm{j} \leqslant \mathrm{r}-1 \end{array}\right\} \Big/\sim.$$

In the above definition we have $\phi = \{\phi_x\}_{x\in D}$, where $\phi_x : \mathcal{O}_X^{\oplus r}|_x \longrightarrow E|_x$ are isomorphisms defining a framing of E over D. Since a framing defines a parabolic structure, there is a natural map

$$q^{\rm par}_1 \ : \ \mathcal{M}^e_{\rm FC}(\boldsymbol{\nu})^\circ \ \longrightarrow \ \mathcal{M}_{\rm PC}(\boldsymbol{\nu})^\circ\,. \tag{4.17}$$

Notice that $\mathcal{M}_{\rm PC}(\boldsymbol{\nu})^\circ$ is non-empty by virtue of the assumption in the proposition, and so is $\mathcal{M}^e_{\rm FC}(\boldsymbol{\nu})^\circ$. Consider the complex

$$\mathcal{D}^{\rm par}_\bullet \ : \ \mathrm{ad}_\phi(E_G) \ \xrightarrow{\ [\nabla,\,\cdot\,]\ } \ \mathrm{ad}^n_\phi(E_G)\otimes K_X(D)$$

for $\{H_x\}_{x\in D}$. Here, $\mathrm{ad}_\phi(E_G)$ and $\mathrm{ad}^n_\phi(E_G)\otimes K_X(D)$ are defined as in (2.16). The tangent space of $\mathcal{M}_{\rm PC}(\boldsymbol{\nu})^\circ$ at $(E,\,\nabla,\,\boldsymbol{l})$ is $\mathbb{H}^1(\mathcal{D}^{\rm par}_\bullet)$. There is also a natural morphism

$$p^{\rm par}_0 \ : \ \mathcal{M}_{\rm PC}(\boldsymbol{\nu})^\circ \ \longrightarrow \ \mathcal{N}_{\rm par}(d)^\circ$$

which is étale locally an affine space bundle whose fiber is isomorphic to $H^0(X,\,\mathrm{ad}^n_\phi(E_G)\otimes K_X(D))$. So there is a non-empty analytic open subset $U \subset \mathcal{N}^{\rm par}(d)^\circ$ with a local section

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$s^{\mathrm{par}} : U \longrightarrow (p_0^{\mathrm{par}})^{-1}(U)$ of p_0^{par}. Consider the commutative diagram

$$\begin{array}{ccccc} \mathcal{M}_{\mathrm{PC}}(\boldsymbol{\nu})^\circ & \xleftarrow{q_1^{\mathrm{par}}} & \mathcal{M}^e_{\mathrm{FC}}(\boldsymbol{\nu})^\circ & \underset{\subset}{\xrightarrow{\iota}} & \mathcal{M}^e_{\mathrm{FC}}(d)^\circ \\ {\scriptstyle p_0^{\mathrm{par}}}\downarrow & & \downarrow{\scriptstyle p_1|_{\mathcal{M}^e_{\mathrm{FC}}(\nu)^\circ}} & & \\ \mathcal{N}^{\mathrm{par}}(d)^\circ & \xleftarrow{q_B} & \mathcal{N}^e(d)^\circ & & \end{array}$$

whose left square is Cartesian. The local section s^{par} of p_0^{par} produces a local section

$$s_1 : q_B^{-1}(U) \longrightarrow (q_B \circ p_1)^{-1}(U)$$

of $p_1|_{\mathcal{M}^e_{\mathrm{FC}}(\boldsymbol{\nu})^\circ}$. Let Θ_{par} be the symplectic structure on $\mathcal{M}^{\boldsymbol{\alpha}}_{\mathrm{PC}}(\boldsymbol{\nu})$ constructed in [Ina]. This symplectic form Θ_{par} is described as follows:

$$\begin{aligned} \Theta_{\mathrm{par}} : \mathbb{H}^1(\mathcal{D}^{\mathrm{par}}_\bullet) \otimes \mathbb{H}^1(\mathcal{D}^{\mathrm{par}}_\bullet) &\longrightarrow \mathbb{H}^2(\mathbb{K}) \cong \mathbb{C} \\ [(\{u_{\alpha\beta}\}, \{v_\alpha\})] \otimes [(\{u'_{\alpha\beta}\}, \{v'_\alpha\})] &\longmapsto [(\{\mathrm{Tr}(u_{\alpha\beta}u'_{\beta\gamma})\}, -\{\mathrm{Tr}(u_{\alpha\beta}v'_\beta) - \mathrm{Tr}(v_\alpha u'_{\alpha\beta})\})] \end{aligned} \tag{4.18}$$

in terms of the Čech cohomology constructed using an affine open covering $\{U_\alpha\}$ (see [Ina, Proposition 7.2]). The symplectic form Θ_{par} is d-closed [Ina, Proposition 7.3]. Since the images of Θ^e and Θ_{par} in $\mathbb{H}^2(\mathbb{K})$ have the same description in terms of Čech cohomology (see (4.4) and (4.18)), it follows that

$$(q_1^{\mathrm{par}})^*\Theta_{\mathrm{par}}|_{\mathcal{M}_{\mathrm{PC}}(\boldsymbol{\nu})^\circ} = \iota^*\Theta^e|_{\mathcal{M}^e_{\mathrm{FC}}(d)^\circ}.$$

Since Θ^{par} is d-closed, so is $\iota^*\Theta^e|_{\mathcal{M}^e_{\mathrm{FC}}(d)^\circ}$. Set $s := \iota \circ s_1 : U \longrightarrow p_1^{-1}(U)$, which is a local section of U_1. Then the pullback $s^*(\Theta^e) = s_1^*\iota^*\Theta^e|_{\mathcal{M}^e_{\mathrm{FC}}(d)^\circ}$ is d-closed and so is $\Theta^e|_{\mathcal{M}^e_{\mathrm{FC}}(d)^\circ}$ by the first remark in this proof. □

4.4 Symplectic structure on $\mathcal{M}^e_{\mathrm{FC}}(d)$

In § 4.2,a 2-form Θ^e on $\mathcal{M}^e_{\mathrm{FC}}(d)$ was constructed. In the previous section, we considered the restriction of Θ^e on $\mathcal{M}^e_{\mathrm{FC}}(d)^\circ \subset \mathcal{M}^e_{\mathrm{FC}}(d)$. It was shown that this restriction is a symplectic form. Note that in the proof of the d-closedness of this restriction, we used irreducibility of $\mathcal{M}^e_{\mathrm{FC}}(d)^\circ$ (Proposition 3.8) implicitly. In this section, we shall show that the 2-form Θ^e on $\mathcal{M}^e_{\mathrm{FC}}(d)$ is a symplectic form. In the proof of the d-closedness of Θ^e, we will use the d-closedness of $\Theta^e|_{\mathcal{M}^e_{\mathrm{FC}}(d)^\circ}$ on $\mathcal{M}^e_{\mathrm{FC}}(d)^\circ$ for another effective divisor $\widetilde{D}$, instead of any argument on the irreducibility of $\mathcal{M}^e_{\mathrm{FC}}(d)$.

Proposition 4.18. *The 2-form Θ^e on $\mathcal{M}^e_{\mathrm{FC}}(d)$ is nondegenerate.*

Proof. Recall that the 2-form Θ^e is defined in (4.4). Let $\xi_{\Theta^H} : \mathbb{H}^1(\mathcal{C}_\bullet) \longrightarrow \mathbb{H}^1(\mathcal{C}_\bullet)^*$ be the homomorphism induced by Θ^H. Set $\mathcal{C}_0 := \mathcal{E}nd(E)(-D)$ and $\mathcal{C}_1 := \mathcal{E}nd(E) \otimes K_X(D)$. For the above defined map ξ_{Θ^H}, we have the following commutative diagram whose rows are exact:

$$\begin{array}{ccccccccc} H^0(\mathcal{C}_0) & \longrightarrow & H^0(\mathcal{C}_1) & \longrightarrow & \mathbb{H}^1(\mathcal{C}_\bullet) & \longrightarrow & H^1(\mathcal{C}_0) & \longrightarrow & H^1(\mathcal{C}_1) \\ \big\downarrow{\scriptstyle b_1} & & \big\downarrow{\scriptstyle b_2} & & \big\downarrow{\scriptstyle \xi_{\Theta^H}} & & \big\downarrow{\scriptstyle b_3} & & \big\downarrow{\scriptstyle b_4} \\ H^1(\mathcal{C}_1)^* & \longrightarrow & H^1(\mathcal{C}_0)^* & \longrightarrow & \mathbb{H}^1(\mathcal{C}_\bullet)^* & \longrightarrow & H^0(\mathcal{C}_1)^* & \longrightarrow & H^0(\mathcal{C}_0)^* \end{array}$$

where b_1, b_2, b_3, b_4 are Serre duality isomorphisms. So, from the five lemma, it follows that ξ_{Θ^H} is an isomorphism. In other words, the 2-form Θ^H is nondegenerate. □

Next, we shall investigate the d-closedness of the 2-form Θ^e on $\mathcal{M}^e_{\mathrm{FC}}(d)$.

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LEMMA 4.19. *Let $(E_0,\, \phi_0,\, \nabla_0)$ be a point on $\mathcal{M}^e_{\mathrm{FC}}(d)$. For this point on $\mathcal{M}^e_{\mathrm{FC}}(d)$, there exist a reduced effective divisor $\widetilde{D}$ and an isomorphism $\widetilde{\phi}_0 :\, \mathcal{O}_{\widetilde{D}}^{\oplus r} \longrightarrow E_0|_{\widetilde{D}}$ such that $\widetilde{D} \supset D$, $\widetilde{\phi}_0|_D = \phi_0$ and $(E_0,\, \widetilde{\phi}_0)$ is simple.*

Proof. Take a reduced effective divisor $\widetilde{D}$ such that $\widetilde{D} \supset D$ and $H^0(X,\, \mathcal{E}nd(E_0)(-\widetilde{D})) = 0$. Moreover, take an isomorphism $\widetilde{\phi}_0 :\, \mathcal{O}_{\widetilde{D}}^{\oplus r} \longrightarrow E_0|_{\widetilde{D}}$ such that $\widetilde{\phi}_0|_D = \phi_0$. We will show that $(E_0,\, \widetilde{\phi}_0)$ is simple. For that, let $\mathbf{g}$ be an automorphism of $(E_0,\, \widetilde{\phi}_0)$, that is, $\mathbf{g}$ is an automorphism of E_0 such that the diagram

$$\begin{array}{ccc} \mathcal{O}_{\widetilde{D}}^{\oplus r} & \xrightarrow{\ \widetilde{\phi}_0\ } & E_0|_{\widetilde{D}} \\ & \searrow^{\widetilde{\phi}_0} & \downarrow{\scriptstyle \mathbf{g}|_{\widetilde{D}}} \\ & & E_0|_{\widetilde{D}} \end{array}$$

is commutative. So the restriction $\mathbf{g}|_{\widetilde{D}}$ is the identity map. Therefore, we have

$$\mathbf{g} - \mathrm{Id}_{E_0} \in H^0(X,\, \mathcal{E}nd(E_0)(-\widetilde{D})).$$

Since $H^0(X,\, \mathcal{E}nd(E_0)(-\widetilde{D})) = 0$, it follows that $\mathbf{g} = \mathrm{Id}_{E_0}$. In other words, $(E_0,\, \widetilde{\phi}_0)$ is simple. □

Take an open covering

$$\mathcal{M}^e_{\mathrm{FC}}(d) = \bigcup_{m_0} \Sigma^d_{m_0}, \tag{4.19}$$

where each $\Sigma^d_{m_0}$ is the open substack of $\mathcal{M}^e_{\mathrm{FC}}(d)$ defined in (3.1). Recall that a very ample line bundle $\mathcal{O}_X(1)$ on the curve X is fixed; set $\theta_d(m) := rd_X m + d + r(1-g)$, where $d_X := \deg \mathcal{O}_X(1)$ and g is the genus of X. The above open substack $\Sigma^d_{m_0}$ is the fibered category whose objects are simple framed $\mathrm{GL}(r, \mathbb{C})$–connections $(E,\, \phi,\, \nabla)$ on $X \times S$ such that:

(a) $H^1(X,\, E_s(m_0 - 1)) = 0$ for each $s \in S$;

(b) $\chi(E_s(m)) = \theta_d(m)$ for each $s \in S$ and all $m \in \mathbb{Z}$.

By the argument in the proof of Proposition 3.5, the substack $\Sigma^d_{m_0}$ is of finite type.

LEMMA 4.20. *There exists a reduced effective divisor $\widetilde{D} \supset D$ such that for any points*

$$(E,\, \phi,\, \nabla) \,\in\, \Sigma^d_{m_0}$$

there is an isomorphism $\widetilde{\phi} :\, \mathcal{O}_{\widetilde{D}}^{\oplus r} \longrightarrow E|_{\widetilde{D}}$ satisfying the conditions that $\widetilde{\phi}|_D = \phi$ and $(E,\, \widetilde{\phi})$ is simple.

Proof. Take a point $s_0 = (E,\, \phi,\, \nabla) \in \Sigma^d_{m_0}$. By Lemma 4.19, there exists a reduced effective divisor $\widetilde{D}_{s_0}$ together with an isomorphism $\widetilde{\phi} :\, \mathcal{O}_{\widetilde{D}_{s_0}}^{\oplus r} \longrightarrow E|_{\widetilde{D}_{s_0}}$ satisfying the following three conditions: $\widetilde{D}_{s_0} \supset D$, $\widetilde{\phi}|_D = \phi$ and $(E,\, \widetilde{\phi})$ is simple.

Take an open substack $U_s \subset \Sigma^d_{m_0}$, where $s_0 \in U_{s_0}$ and U_{s_0} is small enough, and take a universal family $(\widetilde{E},\, \psi,\, \widetilde{\nabla})$ over $X \times U_{s_0}$. Since $\widetilde{E}$ is locally trivial, we may take a lift $\widetilde{\psi} :\, \mathcal{O}_{\widetilde{D}_{s_0} \times U_{s_0}}^{\oplus r} \longrightarrow \widetilde{E}|_{\widetilde{D}_{s_0} \times U_{s_0}}$ such that $\widetilde{\psi}|_{D \times U_{s_0}} = \psi$. Note that $(\widetilde{E},\, \widetilde{\psi})|_{X \times s_0} \cong (E,\, \widetilde{\phi})$, which is

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simple. Since we have the requirement that $H^0(X \times s,\, \mathcal{E}nd(\widetilde{E}|_{X\times s})(-\widetilde{D}_{s_0})) = 0$ is an open condition, we may assume that $(\widetilde{E},\, \widetilde{\psi})$ is a family of simple framed bundles. Consider an open covering $\Sigma^d_{m_0} = \bigcup_{s_0} U_{s_0}$. Since $\Sigma^d_{m_0}$ is of finite type, we may cover $\Sigma^d_{m_0}$ by a finite number of the open substacks $\{U_{s_0}\}_{s_0}$:

$$\Sigma^d_{m_0} = \bigcup_{i=1}^{m} U_{s_i},$$

where $s_1,\, \dots,\, s_m$ are points on $\Sigma^d_{m_0}$. Now take

$$\widetilde{D} := \bigcup_{i=1}^{m} \widetilde{D}_{s_i}.$$

Then, by the construction of $\widetilde{D}$, for any points $(E,\, \phi,\, \nabla) \in \Sigma^d_{m_0}$, there exists an isomorphism $\widetilde{\phi} : \mathcal{O}_{\widetilde{D}}^{\oplus r} \longrightarrow E|_{\widetilde{D}}$ such that $\widetilde{\phi}|_D = \phi$ and $(E,\, \widetilde{\phi})$ is simple. □

THEOREM 4.21. *The nondegenerate 2-form Θ^e on $\mathcal{M}^e_{\mathrm{FC}}(d)$ defined by (4.4) is d-closed.*

Proof. Consider the open covering $\mathcal{M}^e_{\mathrm{FC}}(d) = \bigcup_{m_0} \Sigma^d_{m_0}$ in (4.19). It is enough to prove that the restriction $\Theta^e|_{\Sigma^d_{m_0}}$ is d-closed for each m_0. Take a reduced effective divisor $\widetilde{D}$ as in Lemma 4.20. Let $\mathcal{M}^e_{\mathrm{FC}}(d,\, \widetilde{D})$ be the Deligne–Mumford stack constructed in Proposition 3.5 for $\widetilde{D}$. Let $\mathcal{M}^e_{\mathrm{FC}}(d, \widetilde{D})^\circ$ be the Deligne–Mumford stack whose objects are objects of $\mathcal{M}^e_{\mathrm{FC}}(d, \widetilde{D})$ such that the underlying framed bundles are simple. In other words, we have

$$\mathcal{M}^e_{\mathrm{FC}}(d, \widetilde{D})^\circ = \left\{ \left(\widetilde{E},\, \widetilde{\phi},\, \widetilde{\nabla}\right) \;\middle|\; \begin{array}{l} \widetilde{\mathrm{E}} \text{ is a vector bundle of degree d,} \\ \widetilde{\phi} : \mathcal{O}_{\widetilde{\mathrm{D}}}^{\oplus\, \mathrm{r}} \longrightarrow \mathrm{E}|_{\widetilde{\mathrm{D}}} \text{ is an isomorphism,} \\ \widetilde{\nabla} : \widetilde{\mathrm{E}} \longrightarrow \widetilde{\mathrm{E}} \otimes \mathrm{K_X}(\widetilde{\mathrm{D}}) \text{ is a connection, and} \\ (\widetilde{\mathrm{E}},\, \widetilde{\phi}\,) \text{ is simple} \end{array} \right\} \Big/ \sim_e.$$

Taking the degree of $\widetilde{D}$ to be sufficiently large, the canonical 2-form $\Theta^e|_{\mathcal{M}^e_{\mathrm{FC}}(d,\widetilde{D})^\circ}$ on $\mathcal{M}^e_{\mathrm{FC}}(d, \widetilde{D})^\circ$ is d-closed by Propositions 4.7 and 4.17. Define a moduli space $\mathcal{M}^e_{\mathrm{FC}}(d, \widetilde{D}, D)$ as follows:

$$\mathcal{M}^e_{\mathrm{FC}}(d, \widetilde{D}, D) = \left\{ \left(\widetilde{E},\, \widetilde{\phi},\, \widetilde{\nabla}\right) \in \mathcal{M}^e_{\mathrm{FC}}(d, \widetilde{D})^\circ \;\middle|\; \begin{array}{l} \widetilde{\nabla} \text{ is regular on } \widetilde{\mathrm{D}} \setminus \mathrm{D}, \text{ and} \\ (\widetilde{\mathrm{E}},\, \widetilde{\phi}\, |_{\mathrm{D}},\, \widetilde{\nabla}\,) \text{ is simple} \end{array} \right\} \Big/ \sim_e.$$

Let $\iota : \mathcal{M}^e_{\mathrm{FC}}(d, \widetilde{D}, D) \longrightarrow \mathcal{M}^e_{\mathrm{FC}}(d, \widetilde{D})^\circ$ be the natural inclusion map and π the natural map from $\mathcal{M}^e_{\mathrm{FC}}(d, \widetilde{D}, D)$ to $\mathcal{M}^e_{\mathrm{FC}}(d)$ induced by the restriction of framings to D:

$$\begin{aligned} \pi \;:\; \mathcal{M}^e_{\mathrm{FC}}(d, \widetilde{D}, D) &\longrightarrow \mathcal{M}^e_{\mathrm{FC}}(d) \\ (\widetilde{E},\, \widetilde{\phi},\, \widetilde{\nabla}) &\longmapsto (\widetilde{E},\, \widetilde{\phi}|_D,\, \widetilde{\nabla}). \end{aligned}$$

This map π is smooth. By Lemma 4.20, the open substack $\Sigma^d_{m_0}$ is contained in the image of π. We consider the following maps.

$$\begin{array}{ccc} & \mathcal{M}^e_{\mathrm{FC}}(d, \widetilde{D}, D) & \xrightarrow{\;\iota\;} \mathcal{M}^e_{\mathrm{FC}}(d, \widetilde{D})^\circ \\ & \Big\downarrow \pi & \\ \Sigma^d_{m_0} \xrightarrow{\;\subset\;} & \mathcal{M}^e_{\mathrm{FC}}(d) & \end{array}$$

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Let $\Theta^e_{\widetilde{D}}$ be the 2-form on $\mathcal{M}^e_{\mathrm{FC}}(d,\widetilde{D})^\circ$ defined in (4.4). By the definition of Θ^e and $\Theta^e_{\widetilde{D}}$, which are described by the same formula via the Čech cohomology, we have

$$\pi^*\Theta^e \;=\; \iota^*\Theta^e_{\widetilde{D}}.$$

As $\Theta^e_{\widetilde{D}}$ is d-closed by Propositions 4.7 and 4.17, we conclude that $\pi^*\Theta^e$ is d-closed. Since π is smooth, and the image of π contains the open substack $\Sigma^d_{m_0}$, it follows that $\Theta^e|_{\Sigma^d_{m_0}}$ is d-closed. □

4.5 Symplectic structure on $\mathcal{M}^H_{\mathrm{FC}}(d)$

Fix a complex Lie proper subgroup $H_x \subsetneq \mathrm{GL}(r,\mathbb{C})$ for each $x \in D$.

Consider the complexes $\mathcal{D}_\bullet$ and $\mathbb{K}$ constructed in (2.16) and (4.3) respectively. Note that the pairing $\mathrm{ad}(E_G)\otimes\mathrm{ad}(E_G) \longrightarrow \mathcal{O}_X$ in (2.5) produces a pairing

$$\mathrm{ad}_\phi(E_G)\otimes(\mathrm{ad}^n_\phi(E_G)\otimes K_X(D)) \longrightarrow K_X\,.$$

The restriction of the pairing $\widehat{\sigma}$ (see (2.5))

$$\mathrm{ad}_\phi(E_G)\otimes\mathrm{ad}_\phi(E_G) \longrightarrow \mathcal{O}_X$$

and the homomorphism

$$(\mathrm{ad}_\phi(E_G)\otimes(\mathrm{ad}^n_\phi(E_G)\otimes K_X(D)))\oplus((\mathrm{ad}^n_\phi(E_G)\otimes K_X(D))\otimes\mathrm{ad}_\phi(E_G)) \longrightarrow K_X$$

constructed using $\widehat{\sigma}$, together produce a homomorphism

$$\mathcal{D}_\bullet\otimes\mathcal{D}_\bullet \longrightarrow \mathbb{K}$$

of complexes. Let

$$\mathbb{H}^2(\mathcal{D}_\bullet\otimes\mathcal{D}_\bullet) \longrightarrow \mathbb{H}^2(\mathbb{K})$$

be the homomorphism of hypercohomologies induced by this homomorphism of complexes. Now, the composition of the natural homomorphism

$$\mathbb{H}^1(\mathcal{D}_\bullet)\otimes\mathbb{H}^1(\mathcal{D}_\bullet) \longrightarrow \mathbb{H}^2(\mathcal{D}_\bullet\otimes\mathcal{D}_\bullet)$$

with the above homomorphism of hypercohomologies produces a pairing

$$\Theta^H\,:\,\mathbb{H}^1(\mathcal{D}_\bullet)\otimes\mathbb{H}^1(\mathcal{D}_\bullet) \longrightarrow \mathbb{H}^2(\mathbb{K}) \,=\, \mathbb{C}\,. \tag{4.20}$$

In terms of the Čech cohomology with respect to an affine open covering $\{U_\alpha\}$, the pairing Θ^H in (4.20) is of the form

$$[(\{u_{\alpha\beta}\},\,\{v_\alpha\})]\otimes[(\{u'_{\alpha\beta}\},\,\{v'_\alpha\})] \longmapsto [(\{\mathrm{Tr}(u_{\alpha\beta}u'_{\beta\gamma})\},\,-\{\mathrm{Tr}(u_{\alpha\beta}v'_\beta)-\mathrm{Tr}(v_\alpha u'_{\alpha\beta})\})]\,.$$

This pairing in (4.20) gives a 2-form on $\mathcal{M}^H_{\mathrm{FC}}(d)$. We also denote by Θ^H this 2-form on $\mathcal{M}^H_{\mathrm{FC}}(d)$. Then Θ^H is nondegenerate by the argument as after [BIKS, Theorem 5] by applying [BLP2, Proposition 4.1]. Now it will be shown that Θ^H is d-closed.

Definition 4.22. Let $\mathcal{M}^e_{\mathrm{FC}}(d)_{\mathfrak{h}^\perp}$ be the stack over the category of locally Noetherian schemes whose objects are quadruples $(S,\,E,\,\phi=\{\phi_{x\times S}\}_{x\in D},\,\nabla)$ that satisfy (1), (3) and (5) in Definition 3.1 and the following $(2)''$ and $(4)''$.

$(2)''$ $\phi_{x\times S}$ be a section of the structure map

$$\mathrm{Isom}_S(\mathcal{O}^{\oplus r}_{x\times S},\,E|_{x\times S}) \longrightarrow x\times S\,.$$

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Denote by

$$\varphi_{x\times S} \,:\, \mathcal{O}_{x\times S}^{\oplus r} \xrightarrow{\sim} E|_{x\times S}$$

the isomorphism given by the map $x \times S \longrightarrow \mathrm{Isom}_S(\mathcal{O}_{x\times S}^{\oplus r},\, E|_{x\times S})$.

(4)″ *Let* $\mathrm{res}_{x\times S}(\nabla) \in \mathrm{End}(E)|_{x\times S}$ *be the residue matrix of the connection* ∇ *along* $x \times S$. *Then* $\phi_{x\times S}^{-1} \circ \mathrm{res}_{x\times S}(\nabla) \circ \phi_{x\times S} \in \mathfrak{h}^{\perp} \otimes \mathcal{O}_S$.

A morphism

$$(S,\, E,\, \phi,\, \nabla) \longrightarrow (S',\, E',\, \phi',\, \nabla')$$

in $\mathcal{M}_{\mathrm{FC}}^{e}(d)_{\mathfrak{h}^{\perp}}$ *is a Cartesian square*

$$\begin{array}{ccc} E & \xrightarrow{\sigma} & E' \\ \downarrow & & \downarrow \\ S & \xrightarrow{\widetilde{\sigma}} & S' \end{array}$$

such that the diagram

$$\begin{array}{ccc} E & \xrightarrow{\nabla} & E \otimes K_X(D) \\ \cong\downarrow\sigma & & \cong\downarrow\sigma \\ E' & \xrightarrow{\nabla'} & E' \otimes K_X(D) \end{array}$$

commutes and the composition $(\phi'_{x\times S})^{-1} \circ \sigma|_{x\times S} \circ \phi_{x\times S}$ *coincides with the identity map of* $\mathcal{O}_{x\times S}^{\oplus r}$ *for each* $x \in D$.

Theorem 4.23. *The nondegenerate* 2*-form* Θ^H *on* $\mathcal{M}_{\mathrm{FC}}^{H}(d)$ *defined by (4.20) is* d*-closed.*

Proof. Consider the diagram

$$\begin{array}{ccc} \mathcal{M}_{\mathrm{FC}}^{e}(d)_{\mathfrak{h}^{\perp}} & \xrightarrow{\pi_1} & \mathcal{M}_{\mathrm{FC}}^{e}(d) \\ \pi_2\downarrow & & \\ \mathcal{M}_{\mathrm{FC}}^{H}(d) & & \end{array}$$

where π_1 and π_2 are the natural maps. It is straightforward to check that

$$\pi_1^*\Theta^e \;=\; \pi_2^*\Theta^H.$$

Since Θ^e is d-closed, the form $\pi_2^*\Theta^H$ is also d-closed. This implies that Θ^H is d-closed, because the map π_2 is dominant. □

4.6 Poisson structure

In this subsection, we will see the details of the Poisson structure mentioned in the introduction. This is influenced by a construction done in [BBG].

Let $\mathcal{M}_{\mathrm{C}}(d)$ be the moduli space of pairs $(E,\, \nabla)$, where E is a holomorphic vector bundle on X of rank r and degree d, and ∇ is a logarithmic connection on E whose singular part is contained in D, such that $(E,\, \nabla)$ is simple in the sense that the endomorphisms of E preserving ∇ are just the constant scalar multiplications. In [Nit], Nitsure constructed the moduli space $\mathcal{M}_{\mathrm{C}}^{\mathrm{ss}}(d)$ of semistable logarithmic connections, which contains the moduli space of stable logarithmic

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connections $\mathcal{M}^{\mathrm{s}}_{\mathrm{C}}(d)$ as a Zariski open subset. By its definition, our moduli space $\mathcal{M}_{\mathrm{C}}(d)$ contains $\mathcal{M}^{\mathrm{s}}_{\mathrm{C}}(d)$ as a Zariski open subspace. Recall that a description of the tangent space of this moduli space is given in [Nit]. For $(E,\,\nabla) \in \mathcal{M}_{\mathrm{C}}(d)$, the tangent space of $\mathcal{M}_{\mathrm{C}}(d)$ at (E, ∇) is

$$T_{(E,\nabla)}\mathcal{M}_{\mathrm{C}}(d) \;=\; \mathbb{H}^1(\mathcal{E}nd(E) \to \mathcal{C}_1),$$

where $\mathcal{C}_0 = \mathcal{E}nd(E)(-D)$, $\mathcal{C}_1 = \mathcal{E}nd(E) \otimes K_X(D)$ and the map $\mathcal{E}nd(E) \longrightarrow \mathcal{C}_1$ is defined by $u \longmapsto \nabla \circ u - u \circ \nabla$. The cotangent space is

$$T^*_{(E,\nabla)}\mathcal{M}_{\mathrm{C}}(d) \;=\; \mathbb{H}^1(\mathcal{E}nd(E) \to \mathcal{C}_1)^* \;\cong\; \mathbb{H}^1(\mathcal{C}_0 \to \mathcal{E}nd(E) \otimes K_X),$$

over which there is a canonical pairing

$$\begin{aligned} &T^*_{(E,\nabla)}\mathcal{M}_{\mathrm{C}}(d) \otimes T^*_{(E,\nabla)}\mathcal{M}_{\mathrm{C}}(d) \\ &= \mathbb{H}^1(\mathcal{C}_0 \to \mathcal{E}nd(E) \otimes K_X) \otimes \mathbb{H}^1(\mathcal{C}_0 \to \mathcal{E}nd(E) \otimes K_X) \longrightarrow \mathbb{H}^2(\Omega^\bullet_X) \cong \mathbb{C}. \end{aligned} \tag{4.21}$$

Consider the open subspace

$$\mathcal{M}^e{}_{\mathrm{FC}}(d)' \;=\; \{(E,\,\nabla,\,\phi) \in \mathcal{M}^e{}_{\mathrm{FC}}(d) \mid (E,\,\nabla) \text{ is simple}\}$$

of the moduli space $\mathcal{M}^e{}_{\mathrm{FC}}(d)$ of simple framed connections. Then, there is a natural forgetful map

$$\pi \;:\; \mathcal{M}^e{}_{\mathrm{FC}}(d)' \longrightarrow \mathcal{M}_{\mathrm{C}}(d), \tag{4.22}$$

and the induced map π^* on the cotangent spaces makes the diagram

$$\begin{array}{ccc} T^*_{(E,\nabla)}\mathcal{M}_{\mathrm{C}}(d) \times T^*_{(E,\nabla)}\mathcal{M}_{\mathrm{C}}(d) & \longrightarrow & \mathbb{H}^2(\Omega^\bullet_X) \cong \mathbb{C} \\ {\scriptstyle \pi^*\times\pi^*}\downarrow & & \downarrow \\ T^*_{(E,\nabla,\phi)}\mathcal{M}^e_{\mathrm{FC}}(d) \times T^*_{(E,\nabla,\phi)}\mathcal{M}^e_{\mathrm{FC}}(d) & \longrightarrow & \mathbb{H}^2(\Omega^\bullet_X) \cong \mathbb{C} \end{array}$$

commutative. The bottom horizontal arrow satisfies the Jacobi identity, because it corresponds to the symplectic form on the moduli space $\mathcal{M}^e{}_{\mathrm{FC}}(d)$ given in Theorem 4.21. So the pairing in (4.21) is also skew-symmetric and satisfies the Jacobi identity. Thus, the following corollary is obtained.

Corollary 4.24. *The moduli space $\mathcal{M}_{\mathrm{C}}(d)$ has a Poisson structure defined by the Poisson bracket in (4.21). Furthermore, the morphism π in (4.22) becomes a Poisson map.*

We will see a slightly different view of the Poisson structure on the moduli space $\mathcal{M}_{\mathrm{C}}(d)$. Set

$$A \;:=\; \left\{ \boldsymbol{a} \;=\; (a^{(i)}_j)^{1\leqslant i\leqslant n}_{0\leqslant j\leqslant r-1} \;\middle|\; \sum_i a^{(i)}_{r-1} \;=\; d \right\}.$$

By associating the coefficients of the characteristic polynomial of $\mathsf{res}_{x_i}(\nabla)$ at each point $x_i \in D$, we can define a morphism

$$\mathcal{M}_{\;\mathrm{C}}(d) \longrightarrow A \tag{4.23}$$

whose fiber $\mathcal{M}_{\;\mathrm{C}}(\boldsymbol{a})$ over $\boldsymbol{a} \in A$ is smooth for generic $\boldsymbol{a}$ but has singularities for special $\boldsymbol{a}$. Consider the moduli space of simple parabolic connections

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$$\mathcal{M}_{\mathrm{PC}}(d) = \left\{ (E,\, \nabla,\, \boldsymbol{l}) \;\middle|\; \begin{array}{l} (E,\, \boldsymbol{l} = (l_j^{(i)})) is\ a\ quasi-parabolic\ bundle\ of\ rank\ r\ and\ degree\ d, \\ \nabla : E \longrightarrow E \otimes K_X(D)\ is\ a\ connection\ satisfying \\ \mathsf{res}_{x_i}(\nabla\ )(l_j^{(i)}) \subset l_j^{(i)}\ for\ any\ i,\, j\ and\ (E,\, \nabla,\, \boldsymbol{l})\ is\ simple. \end{array} \right\} \Big/ \sim .$$

For the open subspace

$$\mathcal{M}_{\mathrm{PC}}(d)' = \{ (E,\, \nabla,\, \boldsymbol{l}) \in \mathcal{M}_{\mathrm{PC}}(d) \mid (E,\, \nabla) \text{ is simple} \}$$

of $\mathcal{M}_{\mathrm{PC}}(d)$, there is a canonical morphism

$$\mathcal{M}_{\mathrm{PC}}(d)' \longrightarrow \mathcal{M}_{\mathrm{C}}(d) \tag{4.24}$$

which is generically finite. Set $\Lambda := \left\{ (\nu_j^{(i)})_{0 \leqslant j \leqslant r-1}^{1 \leqslant i \leqslant n} \in \mathbb{C}^{nr} \;\middle|\; d + \sum_{i,j} \nu_j^{(i)} = 0 \right\}$. Then we have a smooth morphism

$$\mathcal{M}_{\mathrm{PC}}(d) \longrightarrow \Lambda \tag{4.25}$$

whose fiber over any $\boldsymbol{\nu} \in \Lambda$ is the moduli space $\mathcal{M}_{\mathrm{PC}}(\boldsymbol{\nu})$ of $\boldsymbol{\nu}$-parabolic connections. The morphism in (4.24) induces a map between the fibers of (4.23) and (4.25)

$$\mathcal{M}_{\mathrm{PC}}(\boldsymbol{\nu})' := \mathcal{M}_{\mathrm{PC}}(\boldsymbol{\nu}) \cap \mathcal{M}_{\mathrm{PC}}(d)' \longrightarrow \mathcal{M}_{\mathrm{C}}(\boldsymbol{a})$$

which is an isomorphism for generic $\boldsymbol{a}$ and a resolution of singularities of $\mathcal{M}_{\mathrm{C}}(\boldsymbol{a})$ for special $\boldsymbol{a}$, where $\boldsymbol{a} = (a_j^{(i)})$ is determined by $\boldsymbol{\nu} = (\nu_j^{(i)})$ as follows:

$$\prod_{j=0}^{r-1} (t - \nu_j^{(i)}) = t^r + a_{r-1}^{(i)} t^{r-1} + \cdots + a_1^{(i)} t + a_0^{(i)}.$$

Roughly speaking, the moduli space $\mathcal{M}_{\mathrm{C}}(\boldsymbol{a})$ for special $\boldsymbol{a}$ gives a partial resolution of singularities of the corresponding character variety, which we will define precisely later in (5.24). The meaning of the singularities of character varieties and their exceptional loci in the moduli space $\mathcal{M}_{\mathrm{PC}}(\boldsymbol{\nu})$ (or precisely, $\mathcal{M}^{\boldsymbol{\alpha}}_{\mathrm{PC}}(\boldsymbol{\nu})$) is explained in [Iw2] and [IIS] from the viewpoint of the isomonodromic deformation, and their classification in the case of Painlevé equations is given in [SaTe].

Setting

$$\widetilde{\mathcal{D}}_0^{\mathrm{par}} = \left\{ u \in \mathcal{E}nd(E) \;\middle|\; \mathrm{u}|_{\mathrm{x_i}}(\mathrm{l_j^{(i)}}) \subset \mathrm{l_{j+1}^{(i)}} \text{ for any } \mathrm{i},\, \mathrm{j} \right\} \subset \mathcal{D}_0^{\mathrm{par}}$$

$$\widetilde{\mathcal{D}}_1^{\mathrm{par}} = \left\{ v \in \mathcal{E}nd(E) \otimes K_X(D) \;\middle|\; \mathsf{res}_{\mathrm{x_i}}(\mathrm{v})(\mathrm{l_j^{(i)}}) \subset \mathrm{l_j^{(i)}} \text{ for any } \mathrm{i},\, \mathrm{j} \right\},$$

we can define a complex $\mathcal{D}_0^{\mathrm{par}} \longrightarrow \widetilde{\mathcal{D}}_1^{\mathrm{par}}$, $u \longmapsto \nabla \circ u - u \circ \nabla$, which induces complexes $\widetilde{\mathcal{D}}_0^{\mathrm{par}} \longrightarrow \mathcal{D}_1^{\mathrm{par}}$ and $\widetilde{\mathcal{D}}_0^{\mathrm{par}} \longrightarrow \widetilde{\mathcal{D}}_1^{\mathrm{par}}$. The tangent space of the moduli space $\mathcal{M}_{\mathrm{PC}}(d)$ is $T\mathcal{M}_{\mathrm{PC}}(d) = \mathbb{H}^1(\mathcal{D}_0^{\mathrm{par}} \to \widetilde{\mathcal{D}}_1^{\mathrm{par}})$ and the cotangent space is its dual

$$T^*\mathcal{M}_{\mathrm{PC}}(d) = \mathbb{H}^1(\mathcal{D}_0^{\mathrm{par}} \to \widetilde{\mathcal{D}}_1^{\mathrm{par}})^* \cong \mathbb{H}^1(\widetilde{\mathcal{D}}_0^{\mathrm{par}} \to \mathcal{D}_1^{\mathrm{par}}).$$

So we can define a canonical pairing

$$(T^*\mathcal{M}_{\mathrm{PC}}(d)) \otimes (T^*\mathcal{M}_{\mathrm{PC}}(d)) = \mathbb{H}^1(\widetilde{\mathcal{D}}_0^{\mathrm{par}} \to \mathcal{D}_1^{\mathrm{par}}) \otimes \mathbb{H}^1(\widetilde{\mathcal{D}}_0^{\mathrm{par}} \to \mathcal{D}_1^{\mathrm{par}}) \longrightarrow \mathbb{H}^2(\Omega_X^{\bullet}) \cong \mathbb{C}. \tag{4.26}$$

Let B be the Borel subgroup of $\mathrm{GL}(r,\mathbb{C})$ consisting of upper triangular matrices, and let U be the subgroup of B consisting of matrices whose diagonal entries are 1. Consider the open subspace

$$\mathcal{M}^{U}_{\mathrm{FC}}(d)' = \left\{ (E,\ \nabla,\ [\phi]) \in \mathcal{M}^{U}_{\mathrm{FC}}(d) \;\middle|\; \begin{array}{l} \text{the parabolic connection } (\mathrm{E},\ \nabla\ ,\ \boldsymbol{l}) \\ \text{induced from } (\mathrm{E},\ \nabla\ ,\ [\phi\ ]) \text{ is simple} \end{array} \right\}$$

of $\mathcal{M}^{U}_{\mathrm{FC}}(d)$, which is the moduli space of framed connections in Definition 3.1 with $H = U$. Associating the corresponding parabolic connection, we can define a morphism

$$\mathcal{M}^{U}_{\mathrm{FC}}(d)' \longrightarrow \mathcal{M}_{\ \mathrm{PC}}(d) \tag{4.27}$$

which becomes a $(\prod_D B/U)/\mathbb{C}^*$-bundle. By construction, the diagram

$$\begin{array}{ccccc} (T^*\mathcal{M}_{\mathrm{PC}}(d)) \otimes (T^*\mathcal{M}_{\mathrm{PC}}(d)) & = & \mathbb{H}^1(\widetilde{\mathcal{D}}^{\mathrm{par}}_0 \to \mathcal{D}^{\mathrm{par}}_1) \otimes \mathbb{H}^1(\widetilde{\mathcal{D}}^{\mathrm{par}}_0 \to \mathcal{D}^{\mathrm{par}}_1) & \longrightarrow & \mathbb{H}^2(\Omega^\bullet_X) \cong \mathbb{C} \\ & & \downarrow & & \downarrow \\ (T^*\mathcal{M}^{U}_{\mathrm{FC}}(d)') \otimes (T^*\mathcal{M}^{U}_{\mathrm{FC}}(d)') & = & \mathbb{H}^1(\widetilde{\mathcal{D}}^{\mathrm{par}}_0 \to \widetilde{\mathcal{D}}^{\mathrm{par}}_1) \otimes \mathbb{H}^1(\widetilde{\mathcal{D}}^{\mathrm{par}}_0 \to \widetilde{\mathcal{D}}^{\mathrm{par}}_1) & \longrightarrow & \mathbb{H}^2(\Omega^\bullet_X) \cong \mathbb{C} \end{array}$$

is commutative. The lower horizontal arrow is the Poisson bracket corresponding to the symplectic form on the moduli space $\mathcal{M}^{U}_{\mathrm{FC}}(d)$ given by Theorem 4.23. So the pairing in (4.26) defines a Poisson structure on the moduli space $\mathcal{M}_{\ \mathrm{PC}}(d)$, and the morphism in (4.27) is a Poisson map.

We can also see that the pairing in (4.26) commutes with the Poisson bracket on $\mathcal{M}_{\ \mathrm{PC}}(\boldsymbol{\nu})$ corresponding to the symplectic form. So the canonical inclusion $\mathcal{M}_{\ \mathrm{PC}}(\boldsymbol{\nu}) \hookrightarrow \mathcal{M}_{\ \mathrm{PC}}(d)$ is also a Poisson map.

The canonical map $\mathbb{H}^1(\mathcal{C}_0 \to \mathcal{E}nd(E) \otimes K_X) \longrightarrow \mathbb{H}^1(\widetilde{\mathcal{D}}_0^{\ \mathrm{par}} \to \mathcal{D}_1^{\ \mathrm{par}})$ coincides with the map

$$T^*\mathcal{M}_{\mathrm{C}}(d) \longrightarrow T^*\mathcal{M}_{\ \mathrm{PC}}(\boldsymbol{\nu})$$

on the cotangent spaces induced by the morphism in (4.24), which means that the Poisson bracket in (4.21) commutes with that in (4.26). Combining the above, the following corollary is obtained.

Corollary 4.25. *The moduli space $\mathcal{M}_{\ \mathrm{PC}}(d)$ of parabolic connections has a Poisson structure defined by the Poisson bracket given in (4.26). Furthermore, the morphism $\mathcal{M}_{\ \mathrm{PC}}(d)' \longrightarrow \mathcal{M}_{\mathrm{C}}(d)$ in (4.24) becomes a Poisson map for this Poisson structure.*

5. The moduli space of parabolic connections is not affine

5.1 Moduli space of parabolic connections and parabolic Higgs bundles

Throughout this section, we assume that k is an algebraically closed field of arbitrary characteristic.

Let

$$(X,\ \boldsymbol{x})\ :=\ (X,\ (x_1,\ \ldots,\ x_n))$$

be an n-pointed smooth projective curve of genus g over k, where $x_1,\ \ldots,\ x_n$ are distinct k-valued points of X. Denote the reduced divisor $x_1 + \cdots + x_n$ on X by D. Take a positive integer r which is not divisible by the characteristic of k, and take an integer d and an element

$$\boldsymbol{\nu}\ =\ (\nu^{(i)}_j)^{1 \leqslant i \leqslant n}_{0 \leqslant j \leqslant r-1} \in k^{nr}$$

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such that the equality $\sum_{i,j} \nu_j^{(i)} = -d$ holds in k. Take a collection of rational numbers

$$\boldsymbol{\alpha} = (\alpha_j^{(i)})_{1 \leqslant j \leqslant r}^{1 \leqslant i \leqslant n} \in \mathbb{Q}^{rn}$$

satisfying the conditions:

(i) $0 < \alpha_1^{(i)} < \cdots < \alpha_r^{(i)} < 1$;

(ii) $\alpha_j^{(i)} \neq \alpha_{j'}^{(i')}$ for $(i,\, j) \neq (i',\, j')$.

An $(\boldsymbol{x},\ \boldsymbol{\nu})$-parabolic connection on X is defined exactly in the same way as Definition 4.9. Although a parabolic connection includes the data of a parabolic weight, we omit it and simply write $(E,\, \nabla,\, \boldsymbol{l})$. The definition of $\boldsymbol{\alpha}$-stability of a parabolic connection is also defined in the same way as Definition 4.10.

In the proof of the existence of the moduli space of stable parabolic connections in [Ina, Theorem 2.2] , we used the embedding to the moduli space of parabolic Λ_D^1-triples ([IIS, Theorem 5.1]; this argument also works over a field of arbitrary characteristic. So we have the following theorem.

Theorem 5.1. *There exists a coarse moduli scheme $\mathcal{M}_{\mathrm{PC}}^{\boldsymbol{\alpha}}(\boldsymbol{\nu})$ of $\boldsymbol{\alpha}$-stable $(\boldsymbol{x},\, \boldsymbol{\nu})$-parabolic connections on a smooth projective curve X over k. Furthermore, $\mathcal{M}_{\mathrm{PC}}^{\boldsymbol{\alpha}}(\boldsymbol{\nu})$ is quasi-projective over k.*

Definition 5.2 [Mu, Lecture 14, page 99]. Let Y be a projective variety over k, and let $\mathcal{O}_Y(1)$ be a very ample line bundle on Y. Take an integer n_0. A coherent sheaf E on Y is called n_0-regular if

$$H^i(Y,\ E \otimes \mathcal{O}_Y(n_0 - i)) \;=\; 0$$

holds for all $i > 0$.

We will denote $E \otimes \mathcal{O}_Y(m)$ by $E(m)$ for an integer m.

Definition 5.3. Let Y be a projective variety over k. A set $\mathcal{T}$ of coherent sheaves on Y is called bounded if there is a scheme S of finite type over k, and a coherent sheaf $\mathcal{E}$ on $Y \times S$ such that for any member $E \in \mathcal{T}$, there is a k-valued point $s \in S$ such that $\mathcal{E}|_{Y \times \{s\}} \cong E$.

The following lemma is a useful tool to show the boundedness of a family of coherent sheaves.

Lemma 5.4 [Kl, Theorem 1.13].*Let Y be a projective variety over k, and let $\mathcal{O}_Y(1)$ be a very ample line bundle on Y. Then a set $\mathcal{T}$ of coherent sheaves on Y is bounded if and only if there is an integer n_0 such that all the members of $\mathcal{T}$ are n_0-regular and the set*

$$\left\{ \chi(E(m)) = \sum_i (-1)^i \dim H^i(X,\ E(m)) \;\middle|\; E \in \mathcal{T} \right\}$$

of Hilbert polynomials $\chi(E(m))$ in m of the members E of $\mathcal{T}$ is finite.

In the same way as Proposition 4.16, the moduli space of simple $(\boldsymbol{x},\, \boldsymbol{\nu})$-parabolic connections $\mathcal{M}_{\mathrm{PC}}(\boldsymbol{\nu})$ is an algebraic space over k, and the moduli space $\mathcal{M}_{\mathrm{PC}}^{\boldsymbol{\alpha}}(\boldsymbol{\nu})$ of $\boldsymbol{\alpha}$-stable $(\boldsymbol{x},\, \boldsymbol{\nu})$-parabolic connections is a Zariski open subspace of $\mathcal{M}_{\mathrm{PC}}(\boldsymbol{\nu})$. Since $\mathcal{M}_{\mathrm{PC}}^{\boldsymbol{\alpha}}(\boldsymbol{\nu})$ is quasi-projective over k, we can take an integer n_0 such that for all $(E,\, \nabla,\, \boldsymbol{l}) \in \mathcal{M}_{\mathrm{PC}}^{\boldsymbol{\alpha}}(\boldsymbol{\nu})$, the underlying vector bundle E is n_0-regular.

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Fix a line bundle L on X and a logarithmic connection

$$\nabla_L : L \longrightarrow L \otimes K_X(D)$$

such that $\mathrm{res}_{x_i}(\nabla_L) = \sum_{j=0}^{r-1} \nu_j^{(i)}$ for all $1 \leqslant i \leqslant n$. Set

$$\mathcal{M}_{\mathrm{PC}}(\boldsymbol{\nu}, \nabla_L) := \{(E, \nabla, \boldsymbol{l}) \in \mathcal{M}_{\mathrm{PC}}(\boldsymbol{\nu}) \mid \det(E, \nabla) \cong (L, \nabla_L)\}\,, \tag{5.1}$$

$$\mathcal{M}^{\boldsymbol{\alpha}}_{\mathrm{PC}}(\boldsymbol{\nu}, \nabla_L) := \{(E, \nabla, \boldsymbol{l}) \in \mathcal{M}^{\boldsymbol{\alpha}}_{\mathrm{PC}}(\boldsymbol{\nu}) \mid \det(E, \nabla) \cong (L, \nabla_L)\}\,. \tag{5.2}$$

These are closed subspaces of $\mathcal{M}_{\mathrm{PC}}(\boldsymbol{\nu})$ and $\mathcal{M}^{\boldsymbol{\alpha}}_{\mathrm{PC}}(\boldsymbol{\nu})$, respectively. Setting

$$\mathcal{M}^{n_0\text{-reg}}_{\mathrm{PC}}(\boldsymbol{\nu}) := \{(E, \nabla, \boldsymbol{l}) \in \mathcal{M}_{\mathrm{PC}}(\boldsymbol{\nu}) \mid \mathrm{E \text{ is } n_0 - regular}\}\,,$$

there is a canonical open immersion

$$\iota : \mathcal{M}^{\boldsymbol{\alpha}}_{\mathrm{PC}}(\boldsymbol{\nu}) \hookrightarrow \mathcal{M}^{n_0\text{-reg}}_{\mathrm{PC}}(\boldsymbol{\nu}).$$

Set

$$\mathcal{M}^{n_0\text{-reg}}_{\mathrm{PC}}(\boldsymbol{\nu}, \nabla_L) := \left\{(E, \nabla, \boldsymbol{l}) \in \mathcal{M}^{n_0\text{-reg}}_{\mathrm{PC}}(\boldsymbol{\nu}) \mid \det(E, \nabla) \cong (L, \nabla_L)\right\}.$$

Then $\mathcal{M}^{n_0\text{-reg}}_{\mathrm{PC}}(\boldsymbol{\nu}, \nabla_L)$ is a closed subspace of $\mathcal{M}^{n_0\text{-reg}}_{\mathrm{PC}}(\boldsymbol{\nu})$ and it contains $\mathcal{M}^{\boldsymbol{\alpha}}_{\mathrm{PC}}(\boldsymbol{\nu}, \nabla_L)$ as a Zariski open subspace.

Under the assumption that the rank r is not divisible by the characteristic of k, the proof of the smoothness of the moduli space given in [Ina, Theorem 2.1] works because the assumption ensures that the Killing form on $\mathrm{sl}(r, k)$ remains nondegenerate. This is elaborated in the following proposition.

Proposition 5.5. *Assume that the rank r is not divisible by the characteristic of k. Then the moduli space $\mathcal{M}_{\mathrm{PC}}(\boldsymbol{\nu}, \nabla_L)$ is smooth over k, and so is its open subspace $\mathcal{M}^{\boldsymbol{\alpha}}_{\mathrm{PC}}(\boldsymbol{\nu}, \nabla_L)$.*

Proof. We use the criterion of smoothness in [Grot2, Proposition 17.14.2]. Let A be an Artinian local ring over k with the maximal ideal $\mathfrak{m}$, and let I be an ideal of A such that $\mathfrak{m}I = 0$. Suppose that we are given a morphism $\mathrm{Spec} A/I \longrightarrow \mathcal{M}_{\mathrm{PC}}(\boldsymbol{\nu}, \nabla_L)$ that corresponds to a flat family $(E, \nabla, \boldsymbol{l})$ of parabolic connections on $X \times \mathrm{Spec} A/I$ over A/I. It suffices to construct a flat family $(\widetilde{E}, \widetilde{\nabla}, \widetilde{\boldsymbol{l}})$ of $\boldsymbol{\nu}$-parabolic connections on $X \times \mathrm{Spec} A$ over $\mathrm{Spec} A$ that is a lift of $(E, \nabla, \boldsymbol{l})$.

There is an isomorphism $\varphi : \det E \xrightarrow{\sim} L \otimes A/I$ such that $(\nabla_L \otimes A/I) \circ \varphi = (\varphi \otimes \mathrm{id}) \circ \mathrm{Tr}(\nabla)$. Take an affine open covering $\{U_\alpha\}$ of X satisfying the condition that there is an isomorphism

$$\phi_\alpha : E|_{U_\alpha \times \mathrm{Spec} A/I} \xrightarrow{\sim} \mathcal{O}^{\oplus r}_{U_\alpha \times \mathrm{Spec} A/I}.$$

Set $\overline{\phi_\alpha} := \phi_\alpha \otimes A/\mathfrak{m}$ and $\overline{\varphi} := \varphi \otimes A/\mathfrak{m}$. After replacing ϕ_α with $(1 + r^{-1}a)\phi_\alpha$ for some $a \in I\mathcal{O}_{U_\alpha}$, we may assume that

$$\det(\phi_\alpha) \circ \varphi^{-1} = (\det(\overline{\phi_\alpha}) \circ \overline{\varphi}^{-1}) \otimes \mathrm{id}_{A/I}$$

as maps from $L \otimes A/I$ to $\mathcal{O}_{U_\alpha \times \mathrm{Spec} A/I}$. Set $E_\alpha := \mathcal{O}^{\oplus r}_{U_\alpha \times \mathrm{Spec} A}$ and put

$$\varphi_\alpha := (\overline{\varphi} \circ \det(\overline{\phi_\alpha})^{-1}) \otimes A : \det(E_\alpha) \xrightarrow{\sim} L \otimes \mathrm{id}_A.$$

Choose a lift

$$\theta_{\beta\alpha} : E_\alpha|_{U_{\alpha\beta} \times \mathrm{Spec} A} \xrightarrow{\sim} E_\beta|_{U_{\alpha\beta} \times \mathrm{Spec} A}$$

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of $\phi_\beta \circ \phi_\alpha^{-1}$. Replacing $\theta_{\beta\alpha}$ with $(1+r^{-1}b)\theta_{\beta\alpha}$ for some $b \in I\mathcal{O}_{U_{\alpha\beta}\times \operatorname{Spec}A}$, we may assume that the following equality holds

$$\det(\theta_{\beta\alpha}) \;=\; \varphi_\beta^{-1} \circ \varphi_\alpha$$

as maps from $\det(E_\alpha)|_{U_{\alpha\beta}\times \operatorname{Spec}A}$ to $\det(E_\beta)|_{U_{\alpha\beta}\times \operatorname{Spec}A}$. If $x_i \in U_\alpha$, then we take a quasi-parabolic structure l_*^α on E_α at $x_i \times \operatorname{Spec}A$ that is a lift of $l_*^{(i)}$. Take a relative connection

$$\nabla_\alpha \,:\, E_\alpha \longrightarrow E_\alpha \otimes \Omega_{X\times \operatorname{Spec}A/\operatorname{Spec}A}(D \times \operatorname{Spec}A)$$

such that $\nabla_\alpha \otimes A/I = \phi_\alpha \circ \nabla|_{U_\alpha \times \operatorname{Spec}A/I} \circ \phi_\alpha^{-1}$ and $(\operatorname{res}_{x_i\times \operatorname{Spec}A}(\nabla_\alpha) - \nu_j^{(i)})(l_j^\alpha) \subset l_{j+1}^\alpha$ for all $0 \leqslant j \leqslant r-1$. After replacing ∇_α with $\nabla_\alpha + r^{-1}\eta \otimes \mathrm{id}_{E_\alpha}$ for some $\eta \in I\,\Omega^1_{U_\alpha\times \operatorname{Spec}A/\operatorname{Spec}A}$, we may assume that $\varphi_\alpha \operatorname{Tr}(\nabla_\alpha)\varphi_\alpha^{-1} = \nabla_L \otimes \mathrm{id}_A$. Put $(\overline{E}, \overline{\nabla}, \bar{\boldsymbol{l}}) := (E, \nabla, \boldsymbol{l}) \otimes A/\mathfrak{m}$ and set

$$\begin{aligned}
\mathcal{D}^{\mathrm{par}}_{\mathfrak{sl},0} &= \left\{ u \in \mathcal{E}nd(\overline{E}) \,\middle|\, \operatorname{Tr}(u) = 0 \text{ and } \mathrm{res_{x_i}(u)(\bar{l}_j^{(i)}) \subset \bar{l}_j^{(i)} \text{ for any i, j}} \right\},\\
\mathcal{D}^{\mathrm{par}}_{\mathfrak{sl},1} &= \left\{ v \in \mathcal{E}nd(\overline{E}) \otimes K_X(D) \,\middle|\, \operatorname{Tr}(v) = 0 \text{ and } \mathrm{res_{x_i}(v)(\bar{l}_j^{(i)}) \subset \bar{l}_{j+1}^{(i)} \text{ for any i, j}} \right\},\\
\nabla_{\mathcal{D}^{\mathrm{par}}_{\mathfrak{sl},\bullet}} &: \mathcal{D}^{\mathrm{par}}_{\mathfrak{sl},0} \longrightarrow \mathcal{D}^{\mathrm{par}}_{\mathfrak{sl},1}, \quad u \longmapsto \overline{\nabla} \circ u - u \circ \overline{\nabla}.
\end{aligned}$$

Then we get a cohomology class $[\{\theta_{\gamma\alpha}^{-1}\theta_{\gamma\beta}\theta_{\beta\alpha} - \mathrm{id}\}, \{\theta_{\beta\alpha}^{-1} \circ \nabla_\beta \circ \theta_{\beta\alpha} - \nabla_\alpha\}] \in \mathbb{H}^2(\mathcal{D}^{\mathrm{par}}_{\mathfrak{sl},\bullet}) \otimes I$ whose vanishing is equivalent to the existence of a lift $(\widetilde{E}, \widetilde{\nabla}, \widetilde{\boldsymbol{l}}) \in \mathcal{M}_{\mathrm{PC}}(\boldsymbol{\nu}, \nabla_L)(A)$ of $(E, \nabla, \boldsymbol{l})$. There is a commutative diagram with exact rows

$$\begin{array}{ccccccc}
H^1(\mathcal{D}^{\mathrm{par}}_{\mathfrak{sl},0}) & \longrightarrow & H^1(\mathcal{D}^{\mathrm{par}}_{\mathfrak{sl},1}) & \longrightarrow & \mathbb{H}^2(\mathcal{D}^{\mathrm{par}}_{\mathfrak{sl},\bullet}) & \longrightarrow & 0\\
\cong\downarrow & & \cong\downarrow & & \cong\downarrow & & \\
H^0(\mathcal{D}^{\mathrm{par}}_{\mathfrak{sl},1})^\vee & \longrightarrow & H^0(\mathcal{D}^{\mathrm{par}}_{\mathfrak{sl},0})^\vee & \longrightarrow & \mathbb{H}^0(\mathcal{D}^{\mathrm{par}}_{\mathfrak{sl},\bullet})^\vee & \longrightarrow & 0
\end{array}$$

induced by the Serre duality. Take any member

$$u \in \mathbb{H}^0(\mathcal{D}^{\mathrm{par}}_{\mathfrak{sl},\bullet}) = \ker\left(H^0(\mathcal{D}^{\mathrm{par}}_{\mathfrak{sl},0}) \xrightarrow{\nabla_{\mathcal{D}^{\mathrm{par}}_{\mathfrak{sl},\bullet}}} \mathbb{H}^0(\mathcal{D}^{\mathrm{par}}_{\mathfrak{sl},1})\right).$$

Since $(\overline{E}, \overline{\nabla}, \bar{\boldsymbol{l}})$ is simple, we can write $u = c \cdot \mathrm{id}_{\overline{E}}$ for some $c \in k$. By the definition of $\mathcal{D}^{\mathrm{par}}_{\mathfrak{sl},0}$, we have $0 = \operatorname{Tr}(u) = \operatorname{Tr}(c\,\mathrm{id}_{\overline{E}}) = rc$. Since $r^{-1} \in k^\times$ by the assumption, we have $c = 0$. Thus, $u = 0$, and we have $\mathbb{H}^0(\mathcal{D}^{\mathrm{par}}_{\mathfrak{sl},\bullet}) = 0$. So the obstruction space $\mathbb{H}^2(\mathcal{D}^{\mathrm{par}}_{\mathfrak{sl},\bullet}) \cong \mathbb{H}^0(\mathcal{D}^{\mathrm{par}}_{\mathfrak{sl},\bullet})^\vee$ vanishes, and there is a lift $(\widetilde{E}, \widetilde{\nabla}, \widetilde{l}) \in \mathcal{M}_{\mathrm{PC}}(\boldsymbol{\nu}, \nabla_L)(A)$ of (E, ∇, l). This means that $\mathcal{M}_{\mathrm{PC}}(\boldsymbol{\nu}, \nabla_L)$ is smooth. □

Using Proposition 5.5 and a similar calculation as in Lemma 2.6, we have the following proposition.

Proposition 5.6 [Ina, Theorem 2.1, Propositions 5.1, 5.2 and 5.3]. *The dimension of the moduli space $\mathcal{M}_{\mathrm{PC}}(\boldsymbol{\nu}, \nabla_L)$ is $2(r^2-1)(g-1) + nr(r-1)$, which is same as the dimension of its open subspace $\mathcal{M}^{\boldsymbol{\alpha}}_{\mathrm{PC}}(\boldsymbol{\nu}, \nabla_L)$.*

We can similarly define the Higgs bundles. As before, $\boldsymbol{\alpha}$ is a parabolic weight. Take a tuple $\boldsymbol{\mu} = (\mu_j^{(i)})^{1\leqslant i\leqslant n}_{0\leqslant j\leqslant r-1} \in k^{nr}$ satisfying the following condition:

$$\sum_{i=1}^{n}\sum_{j=0}^{r-1} \mu_j^{(i)} \;=\; 0.$$

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We say that a tuple $(E,\,\Phi,\,\boldsymbol{l}=\{l^{(i)}_*\}_{1\leqslant i\leqslant n})$ (equipped with a parabolic weight $\boldsymbol{\alpha}$) is an $(\boldsymbol{x},\,\boldsymbol{\mu})$-parabolic Higgs bundle if:

(1) E is an algebraic vector bundle on X of rank r and degree d;

(2) $\Phi:\,E\longrightarrow E\otimes K_X(D)$ is an $\mathcal{O}_X$-linear homomorphism;

(3) $l^{(i)}_*$ is a filtration

$$E|_{x_i}=l^{(i)}_0\supset l^{(i)}_1\supset\cdots\supset l^{(i)}_r=0$$

for every x_i such that $\dim(l^{(i)}_j/l^{(i)}_{j+1})=1$ and $(\mathrm{res}_{x_i}(\Phi)-\mu^{(i)}_j)(l^{(i)}_j)\subset l^{(i)}_{j+1}$ for all $j=0,\,\cdots,\,r-1$.

An $(\boldsymbol{x},\,\boldsymbol{\mu})$-parabolic Higgs bundle $(E,\,\Phi,\,\boldsymbol{l})$ is said to be simple if every endomorphism

$$f:\,E\longrightarrow E$$

that commutes with Φ and preserves $\boldsymbol{l}$ is a constant scalar multiplication $f=c\,\mathrm{Id}_E$ for some $c\in k$. Denote by $\mathcal{M}_{\mathrm{Higgs}}(\boldsymbol{\mu})$ the moduli space of simple $\boldsymbol{\mu}$-parabolic Higgs bundles. Define $\boldsymbol{\alpha}$-stability for parabolic Higgs bundles analogous to Definition 4.10. If we replace n_0 by a sufficiently large integer, we may assume that for every $\boldsymbol{\alpha}$-stable $(\boldsymbol{x},\,\boldsymbol{\mu})$-parabolic Higgs bundle $(E,\,\Phi,\,\boldsymbol{l})$, the underlying vector bundle E is n_0-regular.

Fix a line bundle L on X together with a homomorphism $\Phi_L:\,L\longrightarrow L\otimes K_X$ of $\mathcal{O}_X$–modules such that $\mathsf{res}_{x_i}(\Phi_L)=\sum_{j=0}^{r-1}\mu^{(i)}_j$ for any i. Set

$$\begin{aligned}
\mathcal{M}_{\mathrm{Higgs}}(\boldsymbol{\mu},\,\Phi_L)\;&:=\;\{(E,\,\Phi,\,\boldsymbol{l})\in\mathcal{M}_{\mathrm{Higgs}}(\boldsymbol{\mu})\mid(\det(E),\;\;\mathrm{Tr}(\Phi))\cong(L,\,\Phi_L)\}\,,\\
\mathcal{M}^{n_0-\mathrm{reg}}_{\mathrm{Higgs}}(\boldsymbol{\mu},\,\Phi_L)\;&:=\;\{(E,\,\Phi,\,\boldsymbol{l})\in\mathcal{M}_{\mathrm{Higgs}}(\boldsymbol{\mu},\,\Phi_L)\mid\mathrm{E\ is\ n_0-regular}\}\,,\\
\mathcal{M}^{\boldsymbol{\alpha}}_{\mathrm{Higgs}}(\boldsymbol{\mu},\,\Phi_L)\;&:=\;\{(E,\,\Phi,\,\boldsymbol{l})\in\mathcal{M}_{\mathrm{Higgs}}(\boldsymbol{\mu},\,\Phi_L)\mid(\mathrm{E},\,\Phi\,,\,\boldsymbol{l})\ \mathrm{is}\ \boldsymbol{\alpha}\ -\mathrm{stable}\}\,.
\end{aligned}$$

The same calculations as in the proof of Propositions 5.5 and Proposition 5.6 yield the following proposition.

Proposition 5.7 (See [BoYo, § 2.1], [Yo, Theorem 2.8]). *Assume that r is not divisible by the characteristic of k. Then the moduli space $\mathcal{M}_{\mathrm{Higgs}}(\boldsymbol{\mu},\,\Phi_L)$ is smooth and $\dim\mathcal{M}_{\mathrm{Higgs}}(\boldsymbol{\mu},\,\Phi_L)=2(r^2-1)(g-1)+nr(r-1)$. Furthermore, the open subspace $\mathcal{M}^{\boldsymbol{\alpha}}_{\mathrm{Higgs}}(\boldsymbol{\mu},\,\Phi_L)$ of $\mathcal{M}_{\mathrm{Higgs}}(\boldsymbol{\mu},\,\Phi_L)$ consisting of $\boldsymbol{\alpha}$-stable parabolic Higgs bundles is quasi-projective.*

It is known that there is no non-constant global algebraic function on the moduli space of logarithmic connections with central residues on a compact Riemann surface of genus at least 3 [BiRa]. In the logarithmic case, the same statement was proved in [Ar] in a very special case when $g=0$, $r=2$ and $n=4$. In [BiRa], the Betti number of the moduli space of stable vector bundles assumed one of the key roles. A similar result is proved in [Sin]. We will prove, in this section, a weaker result that the moduli space $\mathcal{M}^{\boldsymbol{\alpha}}_{\mathrm{PC}}(\boldsymbol{\nu},\,\nabla_L)$ of $(\boldsymbol{x},\,\boldsymbol{\nu})$-parabolic connections is not affine for any genus, except for several special cases. We use a part of the ideas in [BiRa] and compare the transcendence degree of the ring of global algebraic functions on the moduli space $\mathcal{M}^{\boldsymbol{\alpha}}_{\mathrm{PC}}(\boldsymbol{\nu},\,\nabla_L)$ of parabolic connections with that on the moduli space $\mathcal{M}^{\boldsymbol{\alpha}}_{\mathrm{Higgs}}(\mathbf{0},\,0)$ of parabolic Higgs bundles. Our argument also works over the base field of positive characteristic, which is consistent with the existence of the Hitchin map on the moduli space of connections ([LaPa], [Groe]).

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5.2 Codimension estimation for non-simple underlying bundle

This subsection provides an improvement of the result of [Ina, § 5]. Throughout this subsection, k is assumed to be an algebraically closed field of arbitrary characteristic.

Now, let X be a smooth projective irreducible curve over $\operatorname{Spec} k$ of genus is g, and let $D = x_1 + \cdots + x_n$ be a reduced divisor on X. Fix a line bundle L of degree d on X. Consider the set

$$\left|\mathcal{N}_{\mathrm{par}}^{n_0\text{-reg}}(L)\right| = \{(E, \boldsymbol{l})\} /\cong$$

of isomorphism classes of quasi-parabolic bundles $(E, \boldsymbol{l})$ on (X, D) such that:

(i) E is an algebraic vector bundle on X of rank r with $\det E \cong L$;

(ii) $\boldsymbol{l}$ is a quasi-parabolic structure consisting of filtrations

$$E|_{x_i} = l_0^{(i)} \supsetneq l_1^{(i)} \supsetneq \cdots \supsetneq l_{r-1}^{(i)} \supsetneq l_r^{(i)} = 0$$

for every $x_i \in D$;

(iii) E_0 is n_0-regular.

By virtue of Lemma 5.4, there is a scheme Z of finite type over $\operatorname{Spec} k$ and a flat family $(\widetilde{E}, \widetilde{\boldsymbol{l}})$ of quasi-parabolic bundles on $X \times Z$ over Z such that every member $(E, \boldsymbol{l}) \in \mathcal{N}_{\mathrm{par}}^{n_0\text{-reg}}(L)$ is isomorphic to $(\widetilde{E}, \widetilde{\boldsymbol{l}})|_{X\times\{p\}}$ for some point $p \in S$. Consider the subset

$$\left|\mathcal{N}_{\mathrm{par}}^{n_0\text{-reg}}(L)^{\circ}\right| = \left\{(E, \boldsymbol{l}) \in \left|\mathcal{N}_{\mathrm{par}}^{n_0\text{-reg}}(L)\right| \;\middle|\; \dim \operatorname{End}(E, \boldsymbol{l}) = 1\right\}$$

of $|\mathcal{N}_{\mathrm{par}}^{n_0\text{-reg}}(L)|$ consisting of simple quasi-parabolic bundles, where $\operatorname{End}(E, \boldsymbol{l})$ is defined by

$$\operatorname{End}(E, \boldsymbol{l}) = \left\{u \in \operatorname{Hom}_{\mathcal{O}_X}(E, E) \;\middle|\; u|_x \colon E|_x \to E|_x \text{ satisfies } u|_x(l_j^x) \subset l_j^x \text{ for any } x \in D \text{ and any } j\right\}.$$

Definition 5.8. Let, X be a smooth projective curve over k. For a vector bundle E on X, we set $\mu(E) := \deg(E)/\operatorname{rank}(E)$, and call it the slope of E.

We will construct a parameter space of $|\mathcal{N}_{\mathrm{par}}^{n_0\text{-reg}}(L)| \setminus |\mathcal{N}_{\mathrm{par}}^{n_0\text{-reg}}(L)^{\circ}|$ whose dimension is at most $(r^2-1)(g-1) + nr(r-1)/2 - 2$. For its proof, we need the following lemma.

Lemma 5.9. *Let X be a smooth projective curve over k of genus $g \geqslant 2$, and let E and F be semistable vector bundles on X satisfying the condition $\mu(E) > \mu(F)$. Then the following inequality holds:*

$$\dim \operatorname{Ext}^1_X(F, E) \leqslant \max\{\operatorname{rank}(E)\operatorname{rank}(F)(2g-3),\ \operatorname{rank}(E)\operatorname{rank}(F)g - 1\}.$$

Proof. By the Serre duality, we have $\dim \operatorname{Ext}^1_X(F, E) = \dim \operatorname{Hom}(E, F \otimes K_X)$. Choose a general point $x \in X$.

First, consider the case where $\deg(E^\vee \otimes F \otimes \mathcal{O}_X(x)) > 0$. In this case, we have $\operatorname{Hom}(F, E \otimes \mathcal{O}_X(-x)) = 0$. Note that we have $\deg(F^\vee \otimes E \otimes \mathcal{O}_X(-x)) > -\operatorname{rank}(E)\operatorname{rank}(F)$, because $\mu(E) > \mu(F)$. By the Riemann–Roch theorem, we have

$$\begin{aligned}
\dim \operatorname{Ext}^1_X(F, E) &= \dim \operatorname{Hom}(E, F \otimes K_X) \leqslant \dim \operatorname{Hom}(E, F \otimes K_X(x)) \\
&= \dim \operatorname{Ext}^1(F, E \otimes \mathcal{O}_X(-x)) \\
&= -\operatorname{rank}(E)\operatorname{rank}(F)(1-g) - \deg(F^\vee \otimes E \otimes \mathcal{O}_X(-x)) \\
&\leqslant \operatorname{rank}(E)\operatorname{rank}(F)g - 1.
\end{aligned}$$

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Second, consider the case where $\deg(E^\vee \otimes F \otimes \mathcal{O}_X(x)) < 0$. Take general points $x_1, \cdots, x_{2g-3}$ of X. Then we get exact sequences

$$0 \longrightarrow H^0(X,\, E^\vee \otimes F \otimes K_X(-x_1 - \cdots - x_i)) \longrightarrow H^0(X,\, E^\vee \otimes F \otimes K_X(-x_1 - \cdots - x_{i-1})) \longrightarrow E^\vee \otimes F \otimes K_X(-x_1 - \cdots - x_{i-1})|_{x_i}$$

for $i = 1, \cdots, 2g-3$. Note that the condition $\deg(E^\vee \otimes F(x)) < 0$ implies that $\mu(E) > \mu(F \otimes K_X(-x_1 - \cdots - x_{2g-3}))$, which yields

$$\mathrm{Hom}(E,\, F \otimes K_X(-x_1 - \cdots - x_{2g-3})) \;=\; 0,$$

because E and F are semistable. So we have

$$\begin{aligned}\dim\ \mathrm{Ext}^1_X(F,\, E) \;&=\; \dim H^0(X,\, E^\vee \otimes F \otimes K_X)\\ &\leqslant \sum_{i=1}^{2g-3} \dim_{\mathbb{C}} \left(E^\vee \otimes F \otimes K_X(-x_1 - \cdots - x_{i-1})|_{x_i}\right) = \mathrm{rank}(E)\ \mathrm{rank}(F)(2g-3).\end{aligned}$$

Consider the remaining case where $\deg(E^\vee \otimes F(t)) = 0$. Take general points $x_1, \cdots, x_{2g-3} \in X$. Then we have $\mu(E) = \mu(F \otimes K_X(-x_1 - \cdots - x_{2g-3}))$. We can write $\mathrm{gr}(E) = \bigoplus_i E_i$ and $\mathrm{gr}(F) = \bigoplus_j F_j$ for stable vector bundles E_i and F_j such that $\mu(E_i) = \mu(E) = \mu(F) = \mu(F_j)$ for any i, j. If we take $x_1, \cdots, x_{2g-3}$ sufficiently generic, then we may assume $E_i \not\cong F_j \otimes K_X(-x_1 - \cdots - x_{2g-3})$ for any i, j. Then we have $\mathrm{Hom}(E,\, F \otimes K_X(-x_1 - \cdots - x_{2g-3})) = 0$. By the same argument as before, we have the inequality $\dim\ \mathrm{Ext}^1_X(E, F) \leqslant\ \mathrm{rank}(E)\ \mathrm{rank}(F)(2g-3)$. □

Proposition 5.10. *Let X be a smooth projective curve over k of genus $g \geqslant 2$, and let L be a line bundle of degree d on X. Assume that the integers r and n satisfy the conditions $r \geqslant 2$ and $n \geqslant 1$. Then there exists a scheme Z of finite type over* $\mathrm{Spec}\, k$ *and a flat family $(\mathcal{E},\, \ell)$ of quasi-parabolic bundles on $X \times Z$ over Z such that:*

(a) $\dim Z \;\leqslant\; (r^2-1)(g-1) + r(r-1)n/2 - 2$;

(b) $\dim \mathrm{End}\left((\mathcal{E}, \ell)|_{X \times \{z\}}\right) \;\geqslant\; 2$ *for any $z \in Z$;*

and each member of the complement $|\mathcal{N}^{n_0\text{-reg}}_{\mathrm{par}}(L)| \setminus |\mathcal{N}^{n_0\text{-reg}}_{\mathrm{par}}(L)^\circ|$ is isomorphic to $(\mathcal{E},\, \ell)|_{X \times \{z\}}$ for some point $z \in Z$.

Proof. Take a quasi-parabolic bundle $(E,\, \boldsymbol{l})$ on $(X,\, D)$ with $\det E \cong L$. Choose a point $x_i \in D$ and $l^{(i)}_j \subset E|_{x_i}$. Then $E' := \ker(E \to E|_{x_i}/l^{(i)}_j)$ has a canonical quasi-parabolic structure $\boldsymbol{l'}$ induced by $\boldsymbol{l}$. The correspondence $(E,\, \boldsymbol{l}) \longmapsto (E',\, \boldsymbol{l'})$ gives a bijection between the set of isomorphism classes of quasi-parabolic bundles; it is called an elementary transformation or a Hecke modification. After applying a finite number of elementary transformations, it may be assumed that r and d are coprime.

Take a member $(E,\, \boldsymbol{l}) \in |\mathcal{N}^{n_0\text{-reg}}_{\mathrm{par}}(L)| \setminus |\mathcal{N}^{n_0\text{-reg}}_{\mathrm{par}}(L)^\circ|$. Since $\dim\ \mathrm{End}(E,\, \boldsymbol{l}) > 1$ by the definition, we have $\dim\ \mathrm{End}(E) > 1$ and E is not a semistable vector bundle. Let

$$E_1 \subset E_2 \subset \cdots \subset E_m = E$$

be the Harder–Narasimhan filtration of E; note that $m \geqslant 2$ because E is not semistable. Set $\overline{E}_1 := E_1$, $\overline{E}_s := E_s/E_{s-1}$ for $s \geqslant 2$ and $r_s :=\ \mathrm{rank}\, \overline{E}_s$. By the definition of a Harder–Narasimhan filtration, each $\overline{E}_s$ is semistable for $1 \leqslant s \leqslant m$ and the inequalities $\mu(\overline{E}_1) >$

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$\mu(\overline{E}_2) > \cdots > \mu(\overline{E}_m)$ hold. Each semistable vector bundle $\overline{E}_s$ has a Jordan–Hölder filtration

$$0 \subset E_s^{(1)} \subset E_s^{(2)} \subset \cdots \subset E_s^{(\gamma_s)} = \overline{E}_s$$

with $\gamma_s \geqslant 1$. Set $\overline{E}_s^{(1)} := E_s^{(1)}$, $\overline{E}_s^{(i)} := E_s^{(i)}/E_s^{(i-1)}$ for $2 \leqslant i \leqslant \gamma_s$, $r_s^{(i)} := \operatorname{rank}\overline{E}_s^{(i)}$ and $d_s^{(i)} := \deg \overline{E}_s^{(i)}$. Then, each $\overline{E}_s^{(i)}$ is a stable bundle on X and

$$L \cong \det\left(\bigoplus_{s=1}^{m}\bigoplus_{i=1}^{\gamma_s}\overline{E}_s^{(i)}\right)$$

holds.

Let us consider the converse. If stable bundles $\{\overline{E}_s^{(i)}\}$ are given, $\{\overline{E}_s\}$ are given by successive extensions

$$0 \longrightarrow E_s^{(i-1)} \longrightarrow E_s^{(i)} \longrightarrow \overline{E}_s^{(i)} \longrightarrow 0 \qquad (2 \leqslant i \leqslant \gamma_s) \tag{5.3}$$

with $E_s^{(\gamma_s)} = \overline{E}_s$. If $\{\overline{E}_s\}$ are given, then E is given by successive extensions

$$0 \longrightarrow E_{s-1} \longrightarrow E_s \longrightarrow \overline{E}_s \longrightarrow 0 \qquad (2 \leqslant s \leqslant m) \tag{5.4}$$

with $E_m = E$. By its definition, $\boldsymbol{l}$ is given by a filtration $E|_{x_i} = l_0^{(i)} \supset l_1^{(i)} \supset \cdots \supset l_{r-1}^{(i)} \supset l_r^{(i)} = 0$ for each $1 \leqslant i \leqslant n$.

We will construct a parameter space of the above data, but we avoid the case of $m = 2$ and $\gamma_1 = \gamma_2 = 1$ and postpone its proof until later. This is because this case needs an extra argument.

Excluding the case where $m = 2$ and $\gamma_1 = \gamma_2 = 1$, we first construct the parameter space of the above data with the further restricted conditions:

(a) $\overline{E}_s^{(i)} \not\cong \overline{E}_s^{(j)}$ for $i \neq j$;

(b) all the extensions in (5.3) and (5.4) do not split.

Set

$$N = \left\{ (\overline{E}_s^{(i)}) \in \prod_{s=1}^{m}\prod_{i=1}^{\gamma_s}\mathcal{N}^e(r_s^{(i)}, d_s^{(i)})^\circ \;\middle|\; \bigotimes_{s=1}^{m}\bigotimes_{i=1}^{\gamma_s}\det(\overline{E}_s^{(i)}) \cong L \right\},$$

where $\mathcal{N}^e(r_s^{(i)}, d_s^{(i)})^\circ$ is the moduli space of stable vector bundles on X of rank $r_s^{(i)}$ and of degree $d_s^{(i)}$. Since $\dim \mathcal{N}^e(r_s^{(i)}, d_s^{(i)})^\circ = (r_s^{(i)})^2(g-1)+1$, we have $\dim N = \sum_{s=1}^{m}\sum_{i=1}^{\gamma_s}((r_s^{(i)})^2(g-1)+1) - g$. Take a quasi-finite covering $N' \longrightarrow N$ whose image consists of those points such that $\overline{E}_s^{(i)} \not\cong \overline{E}_{s'}^{(i')}$ for $(i, s) \neq (i', s')$. We may take a universal family of vector bundles $\{\overline{\mathcal{E}}_s^{(i)}\}_{1\leqslant s\leqslant m}^{1\leqslant i\leqslant\gamma_s}$ on $X \times N'$ over N' such that $\bigotimes_{s=1}^{m}\bigotimes_{i=1}^{\gamma_s}\det(\overline{\mathcal{E}}_s^{(i)}) \cong L \otimes \mathcal{L}'$ for some line bundle $\mathcal{L}'$ on N'. After replacing N' with a disjoint union of locally closed subsets, we may further assume that:

(1) the relative Ext-sheaves $\operatorname{Ext}^p_{X\times N'/N'}(\overline{\mathcal{E}}_s^{(i)}, \overline{\mathcal{E}}_s^{(j)})$ are locally free sheaves on N' for $1 \leqslant s \leqslant m$, $p = 0, 1$ and any $j < i$;

(2) the canonical maps $\operatorname{Ext}^p_{X\times N'/N'}(\overline{\mathcal{E}}_s^{(i)}, \overline{\mathcal{E}}_s^{(j)})|_z \longrightarrow \operatorname{Ext}^p_{X\times\{z\}}(\overline{\mathcal{E}}_s^{(i)}|_{X\times\{z\}}, \overline{\mathcal{E}}_s^{(j)}|_{X\times\{z\}})$ are isomorphisms for all points $z \in N'$.

Set

$$P_s^{(2)} := \mathbb{P}_*\operatorname{Ext}^1_{X\times N'/N'}(\overline{\mathcal{E}}_s^{(2)}, \overline{\mathcal{E}}_s^{(1)}) = \operatorname{ProjSym}\left(\operatorname{Ext}^1_{X\times N'/N'}(\overline{\mathcal{E}}_s^{(2)}, \overline{\mathcal{E}}_s^{(1)})^\vee\right)$$

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for every $1 \leqslant s \leqslant m$, where $\mathrm{Sym}\left(\mathrm{Ext}^1_{X\times N'/N'}(\overline{\mathcal{E}}^{(2)}_s, \overline{\mathcal{E}}^{(1)}_s)^\vee \right)$ is the symmetric algebra of $\mathrm{Ext}^1_{X\times N'/N'}(\overline{\mathcal{E}}^{(2)}_s, \overline{\mathcal{E}}^{(1)}_s)^\vee$ over $\mathcal{O}_{N'}$. Then there is a universal extension

$$0 \longrightarrow \overline{\mathcal{E}}^{(1)}_s \longrightarrow \mathcal{E}^{(2)}_s \longrightarrow \overline{\mathcal{E}}^{(2)}_s \otimes \mathcal{O}_{P^{(2)}_s}(1) \longrightarrow 0$$

on $X \times P^{(2)}_s$. Once $P^{(2)}_s, \dots, P^{(i)}_s$ and $\mathcal{E}^{(2)}_s, \dots, \mathcal{E}^{(i)}_s$ are defined, we set

$$P^{(i+1)}_s = \mathbb{P}_* \, \mathrm{Ext}^1_{X\times_{N'} P^{(i)}_s/P^{(i)}_s}(\overline{\mathcal{E}}^{(i+1)}_s, \mathcal{E}^{(i)}_s).$$

There is a universal extension

$$0 \longrightarrow \mathcal{E}^{(i)}_s \longrightarrow \mathcal{E}^{(i+1)}_s \longrightarrow \overline{\mathcal{E}}^{(i+1)}_s \otimes \mathcal{O}_{P^{(i+1)}_s}(1) \longrightarrow 0$$

on $X \times P^{(i+1)}_s$. Set $P_s := P^{(\gamma_s)}_s$ for $1 \leqslant s \leqslant m$, $P := P_1 \times_{N'} \times \cdots \times_{N'} P_m$ and $\overline{\mathcal{E}}_s := \mathcal{E}^{(\gamma_s)}_s \otimes_{\mathcal{O}_{P_s}} \mathcal{O}_P$. After replacing P with a disjoint union of locally closed subsets, we may assume that:

- the relative Ext-sheaves $\mathrm{Ext}^p_{X\times P/P}(\overline{\mathcal{E}}_s, \overline{\mathcal{E}}_{s'})$ are all locally free sheaves on P for $p = 0, 1$ and $s' < s$;
- the canonical homomorphisms $\mathrm{Ext}^p_{X\times P/P}(\overline{\mathcal{E}}_s, \overline{\mathcal{E}}_{s'})|_z \longrightarrow \mathrm{Ext}^p_{X\times P/P}(\overline{\mathcal{E}}_s|_{X\times\{z\}}, \overline{\mathcal{E}}_{s'}|_{X\times\{z\}})$ are isomorphisms for all points $z \in P$.

Set

$$Q_2 := \mathbb{P}_* \, \mathrm{Ext}^1_{X\times P/P}(\overline{\mathcal{E}}_2, \overline{\mathcal{E}}_1) = \mathrm{ProjSym}\left(\mathrm{Ext}^1_{X\times P/P}(\overline{\mathcal{E}}_2, \overline{\mathcal{E}}_1)^\vee \right).$$

Then there is a universal extension $0 \longrightarrow \overline{\mathcal{E}}_1 \longrightarrow \mathcal{E}_2 \longrightarrow \overline{\mathcal{E}}_2 \otimes \mathcal{O}_{Q_2}(1) \longrightarrow 0$ on $X \times Q_2$. Once $Q_2, \cdots, Q_s$ and $\mathcal{E}_2, \cdots, \mathcal{E}_s$ are defined, set

$$Q_{s+1} := \mathbb{P}_* \, \mathrm{Ext}^1_{X\times Q_s/Q_s}(\overline{\mathcal{E}}_{s+1}, \mathcal{E}_s).$$

Then there are universal extensions $0 \longrightarrow \mathcal{E}_s \longrightarrow \mathcal{E}_{s+1} \longrightarrow \overline{\mathcal{E}}_{s+1} \otimes \mathcal{O}_{Q_{s+1}}(1) \longrightarrow 0$ for $1 \leqslant s \leqslant m-1$. Set

$$Q := \left\{ z \in Q_m \;\middle|\; \mathcal{E}_m|_{X\times\{z\}} \; is n_0 - regular \right\}, \qquad \mathcal{E} := \mathcal{E}_m|_{X\times Q}.$$

Let Y_Q be the flag bundle over Q whose fiber over any $q \in Q$ is the parameter space of the filtrations

$$\mathcal{E}|_{x_i\times q} = l^{(i)}_0 \supset l^{(i)}_1 \supset \cdots \supset l^{(i)}_{r-1} \supset l^{(i)}_r = 0 \qquad (1 \leqslant i \leqslant n).$$

Then there is a universal family of filtrations ℓ so that $(\mathcal{E}, \ell)$ becomes a flat family of quasi-parabolic bundles on $X \times Y_Q$ over Y_Q. Let Z_Q be the reduced closed subscheme of Y_Q consisting of the points y such that $\dim\, \mathrm{End}\,((\mathcal{E}, \ell)|_{X\times y}) \geqslant 2$.

We want to prove that the dimension of Z_Q is at most $(r^2-1)(g-1) + nr(r-1)/2 - 2$. Recall that $\dim N' = -g + \sum_{s=1}^m \sum_{i=1}^{\gamma_s} ((r^{(i)}_s)^2(g-1)+1)$. Since there are exact sequences

$$\mathrm{Ext}^1(\overline{E}^{(i)}_s, E^{(j-1)}_s) \longrightarrow \mathrm{Ext}^1(\overline{E}^{(i)}_s, E^{(j)}_s) \longrightarrow \mathrm{Ext}^1(\overline{E}^{(i)}_s, \overline{E}^{(j)}_s)$$

for $1 \leqslant j < i$, the dimension of $\mathbb{P}_*(\mathrm{Ext}^1(\overline{E}^{(i)}_s, E^{(i-1)}_s))$ is at most $-1 + \sum_{j<i} \dim\, \mathrm{Ext}^1(\overline{E}^{(i)}_s, \overline{E}^{(j)}_s)$. Furthermore, the Riemann–Roch theorem implies that $\dim\, \mathrm{Ext}^1(\overline{E}^{(i)}_s, \overline{E}^{(j)}_s) = r^{(i)}_s r^{(j)}_s (g-1)$, because $\overline{E}^{(i)}_s$ and $\overline{E}^{(j)}_s$ are stable vector bundles

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of the same slope and $E_s^{(i)} \not\cong E_s^{(j)}$. Therefore, the dimension of the fibers of $P_s^{(i+1)} = \mathbb{P}_* \operatorname{Ext}^1_{X\times_{N'} P_s^{(i)}/P_s^{(i)}}(\overline{\mathcal{E}}_s^{(i+1)}, \mathcal{E}_s^{(i)})$ over $P_s^{(i)}$ is at most $-1+\sum_{j=1}^{i-1} r_s^{(i)} r_s^{(j)}(g-1)$, which implies that the dimension of the fibers of $P_s = P_s^{(\gamma_s)}$ over N' is at most

$$\sum_{i=2}^{\gamma_s}\Big(-1+\sum_{j=1}^{i-1} r_s^{(i)} r_s^{(j)}(g-1)\Big) = 1-\gamma_s + \sum_{1\leqslant j<i\leqslant \gamma_s} r_s^{(i)} r_s^{(j)}(g-1). \tag{5.5}$$

Since the extensions in (5.4) do not split, we can see, by an argument similar to the above, that the dimension of the fibers of Q over $P_1 \times_{N'} \times \cdots \times_{N'} P_m$ is at most

$$\sum_{s=2}^{m}\Big(-1+\sum_{t=1}^{s-1}\dim\ \operatorname{Ext}^1(\overline{E}_t, \overline{E}_s)\Big) = 1-m+\sum_{1\leqslant s<t\leqslant m}\dim\ \operatorname{Ext}^1(\overline{E}_t, \overline{E}_s). \tag{5.6}$$

By Lemma 5.9, we have the inequality

$$\dim\ \operatorname{Ext}^1_X(\overline{E}_t,\, \overline{E}_s) \leqslant \max\{r_s r_t(2g-3),\, r_s r_t g-1\} \leqslant 2 r_s r_t (g-1)-1.$$

Using the equality $r_s = r_s^{(1)}+\cdots+r_s^{(\gamma_s)}$ we get the following:

$$\begin{aligned}
\dim Q \ &\leqslant -g+\sum_{s=1}^{m}\sum_{i=1}^{\gamma_s}((r_s^{(i)})^2(g-1)+1)+\sum_{s=1}^{m}\Big(1-\gamma_s+\sum_{1\leqslant j<i\leqslant\gamma_s} r_s^{(i)}r_s^{(j)}(g-1)\Big)\\
&\quad +1-m+\sum_{1\leqslant s<t\leqslant m}(2r_s r_t(g-1)-1)\\
&\leqslant\ -g+m+\sum_{s=1}^{m}(r_s^{(1)}+\cdots+r_s^{(\gamma_s)})^2(g-1)-\sum_{s=1}^{m}(\gamma_s-1)\\
&\quad +1-m+\sum_{1\leqslant s<t\leqslant m}2r_s r_t(g-1)-\frac{m(m-1)}{2}\\
&= (r^2-1)(g-1)-\frac{m(m-1)}{2}-\sum_{s=1}^{m}(\gamma_s-1).
\end{aligned} \tag{5.7}$$

Taking into account the condition $m \geqslant 2$, we have $\dim Q \leqslant (r^2-1)(g-1)-2$, because we avoid the case where $m=2$ and $\gamma_1=\gamma_2=1$. Since the dimension of the fibers of Y_Q over Q is $nr(r-1)/2$, and Z_Q is contained in Y_Q, we have $\dim Z_Q \leqslant \dim Q + nr(r-1)/2 \leqslant (r^2-1)(g-1)+nr(r-1)/2-2$.

Consider the case where one of the extensions (5.3) and (5.4) splits, while again excluding the case of $m=2$ and $\gamma_1=\gamma_2=1$. In this case, we replace $P_s^{(i+1)} = \mathbb{P}_* \operatorname{Ext}^1_{X\times P_s^{(i)}/P_s^{(i)}}(\overline{\mathcal{E}}_s^{(i+1)}, \mathcal{E}_s^{(i)})$ with $P_s^{(i+1)} = P_s^{(i)}$ or replace $Q_{s+1} = \mathbb{P}_* \operatorname{Ext}^1_{X\times Q_s/Q_s}(\overline{\mathcal{E}}_{s+1}, \mathcal{E}_s)$ with $Q_{s+1}=Q_s$ (depending on which extension splits). So, the replacement of the estimation of (5.6) does not affect the calculation in (5.7). Thus, the inequality $\dim Q \leqslant (r^2-1)(g-1)-2$ still holds, and we get that $\dim Z_Q \leqslant (r^2-1)(g-1)+nr(r-1)/2-2$.

Next, consider the case where $\overline{E}_s^{(i)} \cong \overline{E}_s^{(j)}$ for some $i\neq j$. In the calculation of (5.5), we should replace $\dim\ \operatorname{Ext}^1_X(\overline{E}_s^{(i)}, \overline{E}_s^{(j)}) = r_s^{(i)} r_s^{(j)}(g-1)$ with $\dim\ \operatorname{Ext}^1_X(\overline{E}_s^{(i)}, \overline{E}_s^{(j)}) = r_s^{(i)} r_s^{(j)}(g-1)+1$ in the term related to the above pair $(i,\, j)$. However, we replace the condition $\overline{E}_s^{(i)} \not\cong \overline{E}_s^{(j)}$ with the condition $\overline{E}_s^{(i)} \cong \overline{E}_s^{(j)}$ in the definition of N'. So, the calculation of (5.7) is still valid and we get the inequality $\dim Z_Q \leqslant (r^2-1)(g-1)+nr(r-1)-2$.

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Consider now the remaining case where $m = 2$ and $\gamma_1 = \gamma_2 = 1$. In this case, Q is a parameter space of the extensions

$$0 \longrightarrow E_1 \longrightarrow E \longrightarrow E_2 \longrightarrow 0,$$

where $E_1,\, E_2$ are stable vector bundles such that $\mu(E_1) = \mu_1 > \mu_2 = \mu(E_2)$. In the calculation of (5.7), we have $\dim Q \leqslant (r^2-1)(g-1)-1$ in this case. So, we have $\dim Z_Q \leqslant (r^2-1)(g-1) + nr(r-1)/2 - 1$. Note that an automorphism $\mathbf{g}$ of E makes the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & E_1 & \longrightarrow & E & \longrightarrow & E_2 & \longrightarrow & 0 \\ & & {\scriptstyle \mathbf{g}_1}\downarrow & & {\scriptstyle \mathbf{g}}\downarrow & & {\scriptstyle \mathbf{g}_2}\downarrow & & \\ 0 & \longrightarrow & E_1 & \longrightarrow & E & \longrightarrow & E_2 & \longrightarrow & 0 \end{array}$$

commutative and we have $\mathbf{g}_1 = c_1 \mathrm{id}_{E_1}$ and $\mathbf{g}_2 = c_2 \mathrm{id}_{E_2}$ for some $c_1,\, c_2 \in k^\times$.

Consider the case where $\mathrm{Hom}(E_2,\, E_1) = 0$ for generic members $(E_1,\, E_2)$ of N'. In that case, the dimension of the locus $\mathrm{Hom}(E_2,\, E_1) \neq 0$ in Z_Q is at most $(r^2-1)(g-1) + nr(r-1) - 2$. For a general member $(E_1,\, E_2)$ of N', the automorphisms $\mathbf{g}$ of E are given by $(c_1,\, c_2) \in k^\times \times k^\times$ satisfying the conditions $\mathbf{g}_1 = c_1\mathrm{id}_{E_1}$ and $\mathbf{g}_2 = c_2 \mathrm{id}_{E_2}$. Let $v = v_1 e_1 + \cdots + v_r e_r$ be a generator of $l^{(1)}_{r-1}$ with respect to a chosen basis $e_1,\, \ldots,\, e_r$ of $E|_{x_1}$ such that $e_1,\, \ldots,\, e_{r_1}$ generates $E_1|_{x_1}$. Applying the automorphisms of E of the above form, we can normalize $l^{(1)}_{r-1}$ so that a generator $v = v_1e_1 + \cdots + v_r e_r$ of $l^{(1)}_{r-1}$ satisfies $v_i = v_{r_1+j}$ or $v_i v_{r_1+j} = 0$ for some $i,\, j$ with $1 \leqslant i \leqslant r_1$ and $1 \leqslant j \leqslant r_2$. The reduced subscheme of Z_Q defined by this condition is of dimension at most $(r^2-1)(g-1) + nr(r-1) - 2$.

Consider the case where $\mathrm{Hom}(E_2,\, E_1) \neq 0$ for generic members $(E_1,\, E_2)$ of N'. Then there are automorphisms of E of the form $c \cdot \mathrm{id}_E + h$ with $0 \neq h \in \mathrm{Hom}(E_2,\, E_1)$. After replacing $(E_1,\, E_2)$ with $(E_1 \otimes \mathcal{L}^{\otimes r_2},\, E_2 \otimes \mathcal{L}^{\otimes -r_1})$, for a generic member $\mathcal{L} \in \mathrm{Pic}^0_X$, we may assume that $h|_{x_1} \neq 0$, because the locus of N' satisfying $\mathrm{Hom}(E_2,\, E_1(-x_1)) = \mathrm{Hom}(E_2,\, E_1)$ is of dimension less than $\dim N'$. After applying the automorphisms of E, we may normalize a generator $v = v_1 e_1 + \cdots + v_r e_r$ of $l^{(1)}_{r-1}$ such that $v_i = 0$ for some $1 \leqslant i \leqslant r_1$ or $l^{(1)}_{r-1} \subset E_1|_{x_1}$. The locus of Z_Q defined by this condition is of dimension at most $(r^2-1)(g-1) + nr(r-1) - 2$.

The disjoint union of all of the Z_Q in the above arguments and the flat family of quasi-parabolic bundles given by $(\mathcal{E},\, \ell)$ satisfy the assertion of the proposition. ☐

Proposition 5.11. *Let X be an elliptic curve over k, and let L be a line bundle of degree d on X. Assume that one of the following holds:*

- $n \geqslant 3$ *and* $r \geqslant 2$;
- $n = 2$ *and* $r \geqslant 3$.

Then there exists a scheme Z of finite type over k and a flat family $(\widetilde{E},\, \widetilde{\boldsymbol{l}})$ of quasi-parabolic bundles on $X \times Z$ over Z such that:

(a) $\dim Z \leqslant r(r-1)n/2 - 2$;

(b) $\dim\, \mathrm{End}((\widetilde{E},\, \widetilde{\boldsymbol{l}})|_{X \times \{z\}}) \geqslant 2$ *for any* $z \in Z$;

and each member of the complement $|\mathcal{N}^{n_0\text{-reg}}_{\mathrm{par}}(L)| \setminus |\mathcal{N}^{n_0\text{-reg}}_{\mathrm{par}}(L)^\circ|$ is isomorphic to $(\widetilde{E},\, \widetilde{\boldsymbol{l}})|_{X \times \{z\}}$ for some point $z \in Z$.

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Proof. As in the proof of Proposition 5.10, we may assume that r and d are coprime. Take a member $(E, \boldsymbol{l}) \in |\mathcal{N}_{\mathrm{par}}^{n_0\text{-reg}}(L)| \setminus |\mathcal{N}_{\mathrm{par}}^{n_0\text{-reg}}(L)^{\circ}|$. Since $\dim\ \mathrm{End}(E, \boldsymbol{l}) \geqslant 2$, it follows that $\dim\ \mathrm{End}(E) \geqslant 2$. As r and d are coprime, the vector bundle E is not semistable. Let

$$0 \subset E_1 \subset E_2 \subset \cdots \subset E_m = E$$

be the Harder–Narasimhan filtration of E; note that $m \geqslant 2$ because E is not semistable. Setting $\overline{E}_1 = E_1$ and $\overline{E}_s := E_s/E_{s-1}$ for $2 \leqslant s \leqslant m$, the slopes $\mu_t := \mu(\overline{E}_t)$ satisfy the inequalities

$$\mu_1 > \mu_2 > \cdots > \mu_m, \tag{5.8}$$

and we get extensions

$$0 \longrightarrow E_s \longrightarrow E_{s+1} \longrightarrow \overline{E}_{s+1} \longrightarrow 0. \tag{5.9}$$

Note that we have $\mathrm{Ext}^1(\overline{E}_t, \overline{E}_s) \cong \mathrm{Hom}(\overline{E}_s, \overline{E}_t)^{\vee} = 0$ for $s < t$, because $\mu(\overline{E}_t) < \mu(\overline{E}_s)$ and $\overline{E}_s$, $\overline{E}_t$ are semistable. It follows that $\mathrm{Ext}^1(\overline{E}_{s+1}, E_s) = 0$. So the extension (5.9) must split, and we have a decomposition

$$E \cong \bigoplus_{s=1}^{m} \overline{E}_s.$$

Let $\{\overline{F}_s^{(i)}\}_{i=1,\dots,\gamma_s}$ be the set of stable bundles arising in the direct summands of $\mathrm{gr}(\overline{E}_s)$. Fix an index $i \in \{1, \dots, \gamma_s\}$. Let $G_i \subset \overline{E}_s$ be a maximal subbundle satisfying the condition $\mathrm{Hom}(G_i, \overline{F}_s^{(i)}) = 0$. Then we have $\mathrm{Hom}(\overline{F}_s^{(j)}, \overline{E}_s/G_i) = 0$ for any $j \neq i$, because otherwise the pullback of $\overline{F}_s^{(j)} \subset \overline{E}_s/G_i$ by the surjection $\overline{E}_s \longrightarrow \overline{E}_s/G_i$ contradicts the maximality of G_i.

Taking account that $\overline{E}_s$ is semistable, we can see that $\overline{E}_s/G_i$ is semistable of slope $\mu(\overline{E}_s)$ and $\mathrm{gr}(\overline{E}_s/G_i) \cong (\overline{F}_s^{(i)})^{\oplus u}$ for some positive integer u. So, we have

$$\mathrm{Ext}^1(\overline{E}_s/G_i, G_i) \cong \mathrm{Hom}(G, \overline{E}_s/G_i)^{\vee} = 0,$$

and the extension $0 \longrightarrow G_i \longrightarrow \overline{E}_s \longrightarrow \overline{E}_s/G_i \longrightarrow 0$ must split. Applying the same argument to G_i, we finally get a decomposition

$$\overline{E}_s \cong \bigoplus_{i=1}^{\gamma_s} F_s^{(i)},$$

where $F_s^{(i)}$ is a semistable bundle satisfying the condition $\mathrm{gr}(F_s^{(i)}) \cong (\overline{F}_s^{(i)})^{\oplus u}$ for a positive integer u. Note that $\mu(F_s^{(i)}) = \mu_s$ for any i and these satisfy the inequalities in (5.8). We may further assume that $\overline{F}_s^{(i)} \not\cong \overline{F}_s^{(j)}$ for $i \neq j$. Note that we have

$$\bigotimes_{s=1}^{m} \bigotimes_{i=1}^{\gamma_s} \det F_s^{(i)} \cong L. \tag{5.10}$$

The moduli space of stable bundles parameterizing $\overline{F}_s^{(i)}$ is isomorphic to $\mathrm{Pic}_X^0 \cong X$ for all i, s. Since we have $\dim\ \mathrm{Ext}^1(\overline{F}_s^{(i)}, \overline{F}_s^{(i)}) = 1$ for a stable vector bundle $\overline{F}_s^{(i)}$, a successive non-split extension of $\overline{F}_s^{(i)}$ is unique up to an isomorphism. So, once $\overline{F}_s^{(i)}$ is given, the extensions $F_s^{(i)}$ of $\overline{F}_s^{(i)}$ are parameterized by a finite set. Taking into account the relation (5.10), the underlying vector bundles E of $(E, \boldsymbol{l})$ can be parameterized by a scheme W of finite type over $\mathrm{Spec}\, k$ whose dimension is $-1 + \sum_{s=1}^{m} \gamma_s$.

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Let

$$Y \longrightarrow W$$

be the flag bundle parameterizing the quasi-parabolic structures on the vector bundles E corresponding to the points of W. There is a universal family of quasi-parabolic bundles $(\widetilde{E},\, \widetilde{\boldsymbol{l}})$ on $X \times Y$ over Y. Since each fiber of Y over W is of dimension $nr(r-1)/2$, we have

$$\dim Y \,=\, \dim W + \frac{nr(r-1)}{2} \,=\, \sum_{s=1}^{m} \gamma_s + \frac{nr(r-1)}{2} - 1.$$

Write $(E,\, \boldsymbol{l}) \,:=\, (\widetilde{E},\, \widetilde{\boldsymbol{l}})|_{X \times y}$ for each point $y \in Y$.

Case A. Consider the case where the number of components in the decomposition $E = \bigoplus_{s,i} F_s^{(i)}$ is at least three. Choose a basis $e_{x,s,1}^{(i)},\, \cdots,\, e_{x,s,r_{i,s}}^{(i)}$ of $F_s^{(i)}|_x$ at each point $x \in D$ for $1 \leqslant s \leqslant m$ and $1 \leqslant i \leqslant \gamma_s$. Let

$$\sum_{s=1}^{m} \sum_{i=1}^{\gamma_s} \sum_{p=1}^{r_{i,s}} v_{x_2,s,p}^{(i)} e_{x_2,s,p}^{(i)}$$

be a generator of $l_{r-1}^{(2)}$, and let

$$\sum_{s=1}^{m} \sum_{i=1}^{\gamma_s} \sum_{p=1}^{r_{i,s}} w_{x_2,s,p}^{(i)} e_{x_2,s,p}^{(i)}$$

be a representative of a generator of $l_{r-2}^{(2)}/l_{r-1}^{(2)}$. The group $\mathrm{Aut}E$ of automorphisms of E consists of the invertible elements of the ring of endomorphisms of E:

$$\mathrm{End}E = \Bigg(\bigoplus_{s,i} \mathrm{End}(F_s^{(i)}) \Bigg) \bigoplus \Bigg(\bigoplus_{(s,i) \neq (t,j)} \mathrm{Hom}(F_s^{(i)},\, F_t^{(j)}) \Bigg).$$

By the assumption, we can choose $F_s^{(i)}$, $F_{s'}^{(i')}$, $F_t^{(j)}$ and $F_{t'}^{(j')}$ whose indices satisfy $s' < s$, $t' < t$ and $((s,\, i),\, (s',\, i')) \neq ((t,\, j),\, (t',\, j'))$. So $\mathrm{Aut}E$ contains the three types of automorphisms:

$$\prod_{s,i} k^* \mathrm{id}_{F_s^{(i)}}, \quad \mathrm{id}_E + \mathrm{Hom}(F_s^{(i)},\, F_{s'}^{(i')}), \quad \mathrm{id}_E + \mathrm{Hom}(F_t^{(j)},\, F_{t'}^{(j')}).$$

Note that the restriction maps

$$\mathrm{Hom}(F_s^{(i)},\, F_{s'}^{(i')}) \longrightarrow \mathrm{Hom}\left(F_s^{(i)}|_{x_2},\, F_{s'}^{(i')}|_{x_2} \right), \tag{5.11}$$

$$\mathrm{Hom}(F_t^{(j)},\, F_{t'}^{(j')}) \longrightarrow \mathrm{Hom}\left(F_t^{(j)}|_{x_2},\, F_{t'}^{(j')}|_{x_2} \right) \tag{5.12}$$

are not zero for generic choices of $F_s^{(i)}$, $F_{s'}^{(i')}$, $F_t^{(j)}$ and $F_{t'}^{(j')}$.

If $F_s^{(i)} \neq F_t^{(j)}$ and $v_{x_2,s,p}^{(i)} \neq 0$ for some p, then we may normalize a representative of a generator of $l_{r-2}^{(2)}/l_{r-1}^{(2)}$ such that $w_{x_2,s,p}^{(i)} = 0$. Applying the actions of $\mathrm{id}_E + \mathrm{Hom}(F_s^{(i)},\, F_{s'}^{(i')})$ and $\mathrm{id}_E + \mathrm{Hom}(F_t^{(j)},\, F_{t'}^{(j')})$, we may ensure that $v_{x_2,s',p}^{(i')} = 0$ for some p and $w_{x_2,t',q}^{(j')} w_{x_2,t,q'}^{(j)} = 0$ for some q, q'. The Zariski closed subset Y' defined by this condition is of dimension $\dim Y - 2 = \sum_{s=1}^{m} \gamma_s + nr(r-1)/2 - 3$.

Assume that $F_s^{(i)} \neq F_t^{(j)}$ and $v_{x_2,s,p}^{(i)} = 0$ for all p. If, in addition, the condition $v_{x_2,s',p'}^{(i)} = 0$ holds for all p', then such a locus is of dimension at most $\dim Y - 2 = \sum_{s=1}^{m} \gamma_s + nr(r-1)/2 - 3$.

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So, assume that $v^{(i)}_{x_2,s',p'} \neq 0$ for some p'. Then we can normalize a representative of a generator of $l^{(2)}_{r-2}/l^{(2)}_{r-1}$ such that $w^{(i)}_{x_2,s',p'} = 0$. Applying the action of $\mathrm{id}_E + \mathrm{Hom}(F^{(j)}_t, F^{(j')}_{t'})$, we may have $w^{(j')}_{x_2,t',q} w^{(j)}_{x_2,t,q'} = 0$ for some q, q'. The Zariski closed subset of Y' defined by the condition $v^{(i)}_{x_2,s,p} = 0$ for all p and $w^{(j')}_{x_2,t',q} w^{(j)}_{x_2,t,q'} = 0$ for some q, q' is of dimension at most $\dim Y - 2 = \sum_{s=1}^m \gamma_s + nr(r-1)/2 - 3$.

Assume that $F^{(i)}_s = F^{(j)}_t$. Then we have $F^{(i')}_{s'} \neq F^{(j')}_{t'}$ by the choices of

$$(s,\, i),\ (s',\, i'),\ (t,\, j),\ (t',\, j').$$

If $v^{(i)}_{x_2,s,p} \neq 0$ for some p, then applying automorphisms in $\mathrm{id}_E + \mathrm{Hom}(F^{(i)}_s, F^{(i')}_{s'})$ and $\mathrm{id}_E + \mathrm{Hom}(F^{(j)}_t, F^{(j')}_{t'})$, we may ensure that $v^{(i')}_{x_2,s',p'} = v^{(i)}_{x_2,t',q'} = 0$ for some p', q'. The Zariski closed subset of Y defined by this condition is of dimension at most $\dim Y - 2 = \sum_{s=1}^m \gamma_s + nr(r-1)/2 - 3$. Assume that $v^{(i)}_{x_2,s,p} = 0$ for all p, while still assuming $F^{(i)}_s = F^{(j)}_t$. If, in addition, we have $v^{(i')}_{x_2,s',p'} = 0$ for all p', then such a locus in Y is of dimension at most $\dim Y - 2 = \sum_{s=1}^m \gamma_s + nr(r-1)/2 - 3$. So, assume that $v^{(i')}_{x_2,s',p'} \neq 0$ for some p'. Then we may normalize a representative of a generator of $l^{(2)}_{r-2}/l^{(2)}_{r-1}$ so that the condition $w^{(i')}_{x_2,s',p'} = 0$ holds. Applying an automorphism in $\mathrm{id}_E + \mathrm{Hom}(F^{(j)}_t, F^{(j')}_{t'})$, we may have $w^{(j')}_{x_2,t',q'} w^{(j)}_{x_2,t,q} = 0$ for some q, q'. The locus of Y defined by $v^{(i')}_{x_2,s',p'} = 0$ for all p' and $w^{(j')}_{x_2,t',q'} w^{(j)}_{x_2,t,q} = 0$ for some q, q' is of dimension at most $\dim Y - 2 = \sum_{s=1}^m \gamma_s + nr(r-1)/2 - 3$.

Therefore, in all cases we can get a disjoint union Y' of locally closed subsets of Y and a flat family of quasi-parabolic bundles $(\widetilde{E}, \widetilde{\boldsymbol{l}})$ on $X \times Y'$ over Y', such that $\dim Y' \leqslant \sum_{s=1}^m \gamma_s + nr(r-1)/2 - 3$ and every member of $\left|\mathcal{N}^{n_0-\mathrm{reg}}_{\mathrm{par}}(L)\right| \setminus \left|\mathcal{N}^{n_0-\mathrm{reg}}_{\mathrm{par}}(L)^\circ\right|$ can be transformed by the actions of $\mathrm{id}_E + \mathrm{Hom}(F^{(i)}_s, F^{(i')}_{s'})$ and $\mathrm{id}_E + \mathrm{Hom}(F^{(j)}_t, F^{(j')}_{t'})$ to a quasi-parabolic bundle $(\widetilde{E}, \widetilde{\boldsymbol{l}})|_{X\times y}$ for some $y \in Y'$.

Using the action of the group $\prod_{s,i} k^\times \mathrm{id}_{F^{(i)}_s}$ on a generator

$$v^{(1)}_{x_1,1,1} e^{(1)}_{x_1,1,1} + \cdots + v^{(m)}_{x_1,\gamma_m,r_{m,\gamma_m}} e^{(m)}_{x_1,\gamma_m,r_{m,\gamma_m}}$$

of $l^{(1)}_{r-1}$ we may have $(v^{(s)}_{x_1,i,p} - 1) v^{(s)}_{x_1,i,p} = 0$ for $1 \leqslant s \leqslant m$, $1 \leqslant i \leqslant \gamma_s$ and any p. The Zariski closed subset Z of Y' defined by this condition satisfies $\dim Z = \dim Y' - (-1 + \sum_{s=1}^m \gamma_s) \leqslant nr(r-1)/2 - 2$.

Case B. Consider the case where $E = F_1 \oplus F_2$ with $\mu(F_1) > \mu(F_2)$, $r_i = \mathrm{rank}\, F_i$ and each F_i is a successive extension of one stable vector bundle. In this case, we have $m = 2$ and $\gamma_1 = \gamma_2 = 1$. So, we have $\dim W = 1$ and $\dim Y = 1 + nr(r-1)/2$. Since $\mu(F_1) > \mu(F_2)$ and F_1, F_2 are semistable, it follows that $\mathrm{Hom}(F_2, F_1) = 0$. So, we have $\dim\ \mathrm{Hom}(F_2, F_1) = \deg(F_2^\vee \otimes F_1) > 0$ by the Riemann–Roch theorem, and

$$\dim\ \mathrm{Hom}(F_2,\, F_1(-x)) = \begin{cases} \deg(F_2^\vee \otimes F_1(-x)) & \text{if } \mu\ (\mathrm{F}_2) < \ \mu\ (\mathrm{F}_1(-\mathrm{x})), \\ \deg(F_2^\vee \otimes F_1(-x)) & \\ \quad + \dim\ \mathrm{Hom}(F_1(-x), F_2) & \text{if } \mu\ (\mathrm{F}_2) = \mu\ (\mathrm{F}_1(-\mathrm{x})), \\ 0 & \text{if } \mu\ (\mathrm{F}_2) > \ \mu\ (\mathrm{F}_1(-\mathrm{x})), \end{cases} \tag{5.13}$$

for a point x of X. In the case where $\mu(F_2) = \mu(F_1(-x))$, we have either

$$\dim\ \mathrm{Hom}(F_2,\, F_1(-x_1)) = 0$$

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or dim $\mathrm{Hom}(F_2,\, F_1(-x_2)) = 0$, because $x_1 \neq x_2$. So, in all cases of (5.13), at least one of the maps

$$\mathrm{Hom}(F_2,\, F_1) \longrightarrow \mathrm{Hom}(F_2|_{x_1},\, F_1|_{x_1}), \qquad \mathrm{Hom}(F_2,\, F_1) \longrightarrow \mathrm{Hom}(F_2|_{x_2},\, F_1|_{x_2})$$

is not zero. Choose a basis $e_{x_i,1},\, \ldots,\, e_{x_i,r_1}$ of $F_1|_{x_i}$ and a basis $e'_{x_i,1},\, \cdots,\, e'_{x_i,r_2}$ of $F_2|_{x_i}$. Take a generator $v_{x_i,1}e_{x_i,1} + \cdots + v_{x_i,r_1}e_{x_i,r_1} + v'_{x_i,1}e'_{x_i,1} + \cdots + v'_{x_i,r_2}e'_{x_i,r_2}$ of $l^{(i)}_{r-1}$. Applying the action of $1_E +\ \mathrm{Hom}(F_2,\, F_1)$, we may have $v_{x_2,q} = 0$ for some q, or $v'_{x_2,q'} = 0$ for all q'. Moreover, applying the action of $k^\times \mathrm{id}_{F_1} \times k^\times \mathrm{id}_{F_2}$, we may have $v_{x_1,p} = v'_{x_1,p'}$ for some $p,\, p'$, or $v_{x_1,p}v'_{x_1,p'} = 0$ for some $p,\, p'$. Let Y' be a disjoint union of subvarieties of Y where the following two conditions hold: $v'_{x_2,q'}v_{x_2,q} = 0$ for some $q,\, q'$ and $(v_{x_1,p} - v'_{x_1,p'})v_{x_1,p}v'_{x_1,p'} = 0$ for some $p,\, p'$. Then we have

$$\dim Y' \leqslant r(r-1)n/2 - 1,$$

and every member of the complement $|\mathcal{N}^{n_0\text{-reg}}_{\mathrm{par}}(L)| \setminus |\mathcal{N}^{n_0\text{-reg}}_{\mathrm{par}}(L)^\circ|$ can be transformed by the actions of $k^\times \cdot 1_{F_1} \times k^\times \cdot 1_{F_2}$ and $1_E +\ \mathrm{Hom}(F_2,\, F_1)$ to a quasi-parabolic bundle $(\widetilde{E},\, \widetilde{\boldsymbol{l}})|_{X\times y}$ for some $y \in Y'$. Let Y'' be the Zariski closed subset of Y' defined by

$$Y'' := \left\{ y \in Y' \;\middle|\; \dim\ \mathrm{End}((\widetilde{E}, \widetilde{\boldsymbol{l}})|_{X\times y}) \geqslant 2 \right\}.$$

For each point $y \in Y''$, write $(E,\, \boldsymbol{l}) := (\widetilde{E},\, \widetilde{\boldsymbol{l}})|_{X\times y}$. Set

$$H := \left\{ g \in \ \mathrm{Aut}E \;\middle|\; \mathrm{g}|_{\mathrm{x_i}}(\mathrm{l}^{(\mathrm{i})}_{\mathrm{r-1}}) = \mathrm{l}^{(\mathrm{i})}_{\mathrm{r-1}} \text{ for i} = 1,\, 2 \right\}.$$

Then H contains non-scalar automorphisms.

(I) Consider the case where $H \not\subset k^\times \mathrm{id}_E + \mathrm{Hom}(F_2, F_1)$. Take $\mathbf{g} \in H \setminus (k^\times \mathrm{id}_E + \mathrm{Hom}(F_2, F_1))$. Then we can write

$$\mathbf{g} = \begin{pmatrix} c_1 \mathrm{Id}_{F_1} & b \\ 0 & c_2 \mathrm{Id}_{F_2} \end{pmatrix},$$

where $c_1,\, c_2 \in k^\times$, $b \in\ \mathrm{Hom}(F_2,\, F_1)$ and $c_1 \neq c_2$.

(a) Consider the case where $n \geqslant 3$. Since $\mathbf{g}|_{x_3}$ has distinct eigenvalues $c_1,\, c_2$, the condition that $\mathbf{g}|_{x_3}$ preserves $l^{(3)}_{r-1}$ implies that $\dim Y'' \leqslant \dim Y - 1 \leqslant nr(r-1)/2 - 2$.

(b) Consider the case where $r \geqslant 3$. In this case, we have either $r_1 =\ \mathrm{rank}F_1 \geqslant 2$ or $r_2 =\ \mathrm{rank}F_2 \geqslant 2$.

(i) Consider the case where $r_2 \geqslant 2$. If $l^{(i)}_{r-1} \subset F_1|_{x_i}$ for $i = 1$ or $i = 2$, then Y' can be replaced by the locus satisfying this condition and we get that $\dim Y' \leqslant nr(r-1)/2 - 2$. So, we may assume that $l^{(i)}_{r-1} \not\subset F_1|_{x_i}$ for $i = 1$ or $i = 2$. Then we have $\mathbf{g}|_{l^{(i)}_{r-1}} = c_2 \mathrm{id}_{l^{(i)}_{r-1}}$, and $\mathbf{g}$ induces a linear map $\overline{\mathbf{g}}:\ E|_{x_i}/l^{(i)}_{r-1} \longrightarrow E|_{x_i}/l^{(i)}_{r-1}$. Since the eigenvalues of $\overline{\mathbf{g}}$ are $c_1,\, c_2$ and $l^{(i)}_{r-2}/l^{(i)}_{r-1}$ is preserved by $\overline{\mathbf{g}}$, it follows that $\dim Y'' \leqslant \dim Y' - 1 \leqslant nr(r-1)/2 - 2$.

(ii) Consider the case where $r_1 \geqslant 2$. If $l^{(i)}_{r-1} \subset F_2|_{x_i}$, then such a locus in Y' is of dimension at most $nr(r-1)/2 - 2$. So we may assume that $l^{(i)}_{r-1} \not\subset F_2|_{x_i}$. Since the induced map $\overline{\mathbf{g}}:\ E|_{x_i}/l^{(i)}_{r-1} \longrightarrow E|_{x_i}/l^{(i)}_{r-1}$ has distinct eigenvalues $c_1,\, c_2$ and $l^{(i)}_{r-2}/l^{(i)}_{r-1}$ is preserved by $\overline{\mathbf{g}}$, it follows that $\dim Y'' \leqslant \dim Y' - 1 \leqslant nr(r-1)/2 - 2$.

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(II) Consider the case where H is contained in $k^{\times}\mathrm{id}_E + \mathrm{Hom}(F_2, F_1)$.

(a) Assume that $n \geqslant 3$, in addition to $H \subset k^{\times}\mathrm{id}_E + \mathrm{Hom}(F_2, F_1)$. We may assume that the composition of maps

$$\{a \in \mathrm{Hom}(F_2, F_1) \mid \mathrm{id}_E + a \in H\} \hookrightarrow \mathrm{Hom}(F_2, F_1) \longrightarrow \mathrm{Hom}(F_2|_{x_3}, F_1|_{x_3})$$

is injective, because the non-injective locus in Y' is of dimension at most $\dim Y' - 1 \leqslant nr(r-1)/2 - 2$. Choose a basis $e_{x_i,1}, \ldots, e_{x_i,r_1}$ of $F_1|_{x_i}$ and a basis $e'_{x_i,1}, \ldots, e'_{x_i,r_2}$ of $F_2|_{x_i}$ for $i = 1, 2$. Take a generator $v_{x_i,1}e_{x_i,1} + \cdots + v_{x_i,r_1}e_{x_i,r_1} + v'_{x_i,1}e'_{x_i,1} + \cdots + v'_{x_i,r_2}e'_{x_i,r_2}$ of $l^{(i)}_{r-1}$. Applying an automorphism $\mathrm{id}_E + a \in H$ with $a \in \mathrm{Hom}(F_2, F_1)$ satisfying the condition $a|_{x_3} \neq 0$, we can normalize $l^{(3)}_{r-1}$ so that the condition $v_{x_3,p} = 0$ holds for some p or the condition $v'_{x_3,p'} = 0$ holds for all p'. The Zariski closed subset of Y'' defined by this condition is of dimension at most $\dim Y' - 1 \leqslant nr(r-1)/2 - 2$.

(b) Consider the case where $r \geqslant 3$ while $H \subset k^{\times}\mathrm{id}_E + \mathrm{Hom}(F_2, F_1)$ is again assumed. We may assume the injectivity of the homomorphism

$$\mathrm{Hom}(F_2, F_1) \longrightarrow \mathrm{Hom}(F_2|_{x_2}, F_1|_{x_2}),$$

because it holds for a generic point of Y'. Take a basis $f_1, f_2, \ldots, f_r$ of $E|_{x_2}$ such that f_1 is a generator of $l^{(2)}_{r-1}$. If there is an element $1_E + a \in H$ such that $a \in \mathrm{Hom}(F_2, F_1) \setminus \{0\}$ and $\mathrm{Im}(a|_{x_2}) \not\subset l^{(2)}_{r-1}$, then, after applying such an automorphism, we can normalize a representative $a_2f_2 + \cdots + a_rf_r$ of a generator of $l^{(2)}_{r-2}/l^{(2)}_{r-1}$ so that the condition $a_p = 0$ holds for some $p \geqslant 2$. Such a locus in Y is of dimension at most $r(r-1)n/2 - 2$. If the condition $\mathrm{Im}(a|_{x_2}) \subset l^{(2)}_{r-1}$ holds for all $a \in \mathrm{Hom}(F_2, F_1)$ satisfying $\mathrm{id}_E + a \in H$, then we have $l^{(2)}_{r-1} = \mathrm{Im}(a|_{x_2})$ for such an a with $a \neq 0$. So we may replace Y' with a Zariski closed subset whose dimension is at most $r(r-1)n/2 - (r-1) \leqslant r(r-1)n/2 - 2$, because $r \geqslant 3$.

Therefore, in all cases, the disjoint union Z of all the locally closed subsets of Y'' in the above argument and the pullback of flat families $(\widetilde{E}, \widetilde{\boldsymbol{l}})|_{X \times Z}$ satisfy the assertion of the proposition. ☐

Proposition 5.12. *Assume that $X = \mathbb{P}^1_k$, L is a line bundle on $\mathbb{P}^1_k$ and one of the following two holds:*

(I) *$n \geqslant 5$ and $r \geqslant 2$; or*

(II) *$n = 4$ and $r \geqslant 3$.*

Then there exists a scheme Z of finite type over $\mathrm{Spec}\, k$ and a flat family $(\widetilde{E}, \widetilde{\boldsymbol{l}})$ of quasi-parabolic bundles on $\mathbb{P}^1 \times Z$ over Z such that:

- $\dim Z \leqslant -r^2 + r(r-1)n/2 - 1$;
- $\dim \mathrm{End}\left((\widetilde{E}, \widetilde{\boldsymbol{l}})|_{\mathbb{P}^1 \times \{z\}}\right) \geqslant 2$ *for any* $z \in Z$;

and each member of the complement $|\mathcal{N}^{n_0\text{-reg}}_{\mathrm{par}}(L)| \setminus |\mathcal{N}^{n_0\text{-reg}}_{\mathrm{par}}(L)^{\circ}|$ is isomorphic to $(\widetilde{E}, \widetilde{\boldsymbol{l}})|_{\mathbb{P}^1 \times \{z\}}$ for some point $z \in Z$.

Proof. Take a quasi-parabolic bundle $(E, \boldsymbol{l})$. Write

$$E = \mathcal{O}_{\mathbb{P}^1}(a_1)^{\oplus r_1} \oplus \cdots \oplus \mathcal{O}_{\mathbb{P}^1}(a_m)^{\oplus r_m}$$

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with $a_1 < \cdots < a_m$. If $l^{(i)}_{r-1} \not\subset \mathcal{O}_{\mathbb{P}^1}(a_2)^{\oplus r_2}|_{x_i} \oplus \cdots \oplus \mathcal{O}_{\mathbb{P}^1}(a_m)^{\oplus r_m}|_{x_i}$ for some i, set

$$\begin{aligned} E' &:= \ker\left(E \longrightarrow E|_{x_i}/l^{(i)}_{r-1}\right) \otimes \mathcal{O}_{\mathbb{P}^1}(x_i) \\ &\cong \mathcal{O}_{\mathbb{P}^1}(a_1)^{\oplus r_1 -1} \oplus \mathcal{O}_{\mathbb{P}^1}(a_1+1) \oplus \mathcal{O}_{\mathbb{P}^1}(a_2)^{\oplus r_2} \oplus \cdots \oplus \mathcal{O}_{\mathbb{P}^1}(a_m)^{\oplus r_m}. \end{aligned}$$

Repeating such process of elementary transformations and a twist by a line bundle, we may replace $(E, \boldsymbol{l})$ with a quasi-parabolic bundle which satisfies one of the following two conditions:

(A) $E \cong \mathcal{O}_{\mathbb{P}^1}^{\oplus r}$;

(B) $E = \mathcal{O}_{\mathbb{P}^1}(a_1)^{\oplus r_1} \oplus \cdots \oplus \mathcal{O}_{\mathbb{P}^1}(a_m)^{\oplus r_m}$ and $l^{(i)}_{r-1} \subset \mathcal{O}_{\mathbb{P}^1}(a_2)^{\oplus r_2}|_{x_i} \oplus \cdots \oplus \mathcal{O}_{\mathbb{P}^1}(a_m)^{\oplus r_m}|_{x_i}$ for any i, where $a_1 < a_2 < \cdots < a_m$.

Case A. Consider the case where $E \cong \mathcal{O}_{\mathbb{P}^1}^{\oplus r}$.

We will construct a parameter space for non-simple quasi-parabolic bundles $(E, \boldsymbol{l})$ satisfying $E \cong \mathcal{O}_{\mathbb{P}^1}^{\oplus r}$. Let $e_1, \cdots, e_r$ be the basis of E obtained by pulling back the canonical basis of $\mathcal{O}_{\mathbb{P}^1}^{\oplus r}$ via the isomorphism $E \xrightarrow{\sim} \mathcal{O}_{\mathbb{P}^1}^{\oplus r}$. We may assume that $l^{(1)}_*$ is given by $l^{(1)}_j = \langle e_1, \cdots, e_{r-j}\rangle$ for $j = 0, \cdots, r-1$, after applying an automorphism of E. Applying automorphisms of E fixing $l^{(1)}_*$, we may further assume that $l^{(2)}_*$ is given by $l^{(2)}_j = \langle e_{\sigma(1)}, \cdots, e_{\sigma(r-j)}\rangle$ for $j = 0, 1, \cdots, r-1$, where σ is a permutation of $\{1, \cdots, r\}$. Let $w_1e_1 + \cdots + w_re_r$ be a generator of $l^{(3)}_{r-1}$. Applying a diagonal automorphism of E, which automatically fixes $l^{(1)}_*$ and $l^{(2)}_*$, we may assume that either $w_i = 1$ holds or $w_i = 0$ holds for any i. Then the group of automorphisms of E fixing $l^{(1)}_*$, $l^{(2)}_*$ and $l^{(3)}_{r-1}$ becomes

$$B'' = \left\{ (a_{ij}) \in \mathrm{GL}_r(k) \;\middle|\; \begin{array}{l} a_{ij} = 0 \text{ and } a_{\sigma(i)\sigma(j)} = 0 \text{ for } i, > j \text{ and there is } c \in k^\times \\ \text{satisfying } a_{ii}w_i + \sum_{j\neq i} a_{ij}w_j = c\, w_i \text{ for any } i \end{array} \right\}.$$

Since $\dim \operatorname{End}(E, \boldsymbol{l}) \geqslant 2$ by the assumption, it follows that either there is some (i, j) with $i < j$ and $\sigma(i) < \sigma(j)$ or there is some i satisfying the condition $w_i = 0$.

(I) First assume that $n \geqslant 5$ and $r \geqslant 2$.

(i) Consider the case where $w_{i_1} = 0$ for some i_1. Then there are automorphisms (a_{ij}) in B'' such that $a_{i_1i_1} = c \in k^\times$, $a_{ii} = 1$ for $i \neq i_1$ and $a_{ij} = 0$ for all $i \neq j$. Applying these automorphisms to a generator $v = v_1e_1 + \cdots + v_re_r$ of $l^{(4)}_{r-1}$, normalize v so that one of the following holds:

- $v_{i_1} = 0$; or
- $v_{i_1} \neq 0$ and $v_{i'} = 0$ for any $i' \neq i_1$; or
- $v_{i_1} = v_{i'} \neq 0$ for some $i' \neq i_1$.

So there is a parameter space of $l^{(4)}_{r-1}$ whose dimension is at most $r-1-1 = r-2$.

(ii) Consider the case where $w_i = 1$ for every i. Then there are some $i_1 < i_2$ with $\sigma(i_1) < \sigma(i_2)$, because $\dim B'' \geqslant 2$. So there are automorphisms (a_{ij}) in B'' of the form $a_{\sigma(i_1)\sigma(i_1)} = c \in k^\times \setminus \{1\}$, $a_{\sigma(i_1)\sigma(i_2)} = 1-c$, $a_{ii} = 1$ for $i \neq \sigma(i_1)$ and $a_{ij} = 0$ if $i \neq j$ and $(i, j) \neq (\sigma(i_1), \sigma(i_2))$. Applying these automorphisms to a generator $v = v_1e_1 + \cdots + v_re_r$ of $l^{(4)}_{r-1}$, normalize v so that one of the following holds:

- $v_{\sigma(i_2)} = 0$; or
- $v_{\sigma(i_1)} = v_{\sigma(i_2)} \neq 0$; or
- $v_{\sigma(i_1)} = 0$, $v_{\sigma(i_2)} \neq 0$.

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So there is a parameter space of $l^{(4)}_{r-1}$ whose dimension is at most $r-1-1 = r-2$. In both cases A(I)(i) and A(I)(ii), consider the group of automorphisms

$$B''' := \left\{ g \in B'' \;\middle|\; \mathbf{g} \text{ fixes } \mathrm{l}^{(1)}_*,\, \mathrm{l}^{(2)}_*,\, \mathrm{l}^{(3)}_{\mathrm{r-1}} \text{ and } \mathrm{l}^{(4)}_{\mathrm{r-1}} \right\}.$$

Since $(E,\, l)$ is not simple, there is an automorphism $\mathbf{g}$ in B''' other than a scalar endomorphism. Then the parameter space of $l^{(5)}_{r-1}$ preserved by $\mathbf{g}$ is of dimension at most $r-1-1$. Thus, there is a parameter space of $l^{(1)}_*, \ldots, l^{(n)}_*$ whose dimension is at most

$$\sum_{j=1}^{r-2} j + 2\left(r-2+\sum_{j=1}^{r-2} j\right) + (n-5)\sum_{j=1}^{r-1} j = -r^2 + 1 + \frac{1}{2}r(r-1)n - 2.$$

(II) Assume that $n = 4$ and $r \geqslant 3$.

(i) Assume that $w_{i_1} = 0$ for some i_1. Then there are automorphisms (a_{ij}) in B'' of the form $a_{ii} = 1 \in k^\times$ for $i \neq i_1$, $a_{i_1 i_1} = c \in k^\times$ and $a_{ij} = 0$ for all $i \neq j$. For a representative $v = v_1e_1 + \cdots + v_re_r \in l^{(3)}_{r-2}$ of a generator of $l^{(3)}_{r-2}/l^{(3)}_{r-1}$, we may assume, after adding an element of $l^{(3)}_{r-1}$, that $v_{i_2} = 0$ for some $i_2 \neq i_1$. Applying an automorphism in B'' of the above form, normalize v so that one of the following holds:

(1) $v_{i_1} = 0$; or
(2) $v_{i_1} \neq 0$ and $v_{i'} = 0$ for any $i' \neq i_1$; or
(3) $v_{i_1} = v_{i_3} \neq 0$ for some $i_3 \neq i_1,\, i_2$.

So there is a parameter space of $l^{(3)}_{r-2}$ whose dimension is at most $r-2-1 = r-3$.

(ii) Assume that $w_i = 1$ for any i. Then there are some $i_1 < i_2$ with $\sigma(i_1) < \sigma(i_2)$ because $B'' \neq k^\times \mathrm{id}$. Then there are automorphisms (a_{ij}) in B'' of the form $a_{\sigma(i_1)\sigma(i_1)} = c \in k^\times \setminus \{1\}$, $a_{\sigma(i_1)\sigma(i_2)} = 1-c$, $a_{ii} = 1$ for $i \neq \sigma(i_1)$ and $a_{ij} = 0$ if $i \neq j$ and $(i,\, j) \neq (\sigma(i_1),\, \sigma(i_2))$. For a representative $v = v_1e_1 + \cdots + v_re_r \in l^{(3)}_{r-2}$ of a generator of $l^{(3)}_{r-2}/l^{(3)}_{r-1}$, we may assume, after adding an element of $l^{(3)}_{r-1}$, that $v_{i'} = 0$ for some $i' \neq \sigma(i_1),\, \sigma(i_2)$. Applying an automorphism in B'', normalize v so that one of the following holds:

- $v_{\sigma(i_1)} = v_{\sigma(i_2)}$; or
- $v_{\sigma(i_1)} v_{\sigma(i_2)} = 0$.

So there is a parameter space of $l^{(3)}_{r-2}$ whose dimension is at most $r-2-1 = r-3$. In both cases A(II)(i) and A(II)(ii), consider the group of automorphisms

$$B''' := \left\{ \mathbf{g} \in B'' \;\middle|\; \mathrm{g} \text{ fixes } \mathrm{l}^{(1)}_*,\, \mathrm{l}^{(2)}_*,\, \mathrm{l}^{(3)}_{\mathrm{r-1}} \text{ and } \mathrm{l}^{(3)}_{\mathrm{r-2}} \right\}.$$

Since $(E,\, l)$ is not simple, there is an automorphism $\mathbf{g}$ in B''' other than a scalar automorphism. The parameter space of $l^{(4)}_{r-1}$ preserved by $\mathbf{g}$ is of dimension at most $r-1-1 = r-2$. Thus, there is a parameter space of $l^{(1)}_*, \ldots, l^{(n)}_*$ whose dimension is at most

$$(r-3) + \sum_{j=1}^{r-3} j + (r-2) + \sum_{j=1}^{r-2} j + \frac{1}{2}r(r-1)(n-4) = -r^2 - 1 + \frac{1}{2}r(r-1)n.$$

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Case B. Consider the case where $E \cong \mathcal{O}_{\mathbb{P}^1}(a_1)^{\oplus r_1} \oplus \cdots \oplus \mathcal{O}_{\mathbb{P}^1}(a_m)^{\oplus r_m}$ with $a_1 < a_2 < \cdots < a_m$ and $l^{(i)}_{r-1} \subset (\mathcal{O}_{\mathbb{P}^1}(a_2)^{\oplus r_2} \oplus \cdots \oplus \mathcal{O}_{\mathbb{P}^1}(a_m)^{\oplus r_m})|_{x_i}$ for any i.

We choose a basis $e^{(i)}_{1,1}, \cdots, e^{(i)}_{1,r_1}, e^{(i)}_{2,1}, \cdots, e^{(i)}_{2,r_2}, \cdots, e^{(i)}_{m,1}, \cdots, e^{(i)}_{m,r_m}$ of $E|_{x_i}$ corresponding to the given decomposition $E|_{x_i} = \mathcal{O}_{\mathbb{P}^1}(a_1)|^{\oplus r_1}_{x_i} \oplus \cdots \oplus \mathcal{O}_{\mathbb{P}^1}(a_m)|^{\oplus r_m}_{x_i}$. For $1 \leqslant p \leqslant m$, let

$$\pi^{(i)}_p : \ E|_{x_i} = \mathcal{O}_{\mathbb{P}^1}(a_1)^{\oplus r_1}|_{x_i} \oplus \cdots \oplus \mathcal{O}_{\mathbb{P}^1}(a_m)^{\oplus r_m}|_{x_i} \longrightarrow \mathcal{O}_{\mathbb{P}^1}(a_p)^{\oplus r_p}|_{x_i}$$

be the projection to the p-th direct summand. So any element $v \in E|_{x_i}$ can be uniquely written as follows:

$$v = v_1 + \cdots + v_m \qquad (v_p \in \mathcal{O}_{\mathbb{P}^1}(a_p)^{\oplus r_p}|_{x_i} \text{ for } 1 \leqslant p \leqslant m).$$

We want to choose suitable generators $v^{(i)}_{p^{(i)}(1),\, j^{(i)}(1)}, \cdots, v^{(i)}_{p^{(i)}(s),\, j^{(i)}(s)}$ of $l^{(i)}_{r-s}$. First, define a number $p^{(i)}(1)$ with $1 \leqslant p^{(i)}(1) \leqslant m$ by setting

$$p^{(i)}(1) := \min \left\{ p \in \{1, \ldots, m\} \;\middle|\; \begin{array}{l} \text{the p}-\text{th component } \mathrm{v_p} \text{ does not vanish} \\ \text{for a generator } \mathrm{v} = \mathrm{v_1} + \cdots + \mathrm{v_m} \text{ of } \mathrm{l^{(i)}_{r-1}} \end{array} \right\}$$

for each $1 \leqslant i \leqslant n$. So, we can choose an element $v = v_{p^{(i)}(1)} + v_{p^{(i)}(1)+1} + \cdots + v_m \in l^{(i)}_{r-1}$ with $v_{p^{(i)}(1)} \neq 0$. Put $j^{(i)}(1) = 1$ and set $v^{(i)}_{p^{(i)}(1), j^{(i)}(1)} := v$. Consider the projection

$$\begin{aligned} \pi^{(i)}_1 \times \cdots \times \pi^{(i)}_p : \ E|_{x_i} &= \mathcal{O}_{\mathbb{P}^1}(a_1)^{\oplus r_1}|_{x_i} \oplus \cdots \oplus \mathcal{O}_{\mathbb{P}^1}(a_m)^{\oplus r_m}|_{x_i} \\ &\longrightarrow \mathcal{O}_{\mathbb{P}^1}(a_1)^{\oplus r_1}|_{x_i} \oplus \cdots \oplus \mathcal{O}_{\mathbb{P}^1}(a_p)^{\oplus r_p}|_{x_i} \end{aligned}$$

for $1 \leqslant p \leqslant m$. For $2 \leqslant s \leqslant r-1$, define $p^{(i)}(s)$, $j^{(i)}(s)$ and $v^{(i)}_{p^{(i)}(s), j^{(i)}(s)}$ inductively on s. For each integer s with $2 \leqslant s \leqslant r-1$, define $p^{(i)}(s)$ by the condition

$$\begin{cases} (\pi^{(i)}_1 \times \cdots \times \pi^{(i)}_{p^{(i)}(s)})(l^{(i)}_{r-s+1}) \subsetneq (\pi^{(i)}_1 \times \cdots \times \pi^{(i)}_{p^{(i)}(s)})(l^{(i)}_{r-s}) & \text{and} \\ (\pi^{(i)}_1 \times \cdots \times \pi^{(i)}_p)(l^{(i)}_{r-s+1}) = (\pi^{(i)}_1 \times \cdots \times \pi^{(i)}_p)(l^{(i)}_{r-s}) & \text{for } \mathrm{p} < \mathrm{p^{(i)}(s)}. \end{cases}$$

Set

$$j^{(i)}(s) := 1 + \# \left\{ s' \in \{1, \ldots, s-1\} \;\middle|\; p^{(i)}(s') = p^{(i)}(s) \right\}.$$

Then we can take an element $v^{(i)}_{p^{(i)}(s), j^{(i)}(s)}$ of $l^{(i)}_{r-s}$ such that

$$(\pi^{(i)}_1 \times \cdots \times \pi^{(i)}_{p^{(i)}(s)})(v^{(i)}_{p^{(i)}(s), j^{(i)}(s)}) \notin (\pi^{(i)}_1 \times \cdots \times \pi^{(i)}_{p^{(i)}(s)})(l^{(i)}_{r-s+1}).$$

By the construction, it follows that $l^{(i)}_{r-s}$ is generated by $v^{(i)}_{p^{(i)}(1), j^{(i)}(1)}, \cdots, v^{(i)}_{p^{(i)}(s), j^{(i)}(s)}$.

Applying an automorphism of E given by an element of

$$B := \left\{ g = (a^{pq})_{1 \leqslant p,q \leqslant m} \;\middle|\; \begin{array}{l} a^{pq} = (a^{pq}_{jj'}) \in \mathrm{Hom}(\mathcal{O}_{\mathbb{P}^1}(a_q)^{\oplus r_q}, \mathcal{O}_{\mathbb{P}^1}(a_p)^{\oplus r_p}) \textit{ for } p \geqslant q, \\ a^{pp} = (a^{pp}_{jj'}) \in \mathrm{Aut}(\mathcal{O}_{\mathbb{P}^1}(a_p)^{\oplus r_p}) \textit{ for } 1 \leqslant p \leqslant m \\ \text{and } a^{pq} = 0 \text{ for } p < q \end{array} \right\},$$

we may assume that $v^{(1)}_{p,j} = e^{(1)}_{p,j}$ for $1 \leqslant p \leqslant m$ and $1 \leqslant j \leqslant r_p$. Note that the group of automorphisms of E fixing $l^{(1)}_*$ is

$$B' = \left\{ g = (a^{pq}) \in B \;\middle|\; \begin{array}{l} a^{p^{(1)}(s),\, p^{(1)}(s')}_{j^{(1)}(s),\, j^{(1)}(s')}|_{x_1} = 0 \text{ for } s > s' \text{ and for each } 1 \leqslant p \leqslant m, \\ (a^{pp}_{jj'}|_{x_1}) \in \mathrm{Aut}(\mathcal{O}_{\mathbb{P}^1}(a_p)^{\oplus r_p}|_{x_1}) \text{ is an upper triangular matrix} \end{array} \right\}.$$

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If $p > q$, we can always take an element $g = (a^{pq}_{jj'})$ of B' such that $a^{pq}_{jj'}|_{x_2} \neq 0$. So, after applying an automorphism in B' to $l^{(2)}_*$, it may be assumed that the condition $v^{(2)}_{p,j} = e^{(2)}_{p,\sigma_p(j)}$ holds for $1 \leqslant j \leqslant r_p$, where σ_p is a permutation of $\{1, \cdots, r_p\}$. The generator $v^{(3)}_{p^{(3)}(1),p^{(3)}(1)}$ of $l^{(3)}_{r-1}$ can be written as

$$v^{(3)}_{p^{(3)}(1),j^{(3)}(1)} = w_{1,1}e^{(3)}_{1,1} + \cdots + w_{m,r_m}e^{(3)}_{m,r_m}.$$

Consider the diagonal automorphisms $\mathbf{g} = (a^{pq}_{jj'})$ of E given by $a^{pq}_{jj'} = 0$ for $(p, j) \neq (q, j')$ and $a^{pp}_{jj} \in k^\times$ for any (p, j). After applying such automorphisms, normalize $v^{(3)}_{1,1}$ such that either $w_{p,j} = 1$ holds or $w_{p,j} = 0$ holds for any p, j. Note that the conditions $p^{(1)}(1) \geqslant 2$, $p^{(2)}(1) \geqslant 2$ and $w_{1,1} = 0$ hold, because of the assumption that $l^{(i)}_{r-1} \subset (\mathcal{O}_{\mathbb{P}^1}(a_2)^{\oplus r_2} \oplus \cdots \oplus \mathcal{O}_{\mathbb{P}^1}(a_m)^{\oplus r_m})|_{x_i}$ for $i = 1, 2, 3$.

(I) If $n \geqslant 5$, then we can give a parameter space of $l^{(i)}_{r-1}$ whose dimension is at most $r-2$ for each $4 \leqslant i \leqslant n$, because $l^{(i)}_{r-1} \subset (\mathcal{O}_{\mathbb{P}^1}(a_2)^{\oplus r_2} \oplus \cdots \oplus \mathcal{O}_{\mathbb{P}^1}(a_m)^{\oplus r_m})|_{x_i}$. So there is a parameter space of (E, l) whose dimension is at most

$$\begin{aligned}\sum_{j=1}^{r-2} j + (n-3)\left((r-2) + \sum_{j=1}^{r-2} j\right) &= \frac{1}{2}r(r-1)(n-2) - (r-1) - (n-3)\\ &= -r^2 + 1 + \frac{1}{2}r(r-1)n - (n-3) \leqslant -r^2 + 1 + \frac{1}{2}r(r-1)n - 2.\end{aligned}$$

(II) Assume that $n = 4$ and $r \geqslant 3$. Recall that $v^{(3)}_{p^{(3)}(1),p^{(3)}(1)}$ is a generator of $l^{(3)}_{r-1}$ and we can write $v^{(3)}_{p^{(3)}(1),p^{(3)}(1)} = \sum_{p,j} w_{p,j}e^{(3)}_{p,j}$ with $w_{1,1} = 0$. Take a representative $u = \sum_{p,j} u_{p,j}e^{(3)}_{p,j} \in l^{(3)}_{r-2}$ of a generator of $l^{(3)}_{r-2}/l^{(3)}_{r-1}$ with the normalized condition $u_{p^{(3)}(1),j^{(3)}(1)} = 0$. Consider the diagonal automorphisms $\mathbf{g} = (a^{pq}_{jj'})$ of E determined by $a^{pq}_{jj'} = 0$ for $(p, j) \neq (q, j')$, $a^{11}_{1,1} = c \in k^\times$ and $a^{p,p}_{j,j} = 1 \in k^\times$ for $(p, j) \neq (1, 1)$. Then such automorphisms preserve $l^{(1)}_*$, $l^{(2)}_*$ and $l^{(3)}_{r-1}$. Choose an index $(q, j') \neq (p^{(3)}(1), j^{(3)}(1)), (1, 1)$. Applying automorphisms of the above form to u, we may assume that one of the following holds:

(a) $u_{1,1} = 0$; or
(b) $u_{q,j'} = 0$; or
(c) $u_{1,1} = u_{q,j'} \neq 0$.

So we can give a parameter space of such $l^{(3)}_{r-2}$ whose dimension is at most $r-3$. Furthermore, the parameter space of $l^{(4)}_{r-1}$ is at most $r-2$, because of the condition $l^{(4)}_{r-1} \subset (\mathcal{O}_{\mathbb{P}^1}(a_2)^{\oplus r_2} \oplus \cdots \oplus \mathcal{O}_{\mathbb{P}^1}(a_m)^{\oplus r_m})|_{x_4}$. So we can give a parameter space of (E, l) whose dimension is at most

$$r - 3 + \sum_{j=1}^{r-3} j + (r-2) + \sum_{j=1}^{r-2} j = r^2 - 2r - 1,$$

which is equal to $-r^2 + 1 + r(r-1)n/2 - 2$ as $n = 4$. □

Proposition 5.13. *Assume that $X = \mathbb{P}^1_k$, $n = 3$, $r \geqslant 4$ and L is a line bundle on $\mathbb{P}^1_k$. Then there exists a scheme Z of finite type over* $\operatorname{Spec} k$, *and a flat family $(\widetilde{E}, \widetilde{\boldsymbol{l}})$ of quasi-parabolic bundles on $\mathbb{P}^1 \times Z$ over Z, such that:*

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(a) $\dim Z \leqslant (r^2 - 3r + 2)/2 - 2$;

(b) $\dim \operatorname{End}((\widetilde{E}, \widetilde{\boldsymbol{l}})|_{\mathbb{P}^1 \times \{z\}}) \geqslant 2$ *for any* $z \in Z$;

and each member of $|\mathcal{N}_{\mathrm{par}}^{n_0\text{-reg}}(L)| \setminus |\mathcal{N}_{\mathrm{par}}^{n_0\text{-reg}}(L)^{\circ}|$ *is isomorphic to* $(\widetilde{E}, \widetilde{\boldsymbol{l}})|_{\mathbb{P}^1 \times \{z\}}$ *for some point* $z \in Z$.

Proof. First we fix a universal constant

$$\lambda_0 \in k \setminus \{0, 1\}. \tag{5.14}$$

As in the proof of Proposition 5.12, we may assume that the quasi-parabolic bundles $(E, \boldsymbol{l})$ satisfy one of the following conditions:

(A) $E \cong \mathcal{O}_{\mathbb{P}^1}^{\oplus r}$; or

(B) $E = \mathcal{O}_{\mathbb{P}^1}(a_1)^{\oplus r_1} \oplus \cdots \oplus \mathcal{O}_{\mathbb{P}^1}(a_m)^{\oplus r_m}$ and $l^{(i)}_{r-1} \subset \mathcal{O}_{\mathbb{P}^1}(a_2)^{\oplus r_2}|_{x_i} \oplus \cdots \oplus \mathcal{O}_{\mathbb{P}^1}(a_m)^{\oplus r_m}|_{x_i}$ for any i, where $a_1 < a_2 < \cdots < a_m$.

Case A. First, consider the case where $E \cong \mathcal{O}_{\mathbb{P}^1}^{\oplus r}$.

As in the proof of Proposition 5.12, we may assume that $l^{(1)}_*$ is determined by the standard basis $e_1, \cdots, e_r$ and $l^{(2)}_*$ is determined by the basis $e_{\sigma(1)}, \cdots, e_{\sigma(r)}$ for a permutation σ of $\{1, \cdots, r\}$ while $l^{(3)}_{r-1}$ is generated by $w = w_1 e_1 + \ldots + w_r e_r$ with $w_i = 1$ or $w_i = 0$ for each i. Consider the following three cases:

(a) $w_{i_1} = w_{i_2} = 0$ for some $i_1 \neq i_2$;

(b) $w_{i_1} = 0$ for some i_1 and $w_i = 1$ for any $i \neq i_1$;

(c) $w_i = 1$ for any i.

(a) Assume that $w_{i_1} = w_{i_2} = 0$ for $i_1 \neq i_2$. Fix indices i_3, i_4 such that $w_{i_3} = 1$ and $i_4 \neq i_1, i_2, i_3$. Consider the automorphisms (a_{ij}) of E satisfying $a_{i_1 i_1} = c_{i_1} \in k^{\times}$, $a_{i_2 i_2} = c_{i_2} \in k^{\times}$, $a_{ii} = 1$ for $i \neq i_1, i_2$ and $a_{ij} = 0$ for $i \neq j$. Then such automorphisms preserve $l^{(1)}_*$, $l^{(2)}_*$ and $l^{(3)}_{r-1}$. Normalize a representative $v = v_1 e_1 + \ldots + v_r e_r \in l^{(3)}_{r-2}$ of a generator of $l^{(3)}_{r-2}/l^{(3)}_{r-1}$ such that $v_{i_3} = 0$ after adding an element of $l^{(3)}_{r-1}$. Applying the above type of automorphisms to v, we can assume that one of the following statements holds:

(i) $v_{i_1} = v_{i_2} = 0$;
(ii) $v_{i_1} = v_{i_4} = 0$;
(iii) $v_{i_2} = v_{i_4} = 0$;
(iv) $v_{i_1} = 0$ and $v_{i_2} = v_{i_4} \neq 0$;
(v) $v_{i_2} = 0$ and $v_{i_1} = v_{i_4} \neq 0$;
(vi) $v_{i_1} = v_{i_2} \neq 0$ and $v_{i_4} = 0$;
(vii) $v_{i_1} = v_{i_2} = v_{i_4} \neq 0$.

So there is a parameter space of $l^{(3)}_{r-2}$ whose dimension is at most $r - 2 - 2 = r - 4$. Adding the data of $l^{(3)}_{r-3}, \cdots, l^{(3)}_1$, we can get a parameter space of $(E, \boldsymbol{l})$ whose dimension is at most

$$(r-4) + \sum_{j=1}^{r-3} j \;=\; \frac{r^2 - 3r + 2}{2} - 2.$$

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(b) Assume that $w_{i_1} = 0$ for some i_1 and $w_i = 1$ for any $i \neq i_1$. Fix an index i_2 other than i_1. For a representative $v = v_1 e_1 + \cdots + v_r e_r \in l^{(3)}_{r-2}$ of a generator of $l^{(3)}_{r-2}/l^{(3)}_{r-1}$, we may assume, after adding an element of $l^{(3)}_{r-1}$, that $v_{i_2} = 0$. Then we have one of the following three cases:

(i) $v_{i_1} = 0$;
(ii) $v_{i_1} \neq 0$ and $v_i = 0$ for any $i \neq i_1, i_2$;
(iii) $v_{i_1} \neq 0$ and $v_{i_3} \neq 0$ for some i_3 with $i_3 \neq i_1$ and $i_3 \neq i_2$.

(i) Consider the case where $v_{i_1} = 0$. Then we can give a parameter space of $l^{(3)}_{r-2}$ whose dimension is at most $r-2-1 = r-3$. Consider the automorphisms $\mathbf{g} = (a_{ij})$ of E given by $a_{i_1 i_1} = c \in k^{\times}$, $a_{ii} = 1$ for $i \neq i_1$ and $a_{ij} = 0$ for $i \neq j$. Then such automorphisms preserve $l^{(1)}_*$, $l^{(2)}_*$, $l^{(3)}_{r-1}$ and also $l^{(3)}_{r-2}$. Since $v \neq 0$, we may choose an index i_3 such that $v_{i_3} \neq 0$ and $i_3 \neq i_1, i_2$. For a representative $u = u_1 e_1 + \cdots + u_r e_r \in l^{(3)}_{r-3}$ of a generator of $l^{(3)}_{r-3}/l^{(3)}_{r-2}$, we may assume, after adding an element in $l^{(3)}_{r-2}$, that $u_{i_2} = u_{i_3} = 0$. After applying the above type of automorphisms to u, we may assume that one of the following statements holds:

- $u_{i_1} = 0$;
- $u_{i_1} \neq 0$ and $u_i = 0$ for $i \neq i_1, i_2, i_3$;
- $u_{i_1} = u_{i_4} \neq 0$ for some $i_4 \neq i_1, i_2, i_3$.

In all these cases, there is a parameter space of $l^{(3)}_{r-3}$ whose dimension is at most $r-3-1 = r-4$. Adding the data of $l^{(3)}_{r-4}, \cdots, l^{(3)}_1$, we can get a parameter space of $(E, \boldsymbol{l})$ whose dimension is at most

$$r-3+r-4+\sum_{j=1}^{r-4} j \;=\; \frac{r^2-3r+2}{2} - 2.$$

(ii) Consider the case where $v_{i_1} \neq 0$ and $v_i = 0$ for any $i \neq i_1, i_2$. Then $l^{(3)}_{r-2}$ is uniquely determined. So the dimension of the parameter space of such $(E, \boldsymbol{l})$ is at most

$$(r-3)+\sum_{j=1}^{r-4} j \;=\; \frac{r^2-3r+2}{2} - r + 2 \leqslant \frac{r^2-3r+2}{2} - 2.$$

(iii) Consider the case where $v_{i_1} \neq 0$ and $v_{i_3} \neq 0$ for some i_3 with $i_3 \neq i_1$ and $i_3 \neq i_2$. Recall again that we normalize a representative $v \in l^{(3)}_{r-2}$ of a generator of $l^{(3)}_{r-2}/l^{(3)}_{r-1}$ such that $v_{i_2} = 0$. Suppose that the condition $\sigma(i) > \sigma(j)$ holds for any $i < j$. Then the automorphisms of E preserving $l^{(1)}_*$ and $l^{(2)}_*$ are only diagonal automorphisms $\mathbf{g} = (a_{ij})$, which satisfy the condition $a_{ij} = 0$ for $i \neq j$. If $\mathbf{g} = (a_{ij})$ preserves $l^{(3)}_{r-1} = \langle w \rangle$ and $l^{(3)}_{r-2} = \langle w, v \rangle$ in addition, then we have $a_{ii} = a_{jj}$ for $i, j \neq i_1$ and $a_{i_1 i_1} = a_{i_3 i_3}$. So $\mathbf{g}$ must be a constant scalar multiplication, which contradicts the assumption that $\dim\ \mathrm{Aut}(E, \boldsymbol{l}) \geqslant 2$. Thus, we have the following:

- There are $i_0 < j_0$ satisfying $\sigma(i_0) < \sigma(j_0)$.

So, consider the following cases:

(α) $\sigma(j_0) = i_1$ and $\sigma(i_0) = i_2$;
(β) $\sigma(j_0) = i_1$ and $\sigma(i_0) \neq i_2$;

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(γ) $\sigma(j_0) \neq i_1$ and $\sigma(i_0) = i_2$;
(δ) $\sigma(j_0) \neq i_1$ and $\sigma(i_0) \neq i_1, i_2$;
(ϵ) $\sigma(i_0) = i_1$ and $\{j \in \{1, \cdots, r\} \mid j > i, \sigma(j) > \sigma(i)\} = \emptyset$ for any $i \neq i_0$.

More precisely, in the remaining case other than (α), (β), (γ) and (δ), we have $\sigma(i_0) = i_1$. If there are $i' \neq i_0$ and $j' > i'$ satisfying $\sigma(j') > \sigma(i')$, then we replace (i_0, j_0) with (i', j') and reduce to the case (α), (β), (γ) or (δ). Otherwise, we may assume (ϵ).
(α) Assume that $\sigma(i_0) = i_2$ and $\sigma(j_0) = i_1$. Consider the automorphisms $\mathbf{g} = (a_{ij})$ of E given by $a_{i_1 i_1} = c \in k^\times$, $a_{\sigma(i_0)\sigma(j_0)} = a_{i_2 i_1} = a \in k$, $a_{ii} = 1$ for $i \neq i_1 = \sigma(j_0)$ and $a_{ij} = 0$ for $i \neq j$ satisfying $(i, j) \neq (i_2, i_1)$. Then such automorphisms preserve $l_*^{(1)}$, $l_*^{(2)}$ and $l_{r-1}^{(3)}$. The coefficient of e_{i_2} in

$$gv = v_1 e_1 + \cdots + c v_{\sigma(j_0)} e_{i_1} + \cdots + (v_{\sigma(i_0)} + a v_{\sigma(j_0)}) e_{i_2} + \cdots + v_{i_3} e_{i_3} + \cdots + v_r e_r$$

is $v_{\sigma(i_0)} + a v_{\sigma(j_0)} = a v_{\sigma(j_0)}$ because of $v_{\sigma(i_0)} = v_{i_2} = 0$, and hence the normalized representative of a generator of $l_{r-2}^{(3)}/l_{r-1}^{(3)}$ becomes

$$\begin{aligned} &gv - a v_{\sigma(j_0)} w \\ &= (v_1 - a v_{\sigma(j_0)}) e_1 + \cdots + c v_{\sigma(j_0)} e_{i_1} + \cdots + 0 e_{i_2} + (v_{i_3} - a v_{\sigma(j_0)}) e_{i_3} + \cdots + (v_r - a v_{\sigma(j_0)}) e_r. \end{aligned}$$

If we choose an index i_4 other than i_1, i_2, i_3, we may assume that one of the following two holds:

(a) $v_{i_1} = v_{i_3} = v_{i_4} \neq 0$;
(b) $v_{i_1} = v_{i_3} \neq 0$ and $v_{i_4} = 0$.

So we can give a parameter space for such $l_{r-2}^{(3)}$ whose dimension is at most $r - 4$.
(β) Assume that $\sigma(j_0) = i_1$ and $\sigma(i_0) \neq i_2$. Consider the automorphisms $\mathbf{g} = (a_{ij})$ of E given by $a_{i_1 i_1} = c \in k^\times$, $a_{\sigma(i_0)\sigma(j_0)} = a \in k$, $a_{ii} = 1$ for $i \neq i_1 = \sigma(j_0)$ and $a_{ij} = 0$ for $i \neq j$ satisfying $(i, j) \neq (\sigma(i_0), \sigma(j_0))$. Then such automorphisms preserve $l_*^{(1)}$, $l_*^{(2)}$ and $l_{r-1}^{(3)} = \langle w \rangle$. Since

$$\mathbf{g}v = v_1 e_1 + \cdots + (v_{\sigma(i_0)} + a v_{\sigma(j_0)}) e_{\sigma(i_0)} + \cdots + c v_{\sigma(j_0)} e_{i_1} + \cdots + 0 e_{i_2} + \cdots + v_r e_r,$$

we may assume that one of the following holds:

(a) $\sigma(i_0) \neq i_3$ and $v_{\sigma(i_0)} = v_{\sigma(j_0)} = v_{i_3} \neq 0$;
(b) $\sigma(i_0) = i_3$ and $v_{\sigma(i_0)} = v_{\sigma(j_0)} = v_{i_4} \neq 0$ for some i_4 other than i_1, i_3, i_2;
(c) $\sigma(i_0) = i_3$, $v_{\sigma(i_0)} = v_{\sigma(j_0)}$ and $v_i = 0$ for any i other than $i_1 (= \sigma(j_0))$, i_3.

So we can give a parameter space of such $l_{r-2}^{(3)}$ whose dimension is at most $r - 4$.
(γ) Assume that $\sigma(j_0) \neq i_1$ and $\sigma(i_0) = i_2$. In this case, consider the automorphisms $\mathbf{g} = (a_{ij})$ of the form $a_{i_1 i_1} = c \in k^\times$, $a_{i_2 i_2} = a \in k^\times \setminus \{1\}$, $a_{i_2 \sigma(j_0)} = 1 - a$, $a_{ii} = 1$ for $i \neq i_1, i_2$ and $a_{ij} = 0$ for any $i \neq j$ satisfying $(i, j) \neq (i_2, \sigma(j_0))$. Then such a $\mathbf{g}$ preserves $l_*^{(1)}$, $l_*^{(2)}$ and $l_{r-1}^{(3)}$. Since the e_{i_2}-coefficient of

$$\mathbf{g}v = v_1 + \cdots + c v_{i_1} e_{i_1} + \cdots + (a v_{i_2} + (1-a) v_{\sigma(j_0)}) e_{i_2} + \cdots + v_{i_3} e_{i_3} + \cdots + v_r e_r$$

is $a v_{i_2} + (1-a) v_{\sigma(j_0)} = (1-a) v_{\sigma(j_0)}$, we should replace $\mathbf{g}v$ with its normalization

$$\begin{aligned} \mathbf{g}v - (1-a) v_{\sigma(j_0)} w \ &= \ (v_1 - (1-a) v_{\sigma(j_0)}) e_1 + \cdots \\ &\cdots + c v_{i_1} e_{i_1} + \cdots + 0 v_{i_2} + \cdots + a v_{\sigma(j_0)} e_{\sigma(j_0)} + \cdots + (v_r - (1-a) v_{\sigma(j_0)}) e_r. \end{aligned}$$

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Fix an index i_4 other than $\sigma(j_0)$, i_1 and i_2. After applying an automorphism of the above form, we may assume that one of the following holds:

- $\sigma(j_0) \neq i_3$ and $v_{i_1} = v_{i_3} = v_{\sigma(j_0)} \neq 0$;
- $\sigma(j_0) \neq i_3$ and $v_{i_1} = v_{i_3} = \lambda_0 v_{\sigma(j_0)} \neq 0$ (see (5.14) for λ_0);
- $\sigma(j_0) \neq i_3$, $v_{i_1} = v_{i_3} \neq 0$ and $v_{\sigma(j_0)} = 0$;
- $\sigma(j_0) = i_3$ and $v_{i_1} = v_{\sigma(j_0)} = v_{i_4} \neq 0$;
- $\sigma(j_0) = i_3$, $v_{i_1} = v_{\sigma(j_0)}$ and $v_{i_4} = 0$.

So we can give a parameter space of such $l^{(3)}_{r-2}$ whose dimension is at most $r-4$.
(δ) Assume that $\sigma(j_0) \neq i_1$ and $\sigma(i_0) \neq i_1, i_2$. Consider the automorphisms $\mathbf{g} = (a_{ij})$ of E given by $a_{i_1 i_1} = c \in k^\times$, $a_{\sigma(i_0)\sigma(i_0)} = a \in k^\times$, $a_{\sigma(i_0),\sigma(j_0)} = 1-a$, $a_{ii} = 1$ for $i \neq i_1, \sigma(i_0)$ and $a_{ij} = 0$ for $i \neq j$ satisfying $(i, j) \neq (\sigma(i_0), \sigma(j_0))$. Then such automorphisms preserve $l^{(1)}_*$, $l^{(2)}_*$ and $l^{(3)}_{r-1}$, and we have

$$\mathbf{g}v = v_1 e_1 + \cdots + c v_{i_1} e_{i_1} + \cdots + (a v_{\sigma(i_0)} + (1-a) v_{\sigma(j_0)}) e_{\sigma(i_0)} + \cdots + 0 e_{i_2} + \cdots + v_r e_r.$$

In the case where $v_{\sigma(i_0)} = v_{\sigma(j_0)}$, we can normalize v so that the condition $v_{i_1} = v_{i_3}$ holds. In the case where $v_{\sigma(i_0)} \neq v_{\sigma(j_0)}$, we can normalize v so that one of the following holds:

(1) $\sigma(i_0) \neq i_3$ and $v_{i_1} = v_{\sigma(i_0)} = v_{i_3} \neq 0$;
(2) $\sigma(i_0) = i_3$ and $v_{i_1} = v_{\sigma(i_0)} = v_{i_4} \neq 0$ for some i_4 other than i_1, i_2, i_3;
(3) $\sigma(i_0) = i_3$ and $v_{i_1} = v_{\sigma(i_0)} = \lambda_0 v_{i_4} \neq 0$ for some i_4 other than i_1, i_2, i_3 (see (5.14) for λ_0);
(4) $\sigma(i_0) = i_3$, $v_{i_1} = v_{\sigma(i_0)} \neq 0$ and $v_{i_4} = 0$ for some i_4 other than i_1, i_2, i_3.

So we can give a parameter space of such $l^{(3)}_{r-2}$ whose dimension is at most $r-4$.
(ϵ) Assume that $\sigma(i_0) = i_1$ and that $\{j > i \mid \sigma(j) > \sigma(i)\} = \emptyset$ for all $i \neq i_0$. Then the group of automorphisms of E preserving $l^{(1)}_*$, $l^{(2)}_*$ and $l^{(3)}_{r-1} = \langle w \rangle$ becomes

$$B'' = \left\{ g = \begin{pmatrix} a_{11} & \cdots & a_{1r} \\ 0 & \ddots & \vdots \\ 0 & 0 & a_{rr} \end{pmatrix} \;\middle|\; \begin{array}{l} a_{ij} = 0 \; for \; i \neq j \text{ satisfying } i \neq i_1 \\ a_{i_1 i_1} \in k^\times, \; a_{ii} = c \in k^\times \text{ for } i \neq i_1 \text{ and} \\ \displaystyle\sum_{j > i_0,\, \sigma(j) > \sigma(i_0) = i_1} a_{i_1 \sigma(j)} = 0 \end{array} \right\}.$$

Suppose that for any index j_1 satisfying $j_1 \neq j_0$ and $i_0 < j_1$, we have $\sigma(i_0) > \sigma(j_1)$. Then any automorphism $\mathbf{g}$ in B'' becomes diagonal. In other words, $\mathbf{g} = (a_{ij})$ satisfies the following conditions: $a_{ij} = 0$ for $i \neq j$ and there is a $c \in k^\times$ such that $a_{ii} = c$ for $i \neq i_1$. If $\mathbf{g}$ further preserves $l^{(3)}_{r-2}$, then we have $a_{i_1 i_1} = a_{i_3 i_3} = c$, because $v_{i_1} \neq 0$, $v_{i_3} \neq 0$ and $v_{i_2} = 0$. Thus $\mathbf{g}$ must be a constant scalar multiplication, which is a contradiction because (E, l) is not simple.

So there is an index j_1 with $j_1 \neq j_0$ satisfying the conditions $i_0 < j_1$ and $\sigma(i_0) < \sigma(j_1)$. Consider the automorphisms $\mathbf{g} = (a_{ij})$ of the form $a_{i_1 i_1} = c' \in k^\times$, $a_{i_1 \sigma(j_0)} = a = -a_{i_1 \sigma(j_1)} \in k$, $a_{ii} = 1$ for $i \neq i_1$ and $a_{ij} = 0$ for any $i \neq j$ satisfying $(i, j) \neq (i_1, \sigma(j_0)), (i_1, \sigma(j_1))$. Recall that the representative $v = \sum_{i=1}^r v_i e_i$ of a generator of $l^{(3)}_{r-2}/l^{(3)}_{r-1}$ is normalized so that $v_{i_2} = 0$. We further normalize a representative $u = \sum_{i=1}^r u_i e_i \in l^{(3)}_{r-3}$ of a generator of $l^{(3)}_{r-3}/l^{(3)}_{r-2}$ so that $u_{i_2} = u_{i_3} = 0$. We may assume that $\{\sigma(j_0), \sigma(j_1)\} \neq \{i_2, i_3\}$, because otherwise we can replace i_2 or i_3 by another index i_4 other than i_1, i_2, i_3 according to whether $v_{i_4} = 0$ or $v_{i_4} \neq 0$. So assume that $\sigma(j_0) \neq i_2, i_3$. Applying an automorphism $\mathbf{g}$ of the above form to v and u,

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we have

$$\begin{aligned}\mathbf{g}v &= v_1e_1 + \cdots + (c'v_{i_1} + av_{\sigma(j_0)} - av_{\sigma(j_1)})e_{i_1} + \cdots \\ &\quad + 0e_{i_2} + \cdots + v_{\sigma(j_0)}e_{\sigma(j_0)} + \cdots + v_{\sigma(j_1)}e_{\sigma(j_1)} + \cdots + v_re_r, \\ \mathbf{g}u &= u_1e_1 + \cdots + (c'u_{i_1} + au_{\sigma(j_0)} - au_{\sigma(j_1)})e_{i_1} + \cdots \\ &\quad + 0e_{i_2} + \cdots + 0e_{i_3} + \cdots + u_{\sigma(j_0)}e_{\sigma(j_0)} + \cdots + u_{\sigma(j_1)}e_{\sigma(j_1)} + \cdots + u_re_r.\end{aligned}$$

So we may assume that one of the following holds:

(1) $v_{\sigma(j_0)} = v_{\sigma(j_1)}$ and $u_{\sigma(j_0)} = u_{\sigma(j_1)}$;

(2) $v_{\sigma(j_0)} = v_{\sigma(j_1)}$, $u_{\sigma(j_0)} \neq u_{\sigma(j_1)}$ and $u_{i_1} = u_{\sigma(j_0)} - u_{\sigma(j_1)}$;

(3) $u_{\sigma(j_0)} = u_{\sigma(j_1)}$, $v_{\sigma(j_0)} \neq v_{\sigma(j_1)}$ and $v_{i_1} = v_{\sigma(j_0)} - v_{\sigma(j_1)}$;

(4) $v_{\sigma(j_0)} - v_{\sigma(j_1)} \neq 0$, $u_{\sigma(j_0)} - u_{\sigma(j_1)} \neq 0$, $v_{i_1}(u_{\sigma(j_0)} - u_{\sigma(j_1)}) - u_{i_1}(v_{\sigma(j_0)} - v_{\sigma(j_1)}) = 0$ and $v_{i_1} = v_{\sigma(j_0)} - v_{\sigma(j_1)}$;

(5) $v_{\sigma(j_0)} - v_{\sigma(j_1)} \neq 0$, $u_{\sigma(j_0)} - u_{\sigma(j_1)} \neq 0$, $v_{i_1}(u_{\sigma(j_0)} - u_{\sigma(j_1)}) - u_{i_1}(v_{\sigma(j_0)} - v_{\sigma(j_1)}) \neq 0$, $v_{i_1} = v_{\sigma(j_0)} - v_{\sigma(j_1)}$ and $u_{i_1} = \lambda_0(u_{\sigma(j_0)} - u_{\sigma(j_1)})$.

In each of the above cases, we can give a parameter space of $l^{(3)}_{r-2}$ and $l^{(3)}_{r-3}$ whose dimension is at most $r-3+r-4 = 2r-7$.

In all cases of A(b)(iii), we can give a parameter space of $(E, \boldsymbol{l})$ whose dimension is at most

$$(r-2) + (r-3) - 2 + (r-4) + (r-5) + \cdots + 1 = \frac{r^2-3r+2}{2} - 2.$$

(c) Consider the case where $w_\ell = 1$ for any ℓ.

(i) Assume further that there are $i_1 < j_1$ and $i_2 < j_2$ satisfying the conditions $\sigma(i_1) \neq \sigma(i_2)$, $\sigma(i_1) < \sigma(j_1)$ and $\sigma(i_2) < \sigma(j_2)$. Let B'' be the group of automorphisms of E preserving $l^{(1)}_*$, $l^{(2)}_*$ and $l^{(3)}_{r-1}$. Then B'' contains two types of automorphisms (a_{ij}), (b_{ij}) such that:

- $a_{\sigma(i_1)\,\sigma(i_1)} = c \in k^\times$, $a_{\sigma(i_1)\,\sigma(j_1)} = 1-c$, $a_{ii} = 1$ for $i \neq \sigma(i_1)$ and $a_{ij} = 0$ for $i \neq j$ satisfying the condition $(i, j) \neq (\sigma(i_1), \sigma(j_1))$;
- $b_{\sigma(i_2)\,\sigma(i_2)} = c' \in k^\times$, $b_{\sigma(i_2)\,\sigma(j_2)} = 1-c'$, $b_{ii} = 1$ for $i \neq \sigma(i_2)$ and $b_{ij} = 0$ for $i \neq j$ satisfying the condition $(i, j) \neq (\sigma(i_2), \sigma(j_2))$.

For a representative $v = v_1e_1 + \cdots + v_re_r$ of a generator of $l^{(3)}_{r-2}/l^{(3)}_{r-1}$, we may assume, after adding an element of $l^{(3)}_{r-1}$, that $v_{i'_2} = 0$ for some i'_2 such that $i'_2 \neq \sigma(i_1), \sigma(j_1), \sigma(i_2)$. We may further assume that $i'_2 \neq \sigma(j_2)$ if $\sigma(i_1), \sigma(j_1), \sigma(i_2), \sigma(j_2)$ are not distinct. Applying automorphisms of the above type, we may assume that one of the following holds:

(1) $j_1 = j_2$, $v_{\sigma(j_1)} = 0$ and $(v_{\sigma(i_1)} - v_{\sigma(i_2)})v_{\sigma(i_1)}v_{\sigma(i_2)} = 0$;

(2) $j_1 = j_2$, $v_{\sigma(j_1)} \neq 0$ and $(v_{\sigma(j_1)} - v_{\sigma(i_1)})(v_{\sigma(j_1)} - \lambda_0 v_{\sigma(i_1)}) = (v_{\sigma(j_1)} - v_{\sigma(i_2)})(v_{\sigma(j_1)} - \lambda_0 v_{\sigma(i_2)}) = 0$;

(3) $j_1 \neq j_2$ and $v_{\sigma(j_1)}(v_{\sigma(i_1)} - v_{\sigma(j_1)}) = v_{\sigma(j_2)}(v_{\sigma(i_2)} - v_{\sigma(j_2)}) = 0$;

(4) $j_1 \neq j_2$, $0 \neq v_{\sigma(j_1)} = \lambda_0 v_{\sigma(i_1)}$ and $(v_{\sigma(i_2)} - v_{\sigma(j_2)})v_{\sigma(j_2)} = 0$;

(5) $j_1 \neq j_2$, $(v_{\sigma(i_1)} - v_{\sigma(j_1)})v_{\sigma(j_1)} = 0$ and $0 \neq v_{\sigma(j_2)} = \lambda_0 v_{\sigma(i_2)}$;

(6) $j_1 \neq j_2$, $v_{\sigma(i_1)} = \lambda_0 v_{\sigma(j_1)} \neq 0$ and $v_{\sigma(i_2)} = \lambda_0 v_{\sigma(j_2)} \neq 0$.

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So we can give a parameter space for $l^{(3)}_{r-2}$ whose dimension is at most $r-2-2 = r-4$. Adding the data of $l^{(3)}_{r-3}, \cdots, l^{(3)}_1$, we can get a parameter space for $(E, \boldsymbol{l})$ whose dimension is at most

$$(r-4) + \sum_{j=1}^{r-3} j \ = \ \frac{r^2-3r+2}{2} - 2.$$

(ii) Consider the rest case of A-(c). So, there is at most one i_0 such that there is $j > i_0$ for which $\sigma(i_0) < \sigma(j)$. Recall that we assumed that $w_i = 1$ for any i. Then the automorphism group B'' of E preserving $l^{(1)}_*$, $l^{(2)}_*$ and $l^{(3)}_{r-1}$ becomes

$$B'' = \left\{ \mathbf{g} = (a_{ij}) \ \middle| \ \begin{array}{l} \bullet \text{ there is a } c \in \ k^{\times} \text{ such that } a_{ii} = c \ for \ i \neq \sigma(i_0), \\ \bullet \text{ for any } i \neq \sigma(i_0), \ a_{ij} = 0 \text{ for } i \neq j \text{ and} \\ \bullet \ a_{\sigma(i_0)\sigma(i_0)} + \displaystyle\sum_{\substack{j > i_0 \\ \sigma(j) > \sigma(i_0)}} a_{\sigma(i_0)\sigma(j)} = c \end{array} \right\}.$$

Since there are non-scalar automorphisms in B'', there is some $j_0 > i_0$ for which $\sigma(j_0) > \sigma(i_0)$. Choosing i'_2 other than $\sigma(i_0)$ and $\sigma(j_0)$, we can normalize a representative $v = v_1e_1 + \cdots + v_re_r$ of a generator of $l^{(3)}_{r-2}/l^{(3)}_{r-1}$ so that $v_{i'_2} = 0$. Consider the automorphisms $\mathbf{g} = (a_{ij})$ of E given by $a_{ii} = 1$ for $i \neq \sigma(i_0)$, $a_{\sigma(i_0)\sigma(i_0)} = c \in k^{\times}$, $a_{\sigma(i_0)\sigma(j_0)} = 1-c$ and $a_{ij} = 0$ for any $i \neq j$ such that $(i, j) \neq (\sigma(i_0), \sigma(j_0))$. Then such automorphisms preserve $l^{(1)}_*$, $l^{(2)}_*$ and $l^{(3)}_{r-1}$. Choose i'_3 other than $\sigma(i_0)$, $\sigma(j_0)$, i'_2. Since v is sent to

$$v_1e_1 + \cdots + (cv_{\sigma(i_0)} + (1-c)v_{\sigma(j_0)})e_{\sigma(i_0)} + \cdots + v_{\sigma(j_0)}e_{\sigma(j_0)} + \cdots + v_re_r$$

by the automorphism $\mathbf{g}$, we can assume that one of the following holds:

(α) $v_{\sigma(i_0)} = v_{\sigma(j_0)}$;
(β) $v_{\sigma(i_0)} \neq v_{\sigma(j_0)}$ and $v_{\sigma(j_0)} = v_{i'_3}$;
(γ) $v_{\sigma(i_0)} \neq v_{\sigma(j_0)}$, $v_{\sigma(j_0)} \neq v_{i'_3}$ and $v_{\sigma(i_0)} = v_{i'_3}$.

If, in addition, we have $v_{i'_3} = 0$, then we can give a parameter space for such $l^{(3)}_{r-2}$ whose dimension is at most $r-4$. So we assume that $v_{i'_3} \neq 0$.
(α) Assume that the condition $v_{\sigma(i_0)} = v_{\sigma(j_0)}$ holds. Recall that we are assuming that $i'_2 \neq \sigma(i_0), \sigma(j_0)$ and $i'_3 \neq \sigma(i_0), \sigma(j_0), i'_2$. Furthermore, we are normalizing v so that $v_{i'_2} = 0$. Consider the automorphisms $\mathbf{g} = (a_{ij})$ given by $a_{ii} = 1$ for $i \neq \sigma(i_0)$, $a_{\sigma(i_0)\sigma(i_0)} = c \in k^{\times}$, $a_{\sigma(i_0)\sigma(j_0)} = 1-c$ and $a_{ij} = 0$ for any $i \neq j$ such that $(i, j) \neq (\sigma(i_0), \sigma(j_0))$. Then such automorphisms g preserve not only $l^{(1)}_*$, $l^{(2)}_*$ and $l^{(3)}_{r-1}$ but also v. Consider a normalized representative $u = u_1e_1 + \cdots + u_re_r$ of a generator of $l^{(3)}_{r-3}/l^{(3)}_{r-2}$ such that $u_{i'_2} = u_{i'_3} = 0$. Then u is sent to

$$u_1e_1 + \cdots + (cu_{\sigma(i_0)} + (1-c)u_{\sigma(j_0)})e_{\sigma(i_0)} + \cdots + u_{\sigma(j_0)}e_{\sigma(j_0)} + 0u_{i'_2} + 0u_{i'_3} + \cdots + u_re_r$$

by the above automorphism $\mathbf{g}$. Replacing u by some $\mathbf{g}u$, we may assume that one of the following holds:

- $u_{\sigma(i_0)} = u_{\sigma(j_0)}$;
- $u_{\sigma(i_0)} \neq u_{\sigma(j_0)}$ and $u_{\sigma(j_0)} = 0$;
- $u_{\sigma(i_0)} \neq u_{\sigma(j_0)}$, $u_{\sigma(j_0)} \neq 0$ and $u_{\sigma(i_0)} = 0$.

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So we can give a parameter space for $(E,\, \boldsymbol{l})$ whose dimension is at most

$$(r-3)+(r-4)+\sum_{j=1}^{r-4} j \;=\; \frac{r^2-3r+2}{2}-2.$$

(β) Assume that $v_{\sigma(i_0)} \neq v_{\sigma(j_0)}$ and $v_{\sigma(j_0)} = v_{i'_3}$. Recall that we are assuming that $v_{i'_3} \neq 0$. After applying an automorphism in B'', we may assume that $v_{\sigma(i_0)} = \lambda_0 v_{i'_3}$. So we can give a parameter space for such $l^{(3)}_{r-2}$ of dimension at most $r-4$. Then we can give a parameter space for $(E,\, \boldsymbol{l})$ whose dimension is at most

$$(r-4)+\sum_{j=1}^{r-3} j \;=\; \frac{r^2-3r+2}{2}-2.$$

(γ) Assume that $v_{\sigma(i_0)} \neq v_{\sigma(j_0)}$, $v_{\sigma(j_0)} \neq v_{i'_3}$ and $v_{\sigma(i_0)} = v_{i'_3}$. Note that there are non-scalar automorphisms $\mathbf{g} = (a_{ij}) \in B''$ preserving $l^{(3)}_{r-2}$. Recall that there is a $c \in k^\times$ such that $a_{ii} = c$ for $i \neq \sigma(i_0)$. Since $\mathbf{g}v \in \langle v,\, w\rangle$, and the coefficient of $e_{i'_2}$ in

$$\mathbf{g}v \;=\; cv_1e_1+\cdots+\Bigg(a_{\sigma(i_0)\sigma(i_0)}v_{\sigma(i_0)}+\sum_{\substack{j>i_0\\ \sigma(j)>\sigma(i_0)}} a_{\sigma(i_0)\sigma(j)}v_{\sigma(j)}\Bigg)e_{\sigma(i_0)} \\ +\cdots+cv_{\sigma(j_0)}e_{\sigma(j_0)}+\cdots+cv_{i'_3}e_{i'_3}+\cdots+cv_re_r$$

is zero, we must have $\mathbf{g}v = cv$. Comparing the coefficients of $e_{\sigma(i_0)}$, we have

$$a_{\sigma(i_0)\sigma(i_0)}v_{\sigma(i_0)}+\sum_{\substack{j>i_0\\ \sigma(j)>\sigma(i_0)}} a_{\sigma(i_0)\sigma(j)}v_{\sigma(j)} \;=\; c\, v_{\sigma(i_0)}.$$

Combining with the equality $a_{\sigma(i_0)\sigma(i_0)}+\sum_{\substack{j>i_0\\ \sigma(j)>\sigma(i_0)}} a_{\sigma(i_0)\sigma(j)} = c$, it follows that

$$\sum_{\substack{j>i_0\\ \sigma(j)>\sigma(i_0)}} a_{\sigma(i_0)\sigma(j)}(v_{\sigma(j)}-v_{\sigma(i_0)}) \;=\; 0.$$

So there is $j_1 \neq j_0$ for which $j_1 > i_0$ and $\sigma(j_1) > \sigma(i_0)$.

If v satisfies the condition $v_{\sigma(j_0)} = v_{\sigma(j_1)}$, then, taking into account the condition (γ), we can give a parameter space for such $l^{(3)}_{r-2}$ of dimension at most $r-2-2 = r-4$.

So we assume that $v_{\sigma(j_0)} \neq v_{\sigma(j_1)}$. For $a \in k^\times$, we can construct an automorphism $\mathbf{g} = (a'_{ij}) \in B''$ satisfying the following conditions:

(1) $a'_{ii} = 1$ for $i \neq \sigma(i_0)$;
(2) $a'_{ij} = 0$ for any $i \neq j$ for which $(i,\, j) \neq (\sigma(i_0),\, \sigma(j_0)),\ (\sigma(i_0),\, \sigma(j_1))$;
(3) $a'_{\sigma(i_0)\sigma(i_0)} = a$, $a'_{\sigma(i_0)\sigma(j_0)} = b \in k$, $a'_{\sigma(i_0)\sigma(j_1)} = b' \in k$;
(4) $a+b+b' = 1$ and $av_{\sigma(i_0)}+bv_{\sigma(j_0)}+b'v_{\sigma(j_1)} = v_{\sigma(i_0)}$.

Indeed, if $a \in k^\times$ is given, then b' is determined by the equality

$$(a-1)(v_{\sigma(i_0)}-v_{\sigma(j_0)}) \;=\; b'(v_{\sigma(j_0)}-v_{\sigma(j_1)})$$

and b is determined by the condition $b = 1-a-b'$. Recall that we normalized $v_{i'_2} = 0$ and we are assuming that $i'_3 \neq i'_2,\, \sigma(i_0),\, \sigma(j_0)$. Consider a representative $u = u_1e_1+\cdots+u_re_r \in l^{(3)}_{r-3}$ of a

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generator of $l^{(3)}_{r-3}/l^{(3)}_{r-2}$ satisfying the normalized condition $u_{i'_2} = u_{i'_3} = 0$. Then the $e_{i'_2}$-coefficient and the $e_{i'_3}$-coefficient of

$$\mathbf{g}u = u_1e_1 + \cdots + (au_{\sigma(i_0)} + bu_{\sigma(j_0)} + b'u_{\sigma(j_1)})e_{\sigma(i_0)} + \cdots + u_{\sigma(j_0)} + \cdots + 0e_{i'_2} + 0e_{i'_3} + \cdots + u_re_r$$

vanish, and the $e_{\sigma(i_0)}$-coefficient of $\mathbf{g}u$ is

$$\begin{aligned} au_{\sigma(i_0)} + bu_{\sigma(j_0)} + b'u_{\sigma(j_1)} &= au_{\sigma(i_0)} + (1-a-b')u_{\sigma(j_0)} + b'u_{\sigma(j_1)} \\ &= a(u_{\sigma(i_0)} - u_{\sigma(j_0)}) + u_{\sigma(j_0)} - (a-1)\frac{v_{\sigma(i_0)} - v_{\sigma(j_0)}}{v_{\sigma(j_0)} - v_{\sigma(j_1)}}(u_{\sigma(j_0)} - u_{\sigma(j_1)}). \end{aligned}$$

If $u_{\sigma(i_0)} - u_{\sigma(j_0)} \neq \dfrac{v_{\sigma(i_0)} - v_{\sigma(j_0)}}{v_{\sigma(j_0)} - v_{\sigma(j_1)}}(u_{\sigma(j_0)} - u_{\sigma(j_1)})$, then we can normalize u so that $u_{\sigma(i_0)} = u_{\sigma(j_0)}$. So we can give a parameter space for such $l^{(3)}_{r-3}$ whose dimension is at most $r-4$. If the equality $u_{\sigma(i_0)} - u_{\sigma(j_0)} = \dfrac{v_{\sigma(i_0)} - v_{\sigma(j_0)}}{v_{\sigma(j_0)} - v_{\sigma(j_1)}}(u_{\sigma(j_0)} - u_{\sigma(j_1)})$ holds, then we can give a parameter space for such $l^{(3)}_{r-3}$ whose dimension is at most $r-4$. Therefore, in all cases we can give a parameter space of $(E, \boldsymbol{l})$ whose dimension is at most

$$(r-3) + (r-4) + \sum_{j=1}^{r-4} j = \frac{r^2 - 3r + 2}{2} - 2.$$

Case B. Consider the case where $E = \mathcal{O}_{\mathbb{P}^1}(a_1)^{\oplus r_1} \oplus \cdots \oplus \mathcal{O}_{\mathbb{P}^1}(a_m)^{\oplus r_m}$ with $a_1 < a_2 < \cdots < a_m$ and $l^{(i)}_{r-1} \subset \mathcal{O}_{\mathbb{P}^1}(a_2)^{\oplus r_2}|_{x_i} \oplus \cdots \oplus \mathcal{O}_{\mathbb{P}^1}(a_m)^{\oplus r_m}|_{x_i}$ for $1 \leqslant i \leqslant n$.

As in the proof of Proposition 5.12, we choose a basis $e^{(i)}_{j,1}, \cdots, e^{(i)}_{j,r_j}$ of $\mathcal{O}_{\mathbb{P}^1}(a_j)^{\oplus r_j}|_{t_i}$ for each i, j and we choose suitable generators $v^{(i)}_{p^{(i)}(1),j^{(i)}(1)}, \cdots, v^{(i)}_{p^{(i)}(s),j^{(i)}(s)}$ of $l^{(i)}_{r-s}$. We may further assume that $l^{(i)}_{r-s}$ is generated by $e^{(i)}_{p^{(i)}(1),j^{(i)}(1)}, \cdots, e^{(i)}_{p^{(i)}(s),j^{(i)}(s)}$ for $i = 1, 2$. Since diagonal automorphisms $\mathbf{g} = (a^{p,q}_{j,j'})$ of E given by $a^{pp}_{jj} \in k^\times$ and $a^{pq}_{jj'} = 0$ for $(p, j) \neq (q, j')$ preserve $l^{(1)}_*$ and $l^{(2)}_*$, we can normalize the generator

$$v^{(3)}_{p^{(3)}(1),j^{(3)}(1)} = w_{1,1}e^{(3)}_{1,1} + \ldots + w_{m,r_m}e^{(3)}_{m,r_m}$$

of $l^{(3)}_{r-1}$ so that either $w_{p,j} = 1$ or $w_{p,j} = 0$ for any p, j. Note that $w_{1,j} = 0$ for $1 \leqslant j \leqslant r_1$ by the assumption of Case (B). There are the following two possible cases:

(i) $r_1 \geqslant 2$;

(ii) $r_1 = 1$.

(i) Assume that the condition $r_1 \geqslant 2$ holds. After adding an element of $l^{(3)}_{r-1}$, we can assume that a representative $v = v_{1,1}e^{(3)}_{1,1} + \cdots + v_{m,r_m}e^{(3)}_{m,r_m}$ of a generator of $l^{(3)}_{r-2}/l^{(3)}_{r-1}$ satisfies the condition $v_{p^{(3)}(1),j^{(3)}(1)} = 0$. Consider the automorphisms $\mathbf{g} = (a^{p,q}_{j,j'})$ of E given by $a^{1,1}_{j,j} = c_j \in k^\times$ for $1 \leqslant j \leqslant r_1$, $a^{p,p}_{j,j} = c' \in k^\times$ for $p \geqslant 2$ and $1 \leqslant j \leqslant r_p$, $a^{p,q}_{j,j'} = 0$ for $(p, j) \neq (q, j')$. Then such automorphisms preserve $l^{(1)}_*$, $l^{(2)}_*$ and $l^{(3)}_{r-1}$. Since

$$\mathbf{g}v = c_1v_{1,1}e^{(3)}_{1,1} + \cdots + c_{r_1}v_{1,r_1}e^{(3)}_{1,r_1} + c'v_{2,1}e^{(3)}_{2,1} + \cdots + 0e^{(3)}_{p^{(3)}(1),j^{(3)}(1)} + \cdots + c'v_{m,r_m}e^{(3)}_{m,r_m},$$

we can assume that either $v_{1,j} = 1$ or $v_{1,j} = 0$ holds for any p, j. If $r_1 > 2$, then the parameter space for such generators of $l^{(3)}_{r-2}/l^{(3)}_{r-1}$ is of dimension at most $r-4$. If $r_1 = 2$, we may further

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assume that for some $(p',\, j') \neq (p^{(3)}(1),\, j^{(3)}(1))$ with $p' \geqslant 2$ the following holds: either $v_{p',j'} = 1$ or $v_{p',j'} = 0$. So we can give a parameter space for $l^{(3)}_{r-2}$ whose dimension is at most $r-4$. Adding the data $l^{(3)}_{r-3}, \cdots, l^{(3)}_1$, we can give a parameter space for $(E,\, \boldsymbol{l})$ whose dimension is at most

$$(r-4)+(r-3)+(r-4)+(r-5)+\cdots+1 \;=\; \frac{r^2-3r+2}{2}-2.$$

(ii) Assume that $r_1 = 1$. We again take a representative $v = v_{1,1}e^{(3)}_{1,1} + \cdots + v_{m,r_m}e^{(3)}_{m,r_m}$ of a generator of $l^{(3)}_{r-2}/l^{(3)}_{r-1}$ so that $v_{p^{(3)}(1),j^{(3)}(1)} = 0$. We may assume that one of the following holds:

(α) $v_{1,1} = 0$;
(β) $v_{1,1} \neq 0$ and $(p^{(1)}(1),\, j^{(1)}(1)) \neq (p^{(2)}(1), j^{(2)}(1))$;
(γ) $v_{1,1} \neq 0$ and $(p^{(1)}(1),\, j^{(1)}(1)) = (p^{(2)}(1),\, j^{(2)}(1))$.

(α) Assume that $v_{1,1} = 0$ holds. After adding an element of $l^{(3)}_{r-1}$, we can normalize $v = v_{1,1}e^{(3)}_{1,1} + \cdots + v_{m,r_m}e^{(3)}_{m,r_m}$ so that $v_{p^{(3)}(1),j^{(3)}(1)} = 0$. Consider the automorphisms $\mathbf{g} = (a^{p,q}_{j,j'})$ of E given by $a^{1,1}_{11} = c_1 \in k^\times$, $a^{p,p}_{j,j} = c_2 \in k^\times$ for $p \geqslant 2$ and $a^{p,q}_{j,j'} = 0$ for $(p,\, j) \neq (q,\, j')$. Then such automorphisms preserve not only $l^{(1)}_*$, $l^{(2)}_*$ and $l^{(3)}_{r-1}$ but also $l^{(3)}_{r-2}$. Choose $(q_1,\, j_1)$ such that $q_1 \geqslant 2$, $v_{q_1,j_1} \neq 0$ and $(q_1,\, j_1) \neq (p^{(3)}(1),\, j^{(3)}(1))$. We can normalize a representative $u = u_{1,1}e^{(3)}_{1,1} + \ldots + u_{m,r_m}e^{(3)}_{m,r_m}$ of a generator of $l^{(3)}_{r-3}/l^{(3)}_{r-2}$ by adding an element of $l^{(3)}_{r-2}$ such that $u_{p^{(3)}(1),j^{(3)}(1)} = u_{q_1,j_1} = 0$. Take an index $(q_2,\, j_2)$ other than $(1,\, 1)$, $(p^{(3)}(1),\, j^{(3)}(1))$ and $(q_1,\, j_1)$. Since

$$\begin{aligned}\mathbf{g}u \;=\; & c_1u_{1,1}e^{(3)}_{1,1} + c_2u_{2,1}e^{(3)}_{2,1} + \ldots + 0e^{(3)}_{p^{(3)}(1),j^{(3)}(1)} + \cdots \\ & + 0e^{(3)}_{q_1,j_1} + \cdots + c_2u_{q_2,j_2}e^{(3)}_{q_2,j_2} + \ldots + c_2u_{m,r_m}e^{(3)}_{m,r_m},\end{aligned}$$

we may assume that one of the following holds:

- $u_{1,1} = u_{q_2,j_2} \neq 0$;
- $u_{1,1} = 0$;
- $u_{q_2,j_2} = 0$.

So we can give a parameter space for $l^{(3)}_{r-2}$, $l^{(3)}_{r-3}$ whose dimension is at most $(r-3)+(r-4)$. Adding the data $l^{(3)}_{r-4}, \cdots, l^{(3)}_1$, we can give a parameter space for $(E,\, \boldsymbol{l})$ whose dimension is at most

$$(r-3)+(r-4)+\sum_{j=1}^{r-4} j \;=\; \frac{r^2-3r+2}{2}-2.$$

(β) Assume that the conditions $v_{1,1} \neq 0$ and $(p^{(1)}(1),\, j^{(1)}(1)) \neq (p^{(2)}(1),\, j^{(2)}(1))$ hold. After replacing the indices $i = 1$ and 2 if necessary, we may assume that $(p^{(3)}(1),\, j^{(3)}(1)) \neq (p^{(1)}(1),\, j^{(1)}(1))$. Consider the automorphisms $\mathbf{g} = (a^{p,q}_{j,j'})$ of E given by:

- $a^{1,1}_{1,1} = c_1 \in k^\times$ and $a^{p,p}_{j,j} = c_2 \in k^\times$ for $p \geqslant 2$ and $1 \leqslant j \leqslant r_p$;
- $a^{p^{(1)}(1),1}_{j^{(1)}(1),1} \in \operatorname{Hom}(\mathcal{O}_{\mathbb{P}^1}(a_1),\, \mathcal{O}_{\mathbb{P}^1}(a_{p^{(1)}(1)}))$ satisfying $a^{p^{(1)}(1),1}_{j^{(1)}(1),1}|_{x_2} = 0$; and
- $a^{p,q}_{j,j'} = 0$ for any $(p,\, j) \neq (q,\, j')$ satisfying $((p,\, j),\, (q,\, j')) \neq ((p^{(1)}(1),\, j^{(1)}(1)),\, (1,\, 1))$.

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Such automorphisms preserve $l_*^{(1)}$, $l_*^{(2)}$ and $l_{r-1}^{(3)} = \langle v^{(3)}_{p^{(3)}(1),j^{(3)}(1)}\rangle$. We can normalize the representative $v = v_{1,1}e^{(3)}_{1,1} + \ldots + v_{m,r_m}e^{(3)}_{m,r_m}$ of a generator of $l^{(3)}_{r-2}/l^{(3)}_{r-1}$ after adding an element of $l^{(3)}_{r-1}$ such that $v_{p^{(3)}(1),j^{(3)}(1)} = 0$. Choose (q_1, j_1) such that $q_1 \geqslant 2$ and $(q_1, j_1) \neq (p^{(1)}(1), j^{(1)}(1)), (p^{(3)}(1), j^{(3)}(1))$. For $\mathbf{g} = (a^{p,q}_{j,j'}) \in B'$, we have

$$\mathbf{g}v = c_1 v_{1,1}e^{(3)}_{1,1} + \ldots + \left(a^{p^{(1)}(1),1}_{j^{(1)}(1),1}v_{1,1} + c_2 v_{p^{(1)}(1),j^{(1)}(1)}\right)e^{(3)}_{p^{(1)}(1),j^{(1)}(1)} + \ldots$$
$$+ \cdots + 0e^{(3)}_{p^{(3)}(1),j^{(3)}(1)} + \ldots + c_2 v_{q_1,j_1}e^{(3)}_{q_1,j_1} + \ldots + c_2 v_{m,r_m}e^{(3)}_{m,r_m}.$$

So we can normalize v so that one of the following statements holds:

- $v_{1,1} = v_{p^{(1)}(1),j^{(1)}(1)} = v_{q_1,j_1} \neq 0$;
- $v_{1,1} = v_{p^{(1)}(1),j^{(1)}(1)} \neq 0$ and $v_{q_1,j_1} = 0$.

Thus, we can give a parameter space for $l^{(3)}_{r-2}$ whose dimension is at most $r-4$. Adding the data $l^{(3)}_{r-2}, \cdots, l^{(3)}_1$, we can give a parameter space for $(E, \boldsymbol{l})$ whose dimension is at most

$$(r-4) + (r-3) + (r-4) + \cdots + 1 = \frac{r^2 - 3r + 2}{2} - 2.$$

(γ) Assume that the following conditions hold: $v_{1,1} \neq 0$ and $(p^{(1)}(1), j^{(1)}(1)) = (p^{(2)}(1), j^{(2)}(1))$. By the definition, $v^{(3)}_{p^{(3)}(1),j^{(3)}(1)} = w_{2,1}e^{(3)}_{2,1} + \ldots + w_{m,r_m}e^{(3)}_{m,r_m}$ is a fixed generator of $l^{(3)}_{r-1}$, and for any p, j, we have either $w_{p,j} = 1$ or $w_{p,j} = 0$. We can choose (q_1, j_1) such that $w_{q_1,j_1} = 1$. Take (q_2, j_2) such that $q_2 \geqslant 2$ and $(q_2, j_2) \neq (q_1, j_1), (p^{(1)}(1), j^{(1)}(1))$. After replacing (q_1, j_1) and (q_2, j_2) if necessary, we can assume that one of the following statements holds:

(γ-1) $w_{q_2,j_2} = 0$;
(γ-2) $w_{p^{(1)}(1),j^{(1)}(1)} = 0$;
(γ-3) $w_{q_2,j_2} = w_{p^{(1)}(1),j^{(1)}(1)} = 1$ and $w_{p,j} = 0$ for any $(p, j) \neq (q_2, j_2), (p^{(1)}(1), j^{(1)}(1))$;
(γ-4) $w_{q_2,j_2} = w_{p^{(1)}(1),j^{(1)}(1)} = 1$, $(q_1, j_1) \neq (q_2, j_2), (p^{(1)}(1), j^{(1)}(1))$ and $q_2 > p^{(1)}(1)$;
(γ-5) $w_{q_2,j_2} = w_{p^{(1)}(1),j^{(1)}(1)} = 1$, $(q_1, j_1) \neq (q_2, j_2), (p^{(1)}(1), j^{(1)}(1))$ and $q_2 \leqslant p^{(1)}(1)$.

(γ-1) Assume that the condition $w_{q_2,j_2} = 0$ holds. Consider the diagonal automorphisms $\mathbf{g} = (a^{p,q}_{j,j'})$ of E given by $a^{1,1}_{1,1} = c_1 \in k^\times$, $a^{q_2,q_2}_{j_2,j_2} = c_2 \in k^\times$, $a^{p,p}_{j,j} = c_3 \in k^\times$ for $(p, j) \neq (1, 1), (q_2, j_2)$ and $a^{p,q}_{j,j'} = 0$ for $(p, j) \neq (q, j')$. Then such automorphisms preserve $l_*^{(1)}$, $l_*^{(2)}$ and $l^{(3)}_{r-1}$. Consider a representative $v = v_{1,1}e^{(3)}_{1,1} + \ldots + v_{m,r_m}e^{(3)}_{m,r_m}$ of a generator of $l^{(3)}_{r-2}/l^{(3)}_{r-1}$ with the normalizing condition $v_{q_1,j_1} = 0$. Applying the above type of automorphisms to v, we have

$$\mathbf{g}v = c_1 v_{1,1}e^{(3)}_{1,1} + c_3 v_{2,1}e^{(3)}_{2,1} + \cdots + 0e^{(3)}_{q_1,j_1} + \cdots + c_2 v_{q_2,j_2}e^{(3)}_{q_2,j_2} + \cdots + c_3 v_{m,r_m}e^{(3)}_{m,r_m}.$$

So we can normalize v so that one of the following holds:

(a) $v_{1,1} = v_{q_2,j_2} = v_{p^{(1)}(1),j^{(1)}(1)} \neq 0$;
(b) $v_{1,1} = v_{q_2,j_2} \neq 0$ and $v_{p^{(1)}(1),j^{(1)}(1)} = 0$;
(c) $v_{1,1} = v_{p^{(1)}(1),j^{(1)}(1)} \neq 0$ and $v_{q_2,j_2} = 0$;
(d) $v_{q_2,j_2} = v_{p^{(1)}(1),j^{(1)}(1)} = 0$.

So we can give a parameter space for $l^{(3)}_{r-2}$ whose dimension is at most $r-4$.

(γ-2) Assume that the condition $w_{p^{(1)}(1),j^{(1)}(1)} = 0$ holds. In this case, we have $(q_1, j_1) \neq (p^{(1)}(1), j^{(1)}(1))$, because $w_{q_1,j_1} = 1 \neq 0$. Consider the automorphisms $\mathbf{g} = (a^{p,q}_{j,j'})$ of E given by $a^{1,1}_{1,1} = c_1 \in k^\times$, $a^{p^{(1)}(1),p^{(1)}(1)}_{j^{(1)}(1),j^{(1)}(1)} = c_2 \in k^\times$, $a^{p,p}_{j,j} = c_3 \in k^\times$ for $(p, j) \neq (1, 1), (p^{(1)}(1), j^{(1)}(1))$ and $a^{p,q}_{j,j'} = 0$ for $(p, j) \neq (q, j')$. Then such automorphisms $\mathbf{g}$ preserve $l^{(1)}_*$, $l^{(2)}_*$ and $l^{(3)}_{r-1}$. Normalize the representative $v = v_{1,1}e^{(3)}_{1,1} + \ldots + v_{m,r_m}e^{(3)}_{m,r_m}$ of a generator of $l^{(3)}_{r-2}/l^{(3)}_{r-1}$ so that $v_{q_1,j_1} = 0$. Applying the above type of automorphism $\mathbf{g}$ to v, we have

$$\mathbf{g}v = c_1 v_{1,1}e^{(3)}_{1,1} + \cdots + c_2 v_{p^{(1)}(1),j^{(1)}(1)} + \cdots + 0e^{(3)}_{q_1,j_1} + \cdots + c_3 v_{q_2,j_2}e^{(3)}_{q_2,j_2} + \cdots + c_3 v_{m,r_m}e^{(3)}_{m,r_m}.$$

So we can assume that one of the following holds:

(a) $v_{1,1} = v_{p^{(1)}(1),j^{(1)}(1)} = v_{q_2,j_2} \neq 0$;

(b) $v_{1,1} = v_{p^{(1)}(1),j^{(1)}(1)} \neq 0$ and $v_{q_2,j_2} = 0$;

(c) $v_{1,1} = v_{q_2,j_2} \neq 0$ and $v_{p^{(1)}(1),j^{(1)}(1)} = 0$;

(d) $v_{p^{(1)}(1),j^{(1)}(1)} = v_{q_2,j_2} = 0$.

So we can give a parameter space for $l^{(3)}_{r-2}$ whose dimension is at most $r-4$.

(γ-3) Assume that $w_{q_2,j_2} = w_{p^{(1)}(1),j^{(1)}(1)} = 1$ and $w_{p,j} = 0$ for any (p, j) other than (q_2, j_2), $(p^{(1)}(1), j^{(1)}(1))$. In this case, we have $(q_1, j_1) = (p^{(1)}(1), j^{(1)}(1))$ because $w_{q_1,j_1} = 1 \neq 0$. Consider the diagonal automorphisms $\mathbf{g} = (a^{p,q}_{j,j'})$ of E given by $a^{1,1}_{1,1} = c_1 \in k^\times$, $a^{q_2,q_2}_{k_2,j_2} = a^{p^{(1)}(1),p^{(1)}(1)}_{j^{(1)}(1),j^{(1)}(1)} = c_2 \in k^\times$, $a^{p,p}_{j,j} = c_3 \in k^\times$ for $(p, j) \neq (1, 1), (q_2, j_2), (p^{(1)}(1), j^{(1)}(1))$ and $a^{p,q}_{j,j'} = 0$ for $(p, j) \neq (q, j')$. Such an automorphism $\mathbf{g}$ preserves $l^{(1)}_*$, $l^{(2)}_*$ and $l^{(3)}_{r-1}$. We again normalize the representative $v = v_{1,1}e^{(3)}_{1,1} + \ldots + v_{m,r_m}e^{(3)}_{m,r_m}$ of a generator of $l^{(3)}_{r-2}/l^{(3)}_{r-1}$ such that $v_{q_1,j_1} = 0$. Further, fix an index (q_3, j_3) other than $(1, 1)$, (q_1, j_1), (q_2, j_2). Applying the above type of automorphisms to v, we have

$$\mathbf{g}v = c_1 v_{1,1}e^{(3)}_{1,1} + \cdots + 0e^{(3)}_{q_1,j_1} + \cdots + c_2 v_{q_2,j_2}e^{(3)}_{q_2,j_2} + \cdots + c_3 v_{q_3,j_3}e^{(3)}_{q_3,j_3} + \cdots + c_3 v_{m,r_m}e^{(3)}_{m,r_m}.$$

So we may assume that one of the following holds:

- $v_{1,1} = v_{q_2,j_2} = v_{q_3,j_3} \neq 0$;
- $v_{1,1} = v_{q_2,j_2} \neq 0$ and $v_{q_3,j_3} = 0$;
- $v_{1,1} = v_{q_3,j_3} \neq 0$ and $v_{q_2,j_2} = 0$;
- $v_{q_2,j_2} = v_{q_3,j_3} = 0$.

Then we can give a parameter space for $l^{(3)}_{r-2}$ whose dimension is at most $r-4$.

(γ-4) Assume that the following three conditions hold: $w_{q_2,j_2} = w_{p^{(1)}(1),j^{(1)}(1)} = 1$, $(q_1, j_1) \neq (q_2, k_2)$, $(p^{(1)}(1), j^{(1)}(1))$ and $q_2 > p^{(1)}(1)$. In this case, we have $a_1 \leqslant a_{q_2} - 2$ and we can take sections α of $\mathrm{Hom}(\mathcal{O}_{\mathbb{P}^1}(a_1), \mathcal{O}_{\mathbb{P}^1}(a_{q_2}))$ such that $\alpha|_{x_1} = \alpha|_{x_2} = 0$ but $\alpha|_{x_3}$ is arbitrary. Recall that $v = v_{1,1}e^{(3)}_{1,1} + \cdots + v_{m,r_m}e^{(3)}_{m,r_m}$ gives a representative of a generator of $l^{(3)}_{r-2}/l^{(3)}_{r-1}$ with $v_{1,1} \neq 0$. We impose the normalizing condition $v_{q_1,j_1} = 0$ after adding an element of $l^{(3)}_{r-1}$ to v. Consider the automorphisms $\mathbf{g} = (a^{p,q}_{j,j'})$ of E given by:

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- $a^{1,1}_{1,1} = c_1 \in k^{\times}$, $a^{p,p}_{j,j} = c_2 \in k^{\times}$ for $(p,\, j) \neq (1,\, 1)$;
- $a^{q_2,1}_{k_2,1} = \alpha \in \operatorname{Hom}(\mathcal{O}_{\mathbb{P}^1}(a_1),\, \mathcal{O}_{\mathbb{P}^1}(a_{q_2}))$ satisfying $\alpha|_{x_1} = 0$, $\alpha|_{x_2} = 0$; and
- $a^{p,q}_{j,j'} = 0$ for any $(p, j,\, q, k)$ such that $(p,\, j) \neq (q,\, j')$ and $(p,\, j,\, q,\, j) \neq (q_2,\, k_2,\, 1,\, 1)$.

Then such automorphisms preserve $l^{(1)}_*$, $l^{(2)}_*$ and $l^{(3)}_{r-1}$. Applying such an automorphism, the representative v of a generator of $l^{(3)}_{r-2}/l^{(3)}_{r-1}$ is sent to

$$\begin{aligned}\mathbf{g}v = {} & c_1 v_{1,1} e^{(3)}_{1,1} + \cdots + c_2 v_{p^{(1)}(1),j^{(1)}(1)} e^{(3)}_{p^{(1)}(1),j^{(1)}(1)} + \cdots \\ & + 0 e^{(3)}_{q_1,j_1} + \cdots + \left(\alpha|_{x_3} v_{1,1} + c_2 v_{q_2,j_2}\right) e^{(3)}_{q_2,j_2} + \cdots + c_2 v_{m,r_m}.\end{aligned}$$

So we may assume that one of the following two hold:

(1) $v_{1,1} = v_{p^{(1)}(1),j^{(1)}(1)} = v_{q_2,j_2} \neq 0$;
(2) $v_{1,1} = v_{q_2,j_2} \neq 0$ and $v_{p^{(1)}(1),j^{(1)}(1)} = 0$.

So we can give a parameter space for $l^{(3)}_{r-2}$ whose dimension is at most $r-4$.
(γ-5) Assume that the following three conditions hold: $w_{q_2,j_2} = w_{p^{(1)}(1),j^{(1)}(1)} = 1$, $(q_1,\, j_1) \neq (q_2,\, j_2)$, $(p^{(1)}(1),\, j^{(1)}(1))$ and $q_2 \leqslant p^{(1)}(1)$. Consider the automorphisms $\mathbf{g} = (a^{p,q}_{j,j'})$ of E given by:

- $a^{1,1}_{1,1} = c_1 \in k^{\times}$, $a^{p^{(1)}(1),p^{(1)}(1)}_{j^{(1)}(1),j^{(1)}(1)} = c_2 \in k^{\times}$,
$$a^{p,p}_{j,j} = c_3 \in k^{\times}$$
for $(p,\, j) \neq (1,\, 1)$, $(p^{(1)}(1),\, j^{(1)}(1))$;
- $a^{p^{(1)}(1),q_2}_{j^{(1)}(1),j_2} = b \in \operatorname{Hom}(\mathcal{O}_{\mathbb{P}^1}(a_{q_2}),\, \mathcal{O}_{\mathbb{P}^1}(a_{p^{(1)}(1)}))$ such that $c_2 w_{p^{(1)}(1),j^{(1)}(1)} + b|_{x_3} w_{q_2,j_2} = c_3 w_{p^{(1)}(1),j^{(1)}(1)}$; and
- $a^{p,q}_{j,j'} = 0$ for any $(p,\, j,\, q,\, j)$ such that $(p,\, j) \neq (q,\, j')$ and
$$(p,\, j,\, q,\, j') \neq (p^{(1)}(1),\, j^{(1)}(1),\, q_2,\, j_2).$$

Note that we can always choose $b \in H^0(\mathcal{O}_{\mathbb{P}^1}(a_{p^{(1)}(1)} - a_{q_2}))$ satisfying the condition that $b|_{x_3} w_{q_2,j_2} = (c_3 - c_2) w_{p^{(1)}(1),j^{(1)}(1)}$ for any given $c_2,\, c_3 \in k^{\times}$. Such automorphisms preserve $l^{(1)}_*$, $l^{(2)}_*$ and $l^{(3)}_{r-1}$. Applying such an automorphism, the representative $v \in l^{(3)}_{r-2}$ of a generator of $l^{(3)}_{r-2}/l^{(3)}_{r-1}$ is sent to

$$\begin{aligned}\mathbf{g}v = {} & c_1 v_{1,1} e^{(3)}_{1,1} + c_3 v_{2,1} e^{(3)}_{2,1} + \cdots + \left(c_2 v_{p^{(1)}(1),j^{(1)}(1)} + b|_{x_3} v_{q_2,j_2}\right) e^{(3)}_{p^{(1)}(1),j^{(1)}(1)} + \cdots \\ & + 0 e^{(3)}_{q_1,j_1} + \cdots + c_3 v_{q_2,j_2} e^{(3)}_{q_2,j_2} + \cdots + c_3 v_{m,r_m}.\end{aligned}$$

So we can assume that one of the following holds:

(a) $v_{1,1} = v_{p^{(1)}(1),j^{(1)}(1)} = v_{q_2,j_2} \neq 0$;
(b) $v_{1,1} = v_{p^{(1)}(1),j^{(1)}(1)} \neq 0$ and $v_{q_2,j_2} = 0$;
(c) $v_{1,1} = v_{q_2,j_2} \neq 0$ and $v_{p^{(1)}(1),j^{(1)}(1)} = 0$;
(d) $v_{p^{(1)}(1),j^{(1)}(1)} = v_{q_2,j_2} = 0$.

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Then we can give a parameter space for $l^{(3)}_{r-2}$ whose dimension is at most $r-4$.

In all cases of B(ii)(γ), by adding the data $l^{(3)}_{r-3}, \cdots, l^{(3)}_{1}$ to the parameter space of $l^{(3)}_{r-2}$, we can give a parameter space for $(E, \boldsymbol{l})$ whose dimension is at most

$$(r-4)+(r-3)+\sum_{j=1}^{r-4} j = \frac{r^2-3r+2}{2}-2.$$

This completes the proof. □

Define the open subset $\mathcal{M}^{n_0\text{-reg}}_{\mathrm{PC}}(\boldsymbol{\nu}, \nabla_L)^{\circ}$ of $\mathcal{M}^{n_0-\mathrm{reg}}_{\mathrm{PC}}(\boldsymbol{\nu}, \nabla_L)$,

$$\mathcal{M}^{n_0\text{-reg}}_{\mathrm{PC}}(\boldsymbol{\nu}, \nabla_L)^{\circ} := \left\{(E, \nabla, \boldsymbol{l}) \in \mathcal{M}^{n_0-\mathrm{reg}}_{\mathrm{PC}}(\boldsymbol{\nu}, \nabla_L) \;\middle|\; \dim(\mathrm{End}(E, \boldsymbol{l})) = 1\right\}, \tag{5.15}$$

which consists of $\boldsymbol{\nu}$-parabolic connections $(E, \nabla, \boldsymbol{l})$ with the determinant isomorphic to (L, ∇_L) such that the underlying quasi-parabolic bundle $(E, \boldsymbol{l})$ is simple.

Proposition 5.14. *Let X be a smooth projective curve of genus g over an algebraically closed field k, and let L be a line bundle on X. Let r and n be positive integers such that r is not divisible by the characteristic of k and one of the following holds:*

(1) $n \geqslant 1$ and $r \geqslant 2$ are arbitrary if $g \geqslant 2$;
(2) $n \geqslant 2$, $r \geqslant 2$ and $n+r \geqslant 5$ if $g = 1$;
(3) $n \geqslant 3$, $r \geqslant 2$ and $n+r \geqslant 7$ if $g = 0$.

Then the following holds:

$$\mathrm{codim}_{\mathcal{M}^{n_0-\mathrm{reg}}_{\mathrm{PC}}(\boldsymbol{\nu}, \nabla_L)}\left(\mathcal{M}^{n_0-\mathrm{reg}}_{\mathrm{PC}}(\boldsymbol{\nu}, \nabla_L) \setminus \mathcal{M}^{n_0-\mathrm{reg}}_{\mathrm{PC}}(\boldsymbol{\nu}, \nabla_L)^{\circ}\right) \geqslant 2.$$

Proof. By Propositions 5.10, 5.11, 5.12 and Proposition 5.13 there is a scheme Z of finite type over k and a flat family $(\widetilde{E}, \widetilde{\boldsymbol{l}})$ of quasi-parabolic bundles on $X \times Z$ over Z such that:

(i) $\dim \mathrm{End}((\widetilde{E}, \widetilde{\boldsymbol{l}})|_{X\times z}) \geqslant 2$ for any point $z \in Z$;
(ii) $\dim Z \leqslant (r^2-1)(g-1) + nr(r-1)/2 - 2$; and
(iii) each quasi-parabolic bundle in $\left|\mathcal{N}^{n_0-\mathrm{reg}}_{\mathrm{PC}}(L)\right| \setminus \left|\mathcal{N}^{n_0-\mathrm{reg}}_{\mathrm{PC}}(L)^{\circ}\right|$ is isomorphic to $(\widetilde{E}, \widetilde{\boldsymbol{l}})|_{X\times\{z\}}$ for some point $z \in Z$.

We may assume that there is an isomorphism $\varphi : \det(\widetilde{E}) \xrightarrow{\sim} L \otimes \mathcal{L}$ for some line bundle $\mathcal{L}$ on Z.

Let

$$0 \longrightarrow \mathcal{E}nd(\widetilde{E}) \longrightarrow \mathrm{At}(\widetilde{E}) \xrightarrow{\mathrm{symb}_1} T_{X\times Z/Z} \longrightarrow 0 \tag{5.16}$$

be the relative Atiyah exact sequence, where $\mathrm{At}(\widetilde{E})$ is the Atiyah bundle for $\widetilde{E}$. Setting $\mathrm{At}_D(\widetilde{E})$ to be the pullback of $T_{X\times Z/Z}(-D\times Z)$ by the surjection $\mathrm{At}(\widetilde{E}) \longrightarrow T_{X\times Z/Z}$ in (5.16), we get a short exact sequence

$$0 \longrightarrow \mathcal{E}nd(\widetilde{E}) \longrightarrow \mathrm{At}_D(\widetilde{E}) \xrightarrow{\mathrm{symb}_1} T_{X\times Z/Z}(-D\times Z) \longrightarrow 0.$$

By [Grot1, Theorem 7.7.6], there exists a coherent sheaf $\mathcal{H}$ on Z and a functorial isomorphism

$$\pi_{S*}\left(\mathrm{At}_D(\widetilde{E}) \otimes \Omega^1_{X\times Z/Z}(D_Z) \otimes_{\mathcal{O}_Z} \mathcal{Q}\right) \cong \mathcal{H}om_{\mathcal{O}_S}(\mathcal{H} \otimes_{\mathcal{O}_Z} \mathcal{O}_S\mathcal{Q})$$

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for any morphism $S \longrightarrow Z$ and any coherent sheaf $\mathcal{Q}$ on S. Set $\mathcal{V} := \operatorname{Spec}(\operatorname{Sym}^*(\mathcal{H}))$. Then there is a universal section $\widetilde{\Psi} : T_{X\times\mathcal{V}/\mathcal{V}}(-D\times\mathcal{V}) \longrightarrow \operatorname{At}_D(\widetilde{E})$. Note that the composition of maps $\operatorname{symb}\circ\widetilde{\Psi}$ defines a global section of $\mathcal{O}_{X\times\mathcal{V}}$, which is a section of $\mathcal{O}_{\mathcal{V}}$. Let $\mathcal{V}'$ be the closed subscheme of $\mathcal{V}$ defined by the condition $\operatorname{symb}_1\circ\widetilde{\Psi} = 1$. Then the restriction $\widetilde{\Psi}|_{\mathcal{V}'}$ defines a universal relative connection

$$\widetilde{\nabla} : \widetilde{E}_{\mathcal{V}'} \longrightarrow \widetilde{E}_{\mathcal{V}'}\otimes\Omega_{X\times\mathcal{V}'/\mathcal{V}'}(D_{\mathcal{V}'}).$$

Let B be the maximal closed subscheme of $\mathcal{V}'$ such that $(\operatorname{res}_{x_i\times\mathcal{V}'}(\widetilde{\nabla}) - \nu_j^{(i)}\mathrm{id})(\widetilde{l}_j^{(i)})_{\mathcal{V}'} \subset (\widetilde{l}_{j+1}^{(i)})_{\mathcal{V}'}$ for any i, j and $(\varphi\otimes\mathrm{id})\circ\widetilde{\nabla}\circ\varphi^{-1} = \nabla_L\otimes\mathrm{id}_{\mathcal{L}}$. Set

$$\widetilde{\mathcal{D}}^{\mathrm{par}}_{\mathfrak{sl},0} := \left\{ u\in\mathcal{E}nd(\widetilde{E}_B) \,\middle|\, \operatorname{Tr}(u) = 0 \text{ and } u|_{x_i\times B}(\widetilde{l}_j^{(i)})_B \subset (\widetilde{l}_j^{(i)})_B \text{ for any } i,\ j \right\},$$
$$\widetilde{\mathcal{D}}^{\mathrm{par}}_{\mathfrak{sl},1} := \left\{ u\in\mathcal{E}nd(\widetilde{E}_B)\otimes K_X(D) \,\middle|\, \operatorname{Tr}(u) = 0 \text{ and } \operatorname{res}_{x_i\times B}(u)(\widetilde{l}_j^{(i)})_B \subset (\widetilde{l}_{j+1}^{(i)})_B \text{ for any } i,\ j \right\},$$
$$\nabla_{\widetilde{\mathcal{D}}^{\mathrm{par}}_{\mathfrak{sl},\bullet}} : \widetilde{\mathcal{D}}^{\mathrm{par}}_{\mathfrak{sl},0} \longrightarrow \widetilde{\mathcal{D}}^{\mathrm{par}}_{\mathfrak{sl},1},\ u\longmapsto \widetilde{\nabla}\circ u - (u\otimes\mathrm{id})\circ\widetilde{\nabla}.$$

There is a canonically induced morphism

$$B \longrightarrow Z$$

whose fiber over a point z is an affine space isomorphic to $H^0(X,\,\widetilde{\mathcal{D}}^{\mathrm{par}}_{\mathfrak{sl},1}|_{X\times\{z\}})$. Set

$$B^\circ := \left\{ x\in B \,\middle|\, (\widetilde{E},\,\widetilde{\nabla},\,\widetilde{\boldsymbol{l}})|_{X\times x} \text{ is simple} \right\}.$$

Then there is a canonically induced morphism

$$q : B^\circ \longrightarrow \mathcal{M}^{n_0-\mathrm{reg}}_{\mathrm{PC}}(\boldsymbol{\nu},\nabla_L).$$

By the construction, the complement $\mathcal{M}^{n_0-\mathrm{reg}}_{\mathrm{PC}}(\boldsymbol{\nu},\nabla_L)\setminus\mathcal{M}^{n_0-\mathrm{reg}}_{\mathrm{PC}}(\boldsymbol{\nu},\nabla_L)^\circ$ coincides with the image $q(B^\circ)$. So it suffices to show that for every irreducible component B' of B°, the closure $\overline{q(B')}$ has dimension at most $2(r^2-1)(g-1)+r(r-1)n-2$.

For each point $b\in B'$, consider the group $\operatorname{Aut}((\widetilde{E},\,\widetilde{\boldsymbol{l}},\,\det\widetilde{E})|_{X\times\{b\}})$ of automorphisms of $\widetilde{E}|_{X\times\{b\}}$ preserving $\widetilde{\boldsymbol{l}}_{D\times\{b\}}$ and $\det\widetilde{E}|_{X\times\{b\}}$. Then the tangent space of $\operatorname{Aut}((\widetilde{E},\,\widetilde{\boldsymbol{l}},\,\det\widetilde{E})|_{X\times\{b\}})$ is isomorphic to $H^0(X,\,\widetilde{\mathcal{D}}^{\mathrm{par}}_{\mathfrak{sl},0}|_{X\times\{b\}})$. For a point b of B', there is the orbit map

$$\operatorname{Aut}((\widetilde{E},\,\widetilde{\boldsymbol{l}},\,\det\widetilde{E})|_{X\times\{b\}}) \longrightarrow B',\quad g\longmapsto g\cdot b,$$

whose differential

$$H^0(X,\,\widetilde{\mathcal{D}}^{\mathrm{par}}_{\mathfrak{sl},0}|_{X\times\{b\}}) \xrightarrow{\nabla_{\widetilde{\mathcal{D}}^{\mathrm{par}}_{\mathfrak{sl},\bullet}}} H^0(X,\,\widetilde{\mathcal{D}}^{\mathrm{par}}_{\mathfrak{sl},1}|_{X\times\{b\}})$$

is injective because $(\widetilde{E},\,\widetilde{\nabla},\,\widetilde{\boldsymbol{l}})|_{X\times\{b\}}$ is simple. Since the fiber $q^{-1}(x)$ over a point x of

$$\mathcal{M}^{n_0-\mathrm{reg}}_{\mathrm{PC}}(\boldsymbol{\nu},\nabla_L)$$

contains an orbit for the action of $\operatorname{Aut}((\widetilde{E},\,\widetilde{\boldsymbol{l}},\,\det\widetilde{E})|_{X\times\{b\}})$, we have

$$\dim q^{-1}(x) \geqslant \dim H^0(X,\,\widetilde{\mathcal{D}}^{\mathrm{par}}_{\mathfrak{sl},0}|_{X\times\{b\}}).$$

Note that we have $(\widetilde{\mathcal{D}}^{\mathrm{par}}_{\mathfrak{sl},0})^\vee\otimes K_X \cong \widetilde{\mathcal{D}}^{\mathrm{par}}_{\mathfrak{sl},1}$, and

$$\dim H^0(X,\widetilde{\mathcal{D}}^{\mathrm{par}}_{\mathfrak{sl},1}|_{X\times\{b\}}) - \dim H^0(X,\widetilde{\mathcal{D}}^{\mathrm{par}}_{\mathfrak{sl},0}|_{X\times\{b\}}) = (r^2-1)(g-1)+nr(r-1)/2$$

by the Riemann–Roch theorem. If we choose x to be a generic point of $q(B')$, then we have

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$$\begin{aligned}\dim \overline{q(B')} &= \dim B' - \dim q^{-1}(x)\\ &\leqslant \dim B' - \dim H^0(X,\, \widetilde{\mathcal{D}}^{\rm par}_{\mathfrak{sl},0}|_{X\times\{b\}})\\ &\leqslant \dim Z + \dim H^0(\widetilde{\mathcal{D}}^{\rm par}_{\mathfrak{sl},1}|_{X\times\{b\}}) - \dim H^0(\widetilde{\mathcal{D}}^{\rm par}_{\mathfrak{sl},0}|_{X\times\{b\}})\\ &= \dim Z + (r^2-1)(g-1) + r(r-1)n/2\\ &\leqslant 2(r^2-1)(g-1) + nr(r-1) - 2.\end{aligned}$$

Since $q(B^\circ) = \mathcal{M}^{n_0-\mathrm{reg}}_{\rm PC}(\boldsymbol{\nu}, \nabla_L) \setminus \mathcal{M}^{n_0-\mathrm{reg}}_{\rm PC}(\boldsymbol{\nu}, \nabla_L)^\circ$ is a union of the images $q(B')$, the proof is completed. □

Define the open subset $\mathcal{M}^{n_0-\mathrm{reg}}_{\rm Higgs}(\boldsymbol{\mu},\, \Phi_L)^\circ$ of $\mathcal{M}^{n_0-\mathrm{reg}}_{\rm Higgs}(\boldsymbol{\mu},\, \Phi_L)$ by

$$\mathcal{M}^{n_0-\mathrm{reg}}_{\rm Higgs}(\boldsymbol{\mu},\, \Phi_L)^\circ := \left\{(E,\, \Phi,\, \boldsymbol{l}) \in \mathcal{M}^{n_0-\mathrm{reg}}_{\rm Higgs}(\boldsymbol{\mu},\, \Phi_L) \,\middle|\, \dim\left(\mathrm{End}(E,\, \boldsymbol{l})\right) = 1\right\} \tag{5.17}$$

which consists of $\boldsymbol{\mu}$-parabolic Higgs bundles $(E,\, \Phi,\, \boldsymbol{l})$ with the determinant isomorphic to (L, Φ_L) such that the underlying quasi-parabolic bundle $(E,\, \boldsymbol{l})$ is simple.

The proof of the following proposition uses an argument similar to one in the proofs of Proposition 5.14.

Proposition 5.15. *Let X be a smooth projective curve of genus g over an algebraically closed field k, and let L be a line bundle on X with a homomorphism $\Phi_L : L \longrightarrow L \otimes K_X(D)$. Take positive integers r, n and a tuple $\boldsymbol{\mu} = (\mu^{(i)}_j)^{1\leqslant i\leqslant n}_{0\leqslant j\leqslant r-1} \in k^{nr}$ such that $\mathsf{res}_{x_i}(\Phi_L) = \sum_{j=0}^{r-1} \mu^{(i)}_j$ for any i. Assume that r is not divisible by the characteristic of k and one of the following holds:*

(a) $n \geqslant 1$ and $r \geqslant 2$ are arbitrary if $g \geqslant 2$;

(b) $n \geqslant 2$, $r \geqslant 2$ and $n + r \geqslant 5$ if $g = 1$;

(c) $n \geqslant 3$, $r \geqslant 2$ and $n + r \geqslant 7$ if $g = 0$.

Then $\mathrm{codim}_{\mathcal{M}^{n_0-\mathrm{reg}}_{\rm Higgs}(\boldsymbol{\mu},\, \Phi_L)}\left(\mathcal{M}^{n_0-\mathrm{reg}}_{\rm Higgs}(\boldsymbol{\mu},\, \Phi_L) \setminus \mathcal{M}^{n_0-\mathrm{reg}}_{\rm Higgs}(\boldsymbol{\mu},\, \Phi_L)^\circ\right) \geqslant 2.$

Proof. By Propositions 5.10, 5.11, 5.12, 5.13, there is a scheme Z of finite type over $\mathrm{Spec}\, k$ and a flat family $(\widetilde{E},\, \widetilde{\boldsymbol{l}})$ of quasi-parabolic bundles on $X \times Z$ over Z such that

(i) $\dim Z \leqslant (r^2-1)(g-1) + nr(r-1)/2 - 2$;

(ii) $\dim\ \mathrm{End}((\widetilde{E},\, \widetilde{\boldsymbol{l}})|_{X\times z}) \geqslant 2$ for all $z \in Z$; and

(iii) each quasi-parabolic bundle in the complement $\left|\mathcal{N}^{n_0-\mathrm{reg}}_{\rm par}(L)\right| \setminus \left|\mathcal{N}^{n_0-\mathrm{reg}}_{\rm par}(L)^\circ\right|$ is isomorphic to $(\widetilde{E},\, \widetilde{\boldsymbol{l}})|_{X\times\{z\}}$ for some $z \in Z$.

Define

$$\begin{aligned}\widetilde{\mathcal{D}}^{\rm par}_{\mathfrak{sl},0} &:= \left\{u \in \mathcal{E}nd(\widetilde{E}) \,\middle|\, u|_{x_i\times Z_\alpha}(\widetilde{l}^{(i)}_j) \subset \widetilde{l}^{(i)}_j \text{ for any } i,\, j\right\}\\ \widetilde{\mathcal{D}}^{\rm par}_{\mathfrak{sl},1} &:= \left\{u \in \mathcal{E}nd(\widetilde{E}) \otimes K_X(D) \,\middle|\, \mathrm{res}_{x_i\times Z_\alpha}(u)(\widetilde{l}^{(i)}_j) \subset \widetilde{l}^{(i)}_{j+1} \text{ for any } i,\, j\right\}\end{aligned}$$

By [Grot1, Theorem 7.7.6], there is a coherent sheaf $\mathcal{H}$ on Z together with a functorial isomorphism

$$\mathrm{Hom}(\mathcal{H} \otimes_{\mathcal{O}_Z} \mathcal{O}_S,\, Q) \cong H^0(X \times S,\, \widetilde{\mathcal{D}}^{\rm par}_{\mathfrak{sl},1} \otimes_{\mathcal{O}_Z} Q)$$

for any Noetherian scheme S over Z and any coherent sheaf Q on S. For $\mathbb{V}(\mathcal{H}) := \mathrm{Spec}(\mathrm{Sym}^*(\mathcal{H}))$, there is a universal family of Higgs fields $\widetilde{\Phi} \in H^0(X \times S,\, \widetilde{\mathcal{D}}^{\rm par}_{\mathfrak{sl},0} \otimes K_X(D) \otimes_{\mathcal{O}_Z}$

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$\mathcal{O}_{\mathbb{V}(\mathcal{H})}$) on $(\widetilde{E}, \widetilde{\boldsymbol{l}}) \otimes \mathcal{O}_{\mathbb{V}(\mathcal{H})}$. We may assume that $\det(\widetilde{E}) \cong L \otimes \mathcal{P}$ for some line bundle $\mathcal{P}$ on Z. Let B be the maximal locally closed subscheme of $\mathbb{V}(\mathcal{H})$ such that the composition of the homomorphisms

$$L \otimes \mathcal{P}_B \xrightarrow{\sim} \det(\widetilde{E})_B \xrightarrow{\mathrm{Tr}\widetilde{\Phi}} \det(\widetilde{E}) \otimes K_X(D)_B \xrightarrow{\sim} L \otimes \mathcal{P}_B$$

coincides with $\nabla_L \otimes \mathcal{P}_B$ and $(\mathsf{res}_{x_i \times Z}(\widetilde{\Phi}) - \mu_\ell^{(i)})(\widetilde{l}_\ell^{(i)}) \subset \widetilde{l}_{\ell+1}^{(i)}$ for any i, ℓ and also $(\widetilde{E}, \widetilde{\boldsymbol{l}}, \widetilde{\Phi})|_{X \times b}$ is simple for any $b \in B$. Then the family $(\widetilde{E}, \widetilde{\boldsymbol{l}}, \widetilde{\Phi})_B$ defines a morphism

$$B \longrightarrow \mathcal{M}_{\mathrm{Higgs}}^{n_0-\mathrm{reg}}(\boldsymbol{\mu}, \Phi_L) \tag{5.18}$$

whose image coincides with $\mathcal{M}_{\mathrm{Higgs}}^{n_0-\mathrm{reg}}(\boldsymbol{\mu}, \Phi_L) \setminus \mathcal{M}_{\mathrm{Higgs}}^{n_0-\mathrm{reg}}(\boldsymbol{\mu}, \Phi_L)^\circ$. Note that the fibers of the morphism in (5.18) contain orbits of the action by the automorphism group of $(\widetilde{E}, \widetilde{\boldsymbol{l}}, \det(\widetilde{E}))_z$ whose dimension is that of $H^0(X, \widetilde{\mathcal{D}}_{\mathfrak{sl},0}^{\mathrm{par}}|_{X \times z})$. So we have

$$\begin{aligned}\dim \mathrm{Im}\left(B \longrightarrow \mathcal{M}_{\mathrm{Higgs}}^{n_0-\mathrm{reg}}(\boldsymbol{\mu}, \Phi_L)\right) &\leqslant \dim B - \dim H^0(X, \widetilde{\mathcal{D}}_{\mathfrak{sl},0}^{\mathrm{par}}|_{X \times z}) \\ &\leqslant \dim Z + \dim H^0(X, \widetilde{\mathcal{D}}_{\mathfrak{sl},1}^{\mathrm{par}}|_{X \times z}) - \dim H^0(X, \widetilde{\mathcal{D}}_{\mathfrak{sl},0}^{\mathrm{par}}|_{X \times z}) \\ &= \dim Z + (r^2-1)(g-1) + nr(r-1)/2 \\ &\leqslant \dim \mathcal{M}_{\mathrm{Higgs}}^{n_0-\mathrm{reg}}(\boldsymbol{\mu}, \Phi_L) - 2.\end{aligned}$$

Since $\mathcal{M}_{\mathrm{Higgs}}^{n_0-\mathrm{reg}}(\boldsymbol{\mu}, \Phi_L) \setminus \mathcal{M}_{\mathrm{Higgs}}^{n_0-\mathrm{reg}}(\boldsymbol{\mu}, \Phi_L)^\circ$ coincides with the image of the morphism in (5.18), the proof is complete. □

As a corollary of the above theorem, we can also get a result by Boden and Yokogawa [BoYo, Theorem 4.2(c)].

Corollary 5.16. *Under the same assumption as in Propositions 5.14 and 5.15, the moduli spaces $\mathcal{M}_{\mathrm{PC}}^{n_0-\mathrm{reg}}(\boldsymbol{\nu}, \nabla_L)$ and $\mathcal{M}_{\mathrm{Higgs}}^{n_0-\mathrm{reg}}(\boldsymbol{\mu}, \Phi_L)$ are irreducible.*

Proof. We only prove the irreducibility for $\mathcal{M}_{\mathrm{Higgs}}^{n_0-\mathrm{reg}}(\boldsymbol{\mu}, \Phi_L)$ as the proof is same for $\mathcal{M}_{\mathrm{PC}}^{n_0-\mathrm{reg}}(\boldsymbol{\nu}, \nabla_L)$. The open subspace $\mathcal{M}_{\mathrm{Higgs}}^{n_0-\mathrm{reg}}(\boldsymbol{\mu}, \Phi_L)^\circ$ is isomorphic to an affine space bundle over the moduli space $\mathcal{N}_{\mathrm{par}}^{n_0-\mathrm{reg}}(L)$ of n_0-regular simple quasi-parabolic bundles with the determinant L. Since $\mathcal{N}_{\mathrm{par}}^{n_0-\mathrm{reg}}(L)$ is irreducible, it follows that $\mathcal{M}_{\mathrm{Higgs}}^{n_0-\mathrm{reg}}(\boldsymbol{\mu}, \Phi_L)^\circ$ is also irreducible. Recall that the moduli space $\mathcal{M}_{\mathrm{Higgs}}^{n_0-\mathrm{reg}}(\boldsymbol{\mu}, \Phi_L)$ is smooth of equidimension by Proposition 5.7. So $\mathcal{M}_{\mathrm{Higgs}}^{n_0-\mathrm{reg}}(\boldsymbol{\mu}, \Phi_L)$ is connected and thus irreducible, because $\dim\left(\mathcal{M}_{\mathrm{Higgs}}^{n_0-\mathrm{reg}}(\boldsymbol{\mu}, \Phi_L) \setminus \mathcal{M}_{\mathrm{Higgs}}^{n_0-\mathrm{reg}}(\boldsymbol{\mu}, \Phi_L)^\circ\right) < \dim \mathcal{M}_{\mathrm{Higgs}}^{n_0-\mathrm{reg}}(\boldsymbol{\mu}, \Phi_L)$ by Proposition 5.15. □

Remark 5.17. The proof of Corollary 5.16 is in fact valid under a weaker assumption than that of Theorem 5.15. Indeed, it is valid under the same assumption as that of [Ina, Theorem 2.2].

5.3 The moduli space is not affine

We use the notation of § 5.1. In this subsection, k is assumed to be an algebraically closed field of arbitrary characteristic unless otherwise noted.

Let X be a smooth projective curve over k of genus g. Fix a line bundle L of degree d on X equipped with a logarithmic connection $\nabla_L : L \longrightarrow L \otimes K_X(D)$, and also fix a string of local exponents $\boldsymbol{\nu} = (\nu_j^{(i)}) \in k^{nr}$ such that $\mathsf{res}_{x_i}(\nabla_L) = \sum_{j=0}^{r-1} \nu_j^{(i)}$ for any i. We assume the following:

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$$\sum_{i=1}^{n}\sum_{\ell=1}^{s}\nu^{(i)}_{j^{(i)}_\ell} \notin \operatorname{Im}(\mathbb{Z}\to k) \quad \text{for any choice of s elements } \{\, \mathrm{j}_1^{(\mathrm{i})},\, \cdots,\, \mathrm{j}_\mathrm{s}^{(\mathrm{i})}\,\} \text{ in } \{1,\, \cdots,\, \mathrm{r}\}. \tag{5.19}$$

Under the assumption in (5.19), any $\boldsymbol{\nu}$-parabolic connection is irreducible, and hence it is $\boldsymbol{\alpha}$-stable for any parabolic weight $\boldsymbol{\alpha}$. So we have $\mathcal{M}^{\boldsymbol{\alpha}}_{\mathrm{PC}}(\boldsymbol{\nu},\, \nabla_L) = \mathcal{M}_{\mathrm{PC}}(\boldsymbol{\nu},\, \nabla_L)$. In this subsection we will show that the moduli space $\mathcal{M}^{\boldsymbol{\alpha}}_{\mathrm{PC}}(\boldsymbol{\nu},\, \nabla_L)$ is not affine. This will be done by comparing the transcendence degree of the ring of global algebraic functions on the moduli space $\mathcal{M}^{\boldsymbol{\alpha}}_{\mathrm{PC}}(\boldsymbol{\nu}, \nabla_L)$ of parabolic connections with the transcendence degree of the ring of global algebraic functions on the moduli space of parabolic Higgs bundles.

Consider the moduli space

$$\mathcal{M}^{\boldsymbol{\alpha}}_{\mathrm{Higgs}}(d) = \left\{ (E,\, \Phi,\, \boldsymbol{l}) \;\middle|\; \begin{array}{l} (E,\, \boldsymbol{l}) \text{ is a quasi}-\text{parabolic bundle of rank } r \text{ and degree } d, \\ \Phi \,:\, E \to E\otimes K_X(D) \text{ is an } O_X-\text{homomorphism such that} \\ \mathsf{res}_{x_i}(\Phi)(l^{(i)}_j) \subset l^{(i)}_j \text{ for any } i,\, j, \text{ and } (E,\, \Phi,\, \boldsymbol{l}) \text{ is } \boldsymbol{\alpha}-\text{stable} \end{array} \right\}$$

of $\boldsymbol{\alpha}$-stable parabolic Higgs bundles. Setting

$$\Lambda_{\mathrm{Higgs}} = \left\{ \boldsymbol{\mu} = (\mu^{(i)}_j)^{1\leqslant i\leqslant n}_{0\leqslant j\leqslant r-1} \in k^{nr} \;\middle|\; \sum_{i=1}^{n}\sum_{j=0}^{r-1}\mu^{(i)}_j = 0 \right\},$$

we have a canonical morphism

$$\mathcal{M}^{\boldsymbol{\alpha}}_{\mathrm{Higgs}}(d) \longrightarrow \Lambda_{\mathrm{Higgs}}$$

whose fiber over any $\boldsymbol{\mu} \in \Lambda_{\mathrm{Higgs}}$ is the moduli space $\mathcal{M}^{\boldsymbol{\alpha}}_{\mathrm{Higgs}}(\boldsymbol{\mu})$ of $\boldsymbol{\alpha}$-stable $\boldsymbol{\mu}$-parabolic Higgs bundles. For a parabolic Higgs bundle $(E,\, \Phi,\, \boldsymbol{l}) \in \mathcal{M}^{\boldsymbol{\alpha}}_{\mathrm{Higgs}}(d)$, consider the homomorphism

$$T\,\mathrm{Id}_E - \Phi \,:\, E\otimes k[T] \longrightarrow E\otimes \mathrm{Sym}^*(K_X(D))\otimes k[T],$$

where T is an indeterminate. We can write

$$\det(T\,\mathrm{Id}_E - \Phi) = T^r + s_1T^{r-1} + \cdots + s_{r-1}T + s_r$$

with $s_j \in H^0(X,\, K_X^{\otimes j}(jD))$. Note that $s_1 = -\operatorname{Tr}(\Phi)$. Set

$$W := \bigoplus_{j=1}^{r} H^0(X,\, K_X^{\otimes j}(jD)).$$

Using the above constructed $(s_1,\, \cdots,\, s_r)$, we get a morphism

$$H \,:\, \mathcal{M}^{\boldsymbol{\alpha}}_{\mathrm{Higgs}}(d) \longrightarrow W, \tag{5.20}$$

which is called the Hitchin map. A remarkable property of the Hitchin map is that it is proper, which was proved by Hitchin, Simpson and Nitsure. We use the parabolic version of it, which was proved by Yokogawa.

THEOREM 5.18 [Hi, Sim3, Nit, Yo]. *Under the assumption that $\boldsymbol{\alpha}$-semistability implies $\boldsymbol{\alpha}$-stability, the Hitchin map $H \,:\, \mathcal{M}^{\boldsymbol{\alpha}}_{\mathrm{Higgs}}(d) \longrightarrow W$ in (5.20) is a proper morphism.*

Set

$$A_{\mathrm{Higgs}} := \left\{ \boldsymbol{a} = (a^{(i)}_j)^{1\leqslant i\leqslant n}_{1\leqslant j\leqslant r} \in k^{nr} \;\middle|\; \sum_{i=1}^{n} a^{(i)}_1 = 0 \right\}.$$

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Using the correspondence $(s_\ell)_{1\leqslant \ell\leqslant r} \longmapsto (\mathsf{res}_{x_i}(s_\ell))^{1\leqslant i\leqslant n}_{1\leqslant \ell\leqslant r}$, we define a morphism

$$W \longrightarrow A_{\text{Higgs}}$$

which is a linear surjection under any of the following conditions:

(i) $n \geqslant 1$ when $g \geqslant 2$;

(ii) $n \geqslant 2$ when $g = 1$;

(iii) $n \geqslant 3$ when $g = 0$.

There is also a morphism

$$\Lambda_{\text{Higgs}} \longrightarrow A_{\text{Higgs}}$$

that associates the coefficients of $\prod_{j=0}^{r-1}(t-\mu_j^{(i)})$. Then the Hitchin map induces a morphism

$$\mathcal{M}^{\boldsymbol{\alpha}}_{\text{Higgs}}(d) \longrightarrow W \times_{A_{\text{Higgs}}} \Lambda_{\text{Higgs}}, \tag{5.21}$$

which is proper by Theorem 5.18.

Fix a line bundle L on X of degree d, and consider the closed subvariety

$$\mathcal{M}^{\boldsymbol{\alpha}}_{\text{Higgs}}(L) := \left\{(E,\, \Phi,\, \boldsymbol{l}) \in \mathcal{M}^{\boldsymbol{\alpha}}_{\text{Higgs}}(d) \;\middle|\; \det(E) \cong L\right\}$$

of $\mathcal{M}^{\boldsymbol{\alpha}}_{\text{Higgs}}(d)$. Then the restriction of the map in (5.21)

$$\mathcal{M}^{\boldsymbol{\alpha}}_{\text{Higgs}}(L) \longrightarrow W \times_{A_{\text{Higgs}}} \Lambda_{\text{Higgs}} \tag{5.22}$$

is also a proper morphism.

Generic fibers of the Hitchin map were investigated by Logares and Martens in [LaMa, Proposition 2.2]. The following result is likely to be well known to the experts. We give a proof of it using the arguments given by Alfaya and Gómez in [AlGo, Lemma 3.2].

Corollary 5.19. *Assume that $\boldsymbol{\alpha}$-semistability implies $\boldsymbol{\alpha}$-stability. Also, assume that one of the following statements holds:*

(i) $n \geqslant 1$ *if* $g \geqslant 2$;

(ii) $n \geqslant 2$ *if* $g = 1$;

(iii) $n \geqslant 3$ *if* $g = 0$.

Then the morphism $\mathcal{M}^{\boldsymbol{\alpha}}_{\text{Higgs}}(L) \longrightarrow W \times_{A_{\text{Higgs}}} \Lambda_{\text{Higgs}}$ *in (5.22) is surjective.*

Proof. It suffices to prove that the morphism in (5.22) is dominant, because it is proper. Take any $(s = (s_\ell),\, \boldsymbol{\mu}) \in W \times_{A_{\text{Higgs}}} \Lambda_{\text{Higgs}}$. Consider the corresponding spectral curve $X_s \subset \mathbb{P}(\mathcal{O}_X \oplus K_X(D))$ which is defined by the equation

$$y^r + s_1 y^{r-1} + \cdots + s_{r-1} y + s_r = 0,$$

where y is the section of $\mathcal{O}_{\mathbb{P}(\mathcal{O}_X\oplus K_X(D))}(1)$ corresponding to the inclusion map $\mathcal{O}_X \hookrightarrow \mathcal{O}_X \oplus K_X(D)$. Take a section $\tau \in H^0(X,\, K_X^{\otimes r}(rD))$ which has at most simple zeroes; since $K_X^{\otimes r}(rD)$ is very ample by the assumption in the corollary, such a section exists. Then the spectral curve $y^r - \tau = 0$ has no singular points.

Since the smoothness is an open condition, there is an open subset $U \subset W \times_{A_{\text{Higgs}}} \Lambda_{\text{Higgs}}$ such that the spectral curve X_s is smooth for every $s \in U$. Take a line bundle $\mathcal{L}$ on X_s such that the locally free sheaf $E := \pi_*(\mathcal{L})$ has its determinant $\det(E)$ isomorphic to L, where $\pi:\ X_s \longrightarrow X$ is the natural projection. By the Beauville–Narasimhan–Ramanan correspondence

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[BNR, Proposition 3.6], there is a Higgs field $\Phi : E \longrightarrow E \otimes K_X(D)$ induced by the action of y on $\mathcal{L}$. Shrinking U if necessary, we may further assume that $\mu_0^{(i)}, \cdots, \mu_{r-1}^{(i)}$ are mutually distinct for any fixed i. Then we can associate a unique parabolic structure l on E compatible with Φ. Since $(E, \Phi, \boldsymbol{l})$ is irreducible by its construction, it is evidently $\boldsymbol{\alpha}$-stable. So we have $(E, \Phi, \boldsymbol{l}) \in \mathcal{M}^{\boldsymbol{\alpha}}_{\mathrm{Higgs}}(L)$, which is sent to $(s, \boldsymbol{\mu})$ under the morphism in (5.22). Thus, the morphism in (5.22) is dominant because its image contains the dense open subset U of $W \times_{A_{\mathrm{Higgs}}} \Lambda_{\mathrm{Higgs}}$. □

As a consequence of Theorem 5.18 and Corollary 5.19, we can determine the transcendence degree of the ring of global algebraic functions on the moduli space of parabolic Higgs bundles.

Corollary 5.20. *Let L be a line bundle on X with a Higgs field $\Phi_L : L \longrightarrow L \otimes K_X(D)$. Take $\boldsymbol{\mu} = (\mu_j^{(i)}) \in \Lambda_{\mathrm{Higgs}}$ satisfying the condition $\mathsf{res}_{x_i}(\Phi_L) = \sum_{j=0}^{r-1} \mu_j^{(i)}$ for all i. Then, under the same assumption as in Theorem 5.15, the transcendence degree of the ring of global algebraic functions on the moduli space of parabolic Higgs bundles is given by the following:*

$$\mathrm{tr.deg}_k\, \Gamma(\mathcal{M}^{\boldsymbol{\alpha}}_{\mathrm{Higgs}}(\boldsymbol{\mu}, \Phi_L), \mathcal{O}_{\mathcal{M}^{\boldsymbol{\alpha}}_{\mathrm{Higgs}}(\boldsymbol{\mu},\Phi_L)}) = (r^2-1)(g-1) + \tfrac{1}{2}nr(r-1).$$

Proof. The closed subvariety

$$Y := \left\{(s = (s_\ell)_{1 \leqslant \ell \leqslant r-1}, \boldsymbol{\mu}) \in W \times_{A_{\mathrm{Higgs}}} \{\boldsymbol{\mu}\} \;\middle|\; s_1 = -\Phi_L\right\}$$

of $W \times_{A_{\mathrm{Higgs}}} \Lambda_{\mathrm{Higgs}}$ is isomorphic to an affine space. Its inverse image $\mathcal{M}^{\boldsymbol{\alpha}}_{\mathrm{Higgs}}(L) \times_{W \times_{A_{\mathrm{Higgs}}} \Lambda_{\mathrm{Higgs}}} Y$ for the morphism in (5.22) is nothing but the moduli space $\mathcal{M}^{\boldsymbol{\alpha}}_{\mathrm{Higgs}}(\boldsymbol{\mu}, \Phi_L)$ of $\boldsymbol{\alpha}$-stable $\boldsymbol{\mu}$-parabolic Higgs bundles with determinant (L, Φ_L). By Corollary 5.19, the base change

$$H' : \mathcal{M}^{\boldsymbol{\alpha}}_{\mathrm{Higgs}}(\boldsymbol{\mu}, \Phi_L) \longrightarrow Y$$

is also a proper and surjective morphism. So the ring homomorphism $\mathcal{O}_Y \longrightarrow H'_*\mathcal{O}_{\mathcal{M}^{\boldsymbol{\alpha}}_{\mathrm{Higgs}}(\boldsymbol{\mu},\Phi_L)}$ is injective, and $H'_*\mathcal{O}_{\mathcal{M}^{\boldsymbol{\alpha}}_{\mathrm{Higgs}}(\boldsymbol{\mu},\Phi_L)}$ is a finite algebra over $\mathcal{O}_Y$. Therefore,

$$\Gamma(\mathcal{M}^{\boldsymbol{\alpha}}_{\mathrm{Higgs}}(\boldsymbol{\mu}, \Phi_L), \mathcal{O}_{\mathcal{M}^{\boldsymbol{\alpha}}_{\mathrm{Higgs}}(\boldsymbol{\mu},\Phi_L)})$$

is a finite algebra over $\Gamma(Y, \mathcal{O}_Y)$ whose Krull dimension is

$$\dim Y = -n(r-1) + \sum_{j=2}^{r} \dim H^0\left(X, K_X^{\otimes j}(jD)\right)$$

$$= -n(r-1) + \sum_{j=2}^{r} ((2g-2)j + jn + (1-g)) = (r^2-1)(g-1) + \frac{nr(r-1)}{2}.$$

Since $\Gamma(\mathcal{M}^{\boldsymbol{\alpha}}_{\mathrm{Higgs}}(\boldsymbol{\mu}, \Phi_L), \mathcal{O}_{\mathcal{M}^{\boldsymbol{\alpha}}_{\mathrm{Higgs}}(\boldsymbol{\mu},\Phi_L)})$ is a finitely generated algebra over k, its transcendence degree over k coincides with its Krull dimension. □

Proposition 5.21. *There is a projective flat morphism*

$$\overline{\mathcal{M}'} \longrightarrow \mathbb{A}^1 = \operatorname{Spec} k[t],$$

and an $\mathbb{A}^1$-relative very ample divisor $Y \subset \overline{\mathcal{M}'}$ such that the complement $\mathcal{M}' := \overline{\mathcal{M}'} \setminus Y$ satisfies the following:

$$\mathcal{M}'_h \cong \begin{cases} \mathcal{M}^{n_0-\mathrm{reg}}_{\mathrm{PC}}(\boldsymbol{\nu}, \nabla_L)^\circ (0 \neq h \in \mathbb{A}^1), \\ \mathcal{M}^{n_0-\mathrm{reg}}_{\mathrm{Higgs}}(\mathbf{0}, 0)^\circ (h = 0). \end{cases}$$

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Proof. Let $\mathcal{N}_{\mathrm{par}}^{n_0-\mathrm{reg}}(L)$ be the moduli space of simple n_0-regular quasi-parabolic bundles $(E,\boldsymbol{l})$ with $\det E\cong L$. Let $(\widetilde{E},\widetilde{\boldsymbol{l}})$ be the universal family over $X\times\mathcal{N}_{\mathrm{par}}^{n_0-\mathrm{reg}}(L)$. As in the proof of Proposition 3.5, we can construct the relative Atiyah bundle $\mathrm{At}_D(\widetilde{E})$, which fits in the exact sequence

$$0\longrightarrow \mathcal{E}nd(\widetilde{E})\otimes K_X(D)\longrightarrow \mathrm{At}_D(\widetilde{E})\otimes K_X(D)\longrightarrow \mathcal{O}_{X\times\mathcal{N}_{\mathrm{par}}^{n_0-\mathrm{reg}}(L)}\longrightarrow 0.$$

Recall the construction of the homomorphism (3.4) in the proof of Proposition 3.5, which defines a surjection

$$\mathrm{At}_D(\widetilde{E})\otimes K_X(D)\ \longrightarrow\ (\mathrm{At}_D(\widetilde{E})\otimes K_X(D))/(\mathrm{At}(\widetilde{E})\otimes K_X)\ \xrightarrow{\ \sim\ }\ \mathcal{E}nd(\widetilde{E})|_{D\times\mathcal{N}_{\mathrm{par}}^{n_0-\mathrm{reg}}(L)}.$$

Let $\mathrm{At}_D(\widetilde{E},\widetilde{\boldsymbol{l}})\subset\mathrm{At}_D(\widetilde{E})$ be the pullback of the subsheaf

$$\left\{a\in\mathcal{E}nd(\widetilde{E})|_{D\times\mathcal{N}_{\mathrm{par}}^{n_0-\mathrm{reg}}(L)}\ \middle|\ a|_{x_i\times\mathcal{N}_{\mathrm{par}}^{n_0-\mathrm{reg}}(L)}(\widetilde{l}_j^{(i)})\subset\widetilde{l}_j^{(i)}\text{ for any } i,\,j\right\}\subset\mathcal{E}nd(\widetilde{E})|_{D\times\mathcal{N}_{\mathrm{par}}^{n_0-\mathrm{reg}}(L)}$$

by the above surjection.

Since $\det(\widetilde{E})\cong L\otimes\mathcal{P}$ for a line bundle $\mathcal{P}$ on $\mathcal{N}_{\mathrm{par}}^{n_0-\mathrm{reg}}(L)$, it follows that $\mathrm{At}_D(\det(\widetilde{E}))\cong\mathrm{At}_D(L)\otimes\mathcal{O}_{X\times\mathcal{N}_{\mathrm{par}}^{n_0-\mathrm{reg}}(L)}$. There is an exact sequence

$$0\longrightarrow\mathcal{O}_{X\times\mathcal{N}_{\mathrm{par}}^{n_0-\mathrm{reg}}(L)}\longrightarrow\mathrm{At}_D(L)\otimes\mathcal{O}_{X\times\mathcal{N}_{\mathrm{par}}^{n_0-\mathrm{reg}}(L)}\xrightarrow{\mathrm{symb}_1}T_X(-D)\otimes\mathcal{O}_{X\times\mathcal{N}_{\mathrm{par}}^{n_0-\mathrm{reg}}(L)}\longrightarrow 0,$$

which admits a section $T_X(-D)\otimes\mathcal{O}_{X\times\mathcal{N}_{\mathrm{par}}^{n_0-\mathrm{reg}}(L)}\longrightarrow\mathrm{At}_D(L)\otimes\mathcal{O}_{X\times\mathcal{N}_{\mathrm{par}}^{n_0-\mathrm{reg}}(L)}$ induced by ∇_L. So its image determines a subbundle of $\mathrm{At}_D(L)\otimes\mathcal{O}_{X\times\mathcal{N}_{\mathrm{par}}^{n_0-\mathrm{reg}}(L)}$. Let $\mathrm{At}_D(\widetilde{E},\widetilde{\boldsymbol{l}},\nabla_L)$ be the pullback of this subbundle by the homomorphism

$$\mathrm{At}_D(\widetilde{E},\widetilde{\boldsymbol{l}})\longrightarrow\mathrm{At}_D(\det(\widetilde{E}))\cong\mathrm{At}_D(L\otimes\mathcal{O}_{X\times\mathcal{N}_{\mathrm{par}}^{n_0-\mathrm{reg}}(L)})\tag{5.23}$$

defined by $D\longmapsto D\wedge\mathrm{Id}\wedge\cdots\wedge\mathrm{Id}+\cdots+\mathrm{Id}\wedge\cdots\wedge\mathrm{Id}\wedge D$. If we set

$$\widetilde{\mathcal{D}}^{\mathrm{par}}_{\mathfrak{sl},0}\ :=\ \left\{a\in\mathcal{E}nd(\widetilde{E})\ \middle|\ \mathrm{Tr(a)}=0\text{ and }\mathrm{a}|_{\mathrm{x_i}\times\ \mathcal{N}_{\mathrm{par}}^{\mathrm{n_0-reg}}(\mathrm{L})}(\widetilde{\mathrm{l}}_{\mathrm{j}}^{(\mathrm{i})})\ \subset\ \widetilde{\mathrm{l}}_{\mathrm{j}}^{(\mathrm{i})}\text{ for any i, j}\right\},$$

then the subbundle $\mathrm{At}_D(\widetilde{E},\widetilde{\boldsymbol{l}},\nabla_L)\subset\mathrm{At}_D(\widetilde{E},\widetilde{\boldsymbol{l}})$ fits in the exact sequence

$$0\longrightarrow\mathcal{E}nd_{\mathrm{par},\mathfrak{sl}}(\widetilde{E},\widetilde{\boldsymbol{l}})\longrightarrow\mathrm{At}_D(\widetilde{E},\widetilde{\boldsymbol{l}},\nabla_L)\xrightarrow{\mathrm{symb}_1}T_X(-D)\otimes\mathcal{O}_{X\times\mathcal{N}_{\mathrm{par}}^{n_0-\mathrm{reg}}(L)}\longrightarrow 0.$$

If we set

$$\widetilde{\mathcal{D}}^{\mathrm{par}}_{\mathfrak{sl},1}\ :=\ \left\{a\in\mathcal{E}nd(\widetilde{E})\otimes K_X(D)\ \middle|\ \mathrm{Tr}(a)=0\text{ and }\mathrm{res}_{x_i\times\ \mathcal{N}_{\mathrm{par}}^{n_0-\mathrm{reg}}(L)}(a)\ (\widetilde{l}_j^{(i)})\subset\widetilde{l}_{j+1}^{(i)}\text{ for any } i,\,j\right\},$$

then, by Serre duality,

$$H^1(X,\,\widetilde{\mathcal{D}}^{\mathrm{par}}_{\mathfrak{sl},0}|_{X\times\{p\}}\otimes K_X(D))^\vee\cong H^0(X,\,\widetilde{\mathcal{D}}^{\mathrm{par}}_{\mathfrak{sl},1}|_{X\times\{p\}}\otimes T_X(-D))\subset H^0(X,\,\widetilde{\mathcal{D}}^{\mathrm{par}}_{\mathfrak{sl},0}|_{X\times\{p\}}),$$

which in fact becomes zero because the underlying quasi-parabolic bundle $(\widetilde{E},\widetilde{\boldsymbol{l}})|_{X\times\{p\}}$ is simple. Let

$$\pi\ :\ X\times\mathcal{N}_{\mathrm{par}}^{n_0-\mathrm{reg}}(L)\longrightarrow\mathcal{N}_{\mathrm{par}}^{n_0-\mathrm{reg}}(L)$$

be the projection. Then we have $R^1\pi_*\left(\widetilde{\mathcal{D}}^{\mathrm{par}}_{\mathfrak{sl},0}\otimes K_X(D)\right)=0$, and get a short exact sequence

$$\begin{aligned}0\ \longrightarrow\ \pi_*\left(\widetilde{\mathcal{D}}^{\mathrm{par}}_{\mathfrak{sl},0}\otimes K_X(D)\right)\ &\longrightarrow\ \pi_*\left(\mathrm{At}_D(\widetilde{E},\widetilde{\boldsymbol{l}},\nabla_L)\otimes K_X(D)\right)\\ &\xrightarrow{\mathrm{symb}_1\otimes\mathrm{id}}\ \pi_*\left(\mathcal{O}_{X\times\mathcal{N}_{\mathrm{par}}^{n_0-\mathrm{reg}}(L)}\right)\ \longrightarrow\ 0.\end{aligned}$$

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Note that $\pi_*\left(\mathcal{O}_{X\times\mathcal{N}_{\mathrm{par}}^{n_0-\mathrm{reg}}(L)}\right)\cong\mathcal{O}_{\mathcal{N}_{\mathrm{par}}^{n_0-\mathrm{reg}}(L)}$. Consider the homomorphism

$$\Psi_t:\ \left(\pi_*\left(\mathrm{At}_D(\widetilde{E},\,\widetilde{\boldsymbol{l}},\,\nabla_L)\otimes K_X(D)\right)\oplus\mathcal{O}_{\mathcal{N}_{\mathrm{par}}^{n_0-\mathrm{reg}}(L)}\right)\otimes k[t]\ \longrightarrow\ \mathcal{O}_{\mathcal{N}_{\mathrm{par}}^{n_0-\mathrm{reg}}(L)}\otimes k[t]$$

on $\mathcal{N}_{\mathrm{par}}^{n_0-\mathrm{reg}}(L)\times\ \mathrm{Spec}\,k[t]$ defined by

$$(u,\,f)\ \longmapsto\ (\mathrm{symb}_1\otimes\mathrm{id}_{K_X(D)})(u)-tf$$

for $u\in\pi_*\left(\mathrm{At}_D(\widetilde{E},\,\widetilde{\boldsymbol{l}},\,\nabla_L)\otimes K_X(D)\right)\otimes k[t]$ and $f\in\mathcal{O}_{\mathcal{N}_{\mathrm{par}}^{n_0-\mathrm{reg}}(L)}\otimes k[t]$. Then $\ker\Psi_t$ is a locally free sheaf on $\mathcal{N}_{\mathrm{par}}^{n_0-\mathrm{reg}}(L)\times\ \mathrm{Spec}\,k[t]$, and we have

$$\ker\Psi_t\otimes k[t]/(t-h)\ \cong\ \begin{cases}\pi_*\left(\mathrm{At}_D(\widetilde{E},\,\widetilde{\boldsymbol{l}},\,\nabla_L)\otimes K_X(D)\right)(h\neq 0),\\ \pi_*\left(\widetilde{\mathcal{D}}_{\mathfrak{sl},0}^{\mathrm{par}}\otimes K_X(D)\right)\oplus\mathcal{O}_{\mathcal{N}_{\mathrm{par}}^{n_0-\mathrm{reg}}(L)}(h=0).\end{cases}$$

Define the projective bundle

$$\mathbb{P}_*(\ker\Psi_t)\ :=\ \ \mathrm{Proj}\left(\mathrm{Sym}\left((\ker\Psi_t)^\vee\right)\right)$$

over $\mathcal{N}_{\mathrm{par}}^{n_0-\mathrm{reg}}(L)\times\mathrm{Spec}\,k[t]$. There is a tautological line-subbundle

$$\mathcal{O}_{\mathbb{P}_*(\ker\Psi_t)}(-1)\ \hookrightarrow\ \ker\Psi_t\otimes\mathcal{O}_{\mathbb{P}_*(\ker\Psi_t)}.$$

Consider the sections

$$\begin{aligned}y:\ \mathcal{O}_{\mathbb{P}_*(\ker\Psi_t)}(-1)&\hookrightarrow\ker\Psi_t\otimes\mathcal{O}_{\mathbb{P}_*(\ker\Psi_t)}\\ &\hookrightarrow\pi_*\left(\mathrm{At}_D(\widetilde{E},\,\widetilde{\boldsymbol{l}},\,\nabla_L)\otimes K_X(D)\right)\otimes\mathcal{O}_{\mathbb{P}_*(\ker\Psi_t)}\oplus\mathcal{O}_{\mathbb{P}_*(\ker\Psi_t)}\longrightarrow\mathcal{O}_{\mathbb{P}_*(\ker\Psi_t)},\\ \widetilde{\nu}_j^{(i)}:\ \mathcal{O}_{\mathbb{P}_*(\ker\Psi_t)}(-1)&\hookrightarrow\ \ker\Psi_t\otimes\mathcal{O}_{\mathbb{P}_*(\ker\Psi_t)}\\ &\hookrightarrow\ \pi_*\left(\mathrm{At}_D(\widetilde{E},\,\widetilde{\boldsymbol{l}},\,\nabla_L)\otimes K_X(D)\right)\otimes\mathcal{O}_{\mathbb{P}_*(\ker\Psi_t)}\ \oplus\ \mathcal{O}_{\mathbb{P}_*(\ker\Psi_t)}\\ &\longrightarrow\ \pi_*\left(\mathrm{At}_D(\widetilde{E},\,\widetilde{\boldsymbol{l}},\,\nabla_L)\otimes K_X(D)\right)\otimes\mathcal{O}_{\mathbb{P}_*(\ker\Psi_t)}\\ &\xrightarrow{\mathrm{res}_D}\ \pi_*(\widetilde{\mathcal{D}}_{\mathfrak{sl},0}^{\mathrm{par}}|_{D\times\mathbb{P}_*(\ker\Psi_t)})\ \longrightarrow\ \pi_*(\mathcal{E}nd(\widetilde{l}_j^{(i)}/\widetilde{l}_{j+1}^{(i)})\otimes\mathcal{O}_{\mathbb{P}_*(\ker\Psi_t)})\\ &=\ \mathcal{O}_{\mathbb{P}_*(\ker\Psi_t)}.\end{aligned}$$

Let I be the ideal sheaf of the graded algebra $\mathrm{Sym}\,((\ker\Psi_t)^\vee)$ over $\mathcal{N}_{\mathrm{par}}^{n_0-\mathrm{reg}}(L)$, which is generated by $\left\{\widetilde{\nu}_j^{(i)}-\nu_j^{(i)}ty\ \middle|\ 1\leqslant i\leqslant n,0\leqslant j\leqslant r-1\right\}$. Set

$$\overline{\mathcal{M}'}\ :=\ \ \mathrm{Proj}(\mathrm{Sym}\left(\ker\Psi_t^\vee\right)/I)\subset\mathbb{P}_*(\ker\Psi_t).$$

Then there is a canonical structure morphism

$$\overline{\mathcal{M}'}\ \longrightarrow\ \ \mathrm{Spec}\,k[t].$$

Let $Y\subset\overline{\mathcal{M}'}$ be the effective divisor defined by the equation $y=0$. Setting $\mathcal{M}':=\overline{\mathcal{M}'}\setminus Y$, we see by the construction that $\mathcal{M}'_h\cong\mathcal{M}_{\mathrm{PC}}^{n_0\text{-reg}}(\boldsymbol{\nu},\nabla_L)^\circ$ for $h\neq 0$ and $\mathcal{M}'_0\cong\mathcal{M}_{\mathrm{Higgs}}^{n_0-\mathrm{reg}}(\mathbf{0},\,0)^\circ$. □

Theorem 5.22. *Let X be a smooth projective curve of genus g over an algebraically closed field k of arbitrary characteristic, and let $D=\sum_{i=1}^n x_i$ be a reduced effective divisor on X. Fix a line bundle L over X with a connection $\nabla_L:\ L\longrightarrow L\otimes K_X(D)$. Take positive integers r and d such that $r\geqslant 2$, $n\geqslant 1$ and r is not divisible by the characteristic of k. Assume that one of the following statements holds:*

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(a) $n \geqslant 1$ *and* $r \geqslant 2$ *are arbitrary if* $g \geqslant 2$;
(b) $n \geqslant 2$ *and* $n+r \geqslant 5$ *if* $g=1$;
(c) $n \geqslant 3$ *and* $n+r \geqslant 7$ *if* $g=0$.

Also, assume that the exponent $\boldsymbol{\nu} = (\nu^{(i)}_j)^{1\leqslant i\leqslant n}_{0\leqslant j\leqslant r-1}$ *satisfies the condition* $\mathrm{res}_{x_i}(\nabla_L) = \sum_{j=0}^{r-1}\nu^{(i)}_j$ *for any* i, *and, furthermore,* $\sum_{i=1}^n\sum_{\ell=1}^s \nu^{(i)}_{j^{(i)}_\ell}$ *is not contained in the image of* $\mathbb{Z}$ *in* k *for any integer* $1 \leqslant s < r$ *and any choice of* s *elements* $\{j^{(i)}_1, \cdots, j^{(i)}_s\}$ *in* $\{1, \ldots, r\}$ *for each* $1 \leqslant i \leqslant n$. *Then, the transcendence degree of the global algebraic functions on the moduli space* $\mathcal{M}^{\boldsymbol{\alpha}}_{\mathrm{PC}}(\boldsymbol{\nu})$ *of* $\boldsymbol{\alpha}$*-stable* $\boldsymbol{\nu}$*-parabolic connections satisfies the inequality*

$$\mathrm{tr.deg}_k\,\Gamma(\mathcal{M}^{\boldsymbol{\alpha}}_{\mathrm{PC}}(\boldsymbol{\nu}),\, \mathcal{O}_{\mathcal{M}^{\boldsymbol{\alpha}}_{\mathrm{PC}}(\boldsymbol{\nu})}) \;\leqslant\; r^2(g-1)-g+1+\frac{nr(r-1)}{2}.$$

Proof. Note that

$$\Gamma\left(\mathcal{M}^{n_0-\mathrm{reg}}_{\mathrm{Higgs}}(\mathbf{0},\,0)^\circ,\, \mathcal{O}_{\mathcal{M}^{n_0-\mathrm{reg}}_{\mathrm{Higgs}}(\mathbf{0},0)^\circ}\right) = \Gamma\left(\mathcal{M}^{n_0-\mathrm{reg}}_{\mathrm{Higgs}}(\mathbf{0},\,0),\, \mathcal{O}_{\mathcal{M}^{n_0-\mathrm{reg}}_{\mathrm{Higgs}}(\mathbf{0},0)}\right)$$

by Proposition 5.15. Since we can extend the Hitchin map in (5.20) to a morphism $\mathcal{M}^{n_0-\mathrm{reg}}_{\mathrm{Higgs}}(\mathbf{0},\,0) \longrightarrow W$, we have the inclusion maps

$$\Gamma(W,\,\mathcal{O}_W) \subset \Gamma\left(\mathcal{M}^{n_0-\mathrm{reg}}_{\mathrm{Higgs}}(\mathbf{0},\,0),\, \mathcal{O}_{\mathcal{M}^{n_0-\mathrm{reg}}_{\mathrm{Higgs}}(\mathbf{0},0)}\right) \subset \Gamma\left(\mathcal{M}^{\boldsymbol{\alpha}'}_{\mathrm{Higgs}}(\mathbf{0},\,0),\, \mathcal{O}_{\mathcal{M}^{\boldsymbol{\alpha}'}_{\mathrm{Higgs}}(\mathbf{0},0)}\right),$$

where we take $\boldsymbol{\alpha}'$ generic so that $\boldsymbol{\alpha}'$-semistability implies $\boldsymbol{\alpha}'$-stability. Then, using Corollary 5.20, it follows that $\Gamma\left(\mathcal{M}^{n_0-\mathrm{reg}}_{\mathrm{Higgs}}(\mathbf{0},\,0)^\circ,\, \mathcal{O}_{\mathcal{M}^{n_0-\mathrm{reg}}_{\mathrm{Higgs}}(\mathbf{0},0)^\circ}\right) = \Gamma\left(\mathcal{M}^{n_0-\mathrm{reg}}_{\mathrm{Higgs}}(\mathbf{0},\,0),\, \mathcal{O}_{\mathcal{M}^{n_0-\mathrm{reg}}_{\mathrm{Higgs}}(\mathbf{0},0)}\right)$ is a finitely generated k-algebra whose Krull dimension is $r^2(g-1)-g+1+nr(r-1)/2$.

We use the notation in the proof of Proposition 5.21. Note that $\mathcal{M}^{n_0-\mathrm{reg}}_{\mathrm{Higgs}}(\mathbf{0},\,0)^\circ$ is isomorphic to the cotangent bundle over $\mathcal{N}^{n_0-\mathrm{reg}}_{\mathrm{par}}(L)$. So we have $\mathcal{M}^{n_0-\mathrm{reg}}_{\mathrm{Higgs}}(\mathbf{0},\,0)^\circ \cong \mathrm{Spec}\Big(\mathrm{Sym}^*\Big(\pi_*(\widetilde{\mathcal{D}}^{\mathrm{par}}_{\mathfrak{sl},1})^\vee\Big)\Big)$, which implies that

$$\Gamma\left(\mathcal{M}^{n_0-\mathrm{reg}}_{\mathrm{Higgs}}(\mathbf{0},\,0)^\circ,\, \mathcal{O}_{\mathcal{M}^{n_0-\mathrm{reg}}_{\mathrm{Higgs}}(\mathbf{0},0)^\circ}\right) \cong \bigoplus_{m=0}^{\infty} H^0\left(\mathcal{N}^{n_0-\mathrm{reg}}_{\mathrm{par}}(L),\, \mathrm{Sym}^m\left(\pi_*(\widetilde{\mathcal{D}}^{\mathrm{par}}_{\mathfrak{sl},1})^\vee\right)\right).$$

Note that there is a short exact sequence

$$0 \longrightarrow \pi_*(\widetilde{\mathcal{D}}^{\mathrm{par}}_{\mathfrak{sl},1}) \longrightarrow \pi_*(\widetilde{\mathcal{D}}^{\mathrm{par}}_{\mathfrak{sl},0}\otimes K_X(D)) \xrightarrow{\;q\;} \bigoplus_{i,j}\pi_*(\mathcal{E}nd(\widetilde{l}^{(i)}_j/\widetilde{l}^{(i)}_{j+1})) \longrightarrow 0.$$

We can see that the above homomorphism q determines the equalities $(\widetilde{\nu}^{(i)}_j - \nu^{(i)}_j ty)|_{t=0}$ on the fiber $\mathbb{P}(\ker\Psi_t^\vee\otimes\mathbb{C}[t]/(t))$ over $t=0$. Taking the dual of the above exact sequence,

$$\begin{aligned}\left(\mathrm{Sym}\left(\ker\Psi_t^\vee\right)/I\right)\otimes k[t]/(t) \;&\cong\; \mathrm{Sym}\left(\pi_*(\widetilde{\mathcal{D}}^{\mathrm{par}}_{\mathfrak{sl},1})^\vee \oplus \mathcal{O}_{\mathcal{N}^{n_0-\mathrm{reg}}_{\mathrm{par}}(L)}\right)\\ &\cong \bigoplus_{d=0}^{\infty}\bigoplus_{d_1+d_2=d}\mathrm{Sym}^{d_1}\left(\pi_*(\widetilde{\mathcal{D}}^{\mathrm{par}}_{\mathfrak{sl},1})^\vee\right)T^{d_2},\end{aligned}$$

where T is a variable corresponding to the second component of $\ker\Psi_t\otimes k[t]/(t) = \pi_*(\widetilde{\mathcal{D}}^{\mathrm{par}}_{\mathfrak{sl},1})^\vee \oplus \mathcal{O}_{\mathcal{N}^{n_0-\mathrm{reg}}_{\mathrm{par}}(L)}$. So the ring of global sections of this sheaves of algebras over $\mathcal{N}^{n_0-\mathrm{reg}}_{\mathrm{par}}(L)$ becomes a polynomial ring

$$\Gamma\Big(\mathcal{N}^{n_0-\mathrm{reg}}_{\mathrm{par}}(L),\, \left(\mathrm{Sym}\left(\ker\Psi_t^\vee\right)/I\right)\otimes k[t]/(t)\Big) \cong \Gamma\left(\mathcal{M}^{n_0-\mathrm{reg}}_{\mathrm{Higgs}}(\mathbf{0},\,0)^\circ,\, \mathcal{O}_{\mathcal{M}^{n_0-\mathrm{reg}}_{\mathrm{Higgs}}(\mathbf{0},0)^\circ}\right)[T]$$

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over $\Gamma\left(\mathcal{M}_{\mathrm{Higgs}}^{n_0-\mathrm{reg}}(\mathbf{0},\,0)^\circ,\,\mathcal{O}_{\mathcal{M}_{\mathrm{Higgs}}^{n_0-\mathrm{reg}}(\mathbf{0},\,0)^\circ}\right)$. In particular, $\dim((\mathrm{Sym}^m(\ker\Phi_t^\vee)/I_m)\otimes k[t]/(t))$ becomes a polynomial in m of degree

$$\mathrm{Krull}-\dim\Gamma\left(\mathcal{M}_{\mathrm{Higgs}}^{n_0-\mathrm{reg}}(\mathbf{0},\,0)^\circ,\,\mathcal{O}_{\mathcal{M}_{\mathrm{Higgs}}^{n_0-\mathrm{reg}}(\mathbf{0},\,0)^\circ}\right) = r^2(g-1)-g+1+\frac{nr(r-1)}{2}.$$

Let $(\mathrm{Sym}\,(\ker\Psi_t^\vee)\,/I)_{(y)}$ be the subalgebra of the localized graded algebra $(\mathrm{Sym}\,(\ker\Psi_t^\vee)\,/I)_y$ consisting of homogeneous elements of degree zero. Then we have

$$\mathcal{M}_{\mathrm{PC}}^{n_0-\mathrm{reg}}(\boldsymbol{\nu},\nabla_L)^\circ \cong \mathrm{Spec}\left(\left(\mathrm{Sym}\left(\ker\Psi_t^\vee\right)/I\right)_{(y)}\otimes k[t]/(t-h)\right)$$

for $h\neq 0$. By the assumption in (5.19) on the choice of the exponent $\boldsymbol{\nu}$, and by Proposition 5.14, we have

$$\begin{aligned}\Gamma(\mathcal{M}_{\mathrm{PC}}^{\boldsymbol{\alpha}}(\boldsymbol{\nu},\nabla_L),\,\mathcal{O}_{\mathcal{M}_{\mathrm{PC}}^{\boldsymbol{\alpha}}(\boldsymbol{\nu},\nabla_L)}) &= \Gamma(\mathcal{M}_{\mathrm{PC}}^{n_0-\mathrm{reg}}(\boldsymbol{\nu},\nabla_L),\,\mathcal{O}_{\mathcal{M}_{\mathrm{PC}}^{n_0-\mathrm{reg}}(\boldsymbol{\nu},\nabla_L)})\\ &= \Gamma(\mathcal{M}_{\mathrm{PC}}^{n_0-\mathrm{reg}}(\boldsymbol{\nu},\nabla_L)^\circ,\,\mathcal{O}_{\mathcal{M}_{\mathrm{PC}}^{n_0-\mathrm{reg}}(\boldsymbol{\nu},\nabla_L)^\circ})\\ &= \Gamma\left(\mathcal{N}_{\mathrm{par}}^{n_0-\mathrm{reg}}(L),\,\left(\mathrm{Sym}\left(\ker\Psi_t^\vee\right)/I\right)_{(y)}\otimes k[t]/(t-h)\right).\end{aligned}$$

By Lemma A.1, which is proved later in § 6, the function

$$\begin{aligned}h\longmapsto{}&\dim H^0\left(\mathcal{N}_{\mathrm{par}}^{n_0-\mathrm{reg}}(L),\,\mathrm{Sym}^m(\ker\Psi_t^\vee)/I_m|_{t=h}\right)\\ &=\dim H^0\left(\mathcal{N}_{\mathrm{par}}^{n_0-\mathrm{reg}}(L),\,\mathrm{Sym}^m(\ker\Psi_t^\vee)/I_m\otimes k[t]/(t-h)\right)\end{aligned}$$

is upper semi-continuous in h. So we have

$$\dim H^0\left(\mathcal{N}_{\mathrm{par}}^{n_0-\mathrm{reg}}(L),\,\mathrm{Sym}^m(\ker\Psi_t^\vee)/I_m|_{t=h}\right)\leqslant\dim H^0\left(\mathcal{N}_{\mathrm{par}}^{n_0-\mathrm{reg}}(L),\,\mathrm{Sym}^m(\ker\Psi_t^\vee)/I_m|_{t=0}\right)$$

for $h\neq 0$.

Let d be the transcendence degree of $\Gamma(\mathcal{M}_{\mathrm{PC}}^{\boldsymbol{\alpha}}(\boldsymbol{\nu},\nabla_L),\,\mathcal{O}_{\mathcal{M}_{\mathrm{PC}}^{\boldsymbol{\alpha}}(\boldsymbol{\nu},\nabla_L)})$ over k. Then we have

$$\begin{aligned}d &= \mathrm{tr.deg}_k\Gamma\left(\mathcal{N}_{\mathrm{par}}^{n_0-\mathrm{reg}}(L),\,(\mathrm{Sym}\,(\ker\Psi_t)\,/I)_{(y)}\otimes k[t]/(t-h)\right)\\ &= \mathrm{tr.deg}_k\Gamma\left(\mathcal{N}_{\mathrm{par}}^{n_0-\mathrm{reg}}(L),\,(\mathrm{Sym}\,(\ker\Psi_t)\,/I)\otimes k[t]/(t-h)\right)-1.\end{aligned}$$

Take homogeneous elements $x_1,\,\cdots,\,x_d$ of

$$\Gamma\left(\mathcal{N}_{\mathrm{par}}^{n_0-\mathrm{reg}}(L),\,(\mathrm{Sym}\,(\ker\Psi_t)\,/I)\,|_{t=h}\right)$$

such that $\{x_1,\,\cdots,\,x_d,\,y\}$ is a transcendence basis of $\Gamma\left(\mathcal{N}_{\mathrm{par}}^{n_0-\mathrm{reg}}(L),\,(\mathrm{Sym}\,(\ker\Psi_t)\,/I)\,|_{t=h}\right)$ over k. Let S be the graded subalgebra of

$$\Gamma\left(\mathcal{N}_{\mathrm{par}}^{n_0-\mathrm{reg}}(L),\,(\mathrm{Sym}\,(\ker\Psi_t)\,/I)\,|_{t=h}\right)$$

generated by $x_1,\,\cdots,x_d,\,y$. Then

$$\begin{aligned}\dim S_m &\leqslant \dim H^0\left(\mathcal{N}_{\mathrm{par}}^{n_0-\mathrm{reg}}(L),\,\mathrm{Sym}^m(\ker\Psi_t^\vee)/I_m|_{t=h}\right)\\ &\leqslant \dim H^0\left(\mathcal{N}_{\mathrm{par}}^{n_0-\mathrm{reg}}(L),\,\mathrm{Sym}^m(\ker\Psi_t^\vee)/I_m|_{t=0}\right).\end{aligned}$$

Since S_m is a polynomial in m of degree d for $m\gg 0$, it follows that $d\leqslant r^2(g-1)-g+1+nr(r-1)/2$. □

Remark 5.23. A statement similar to Theorem 5.22 can be considered for connections without pole. When X is a curve over the field of complex numbers whose genus is greater than 2, then there are only constant global algebraic functions on the de Rham moduli space of connections without pole by [BiRa, Corollary 4.4]. So the inequality similar to Theorem 5.22 becomes strict

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in that case. On the other hand, if X is defined over the base field of positive characteristic, it is proved in [Groe, Theorem 1.1] that the Hitchin map for the de Rham moduli space connections without pole is étale locally equivalent to that on the Dolbeault moduli space. So, the ring of global algebraic functions on the de Rham moduli space has the same transcendence degree as that of the ring of global algebraic functions on the Dolbeault moduli space in that case. The Hitchin map for the logarithmic de Rham moduli space over the base field of positive characteristic is introduced in [dCHZ].

The following is an immediate consequence of Theorem 5.22.

Corollary 5.24. *The moduli space $\mathcal{M}^{\boldsymbol{\alpha}}_{\mathrm{PC}}(\boldsymbol{\nu}, \nabla_L)$ of $\boldsymbol{\alpha}$-stable $\boldsymbol{\nu}$-parabolic connections is not affine.*

From now on, consider the case of $k = \mathbb{C}$.

Since the fundamental group $\pi_1(X \setminus D, *)$ is finitely presented, the space of representations

$$\mathrm{Hom}(\pi_1(X \setminus D, *),\ \mathrm{GL}(r, \mathbb{C}))$$

can be realized as an affine variety. Take generators $\alpha_1, \beta_1, \cdots, \alpha_g, \beta_g$ of the fundamental group $\pi_1(C, *)$, and choose a loop γ_i around each x_i with respect to the base point $*$. Then the fundamental group $\pi_1(X \setminus D, *)$ is generated by $\alpha_1, \beta_1, \cdots, \alpha_g, \beta_g, \gamma_1, \cdots, \gamma_n$ with the single relation $[\alpha_1, \beta_1] \cdots [\alpha_g, \beta_g]\gamma_1 \cdots \gamma_n = 1$. The space of representations of $\pi_1(X \setminus D, *)$ can be realized as the affine variety

$$\begin{aligned}
&\mathrm{Hom}(\pi_1(X \setminus D, *),\ \mathrm{GL}(r, \mathbb{C})) \\
&= \left\{ (A_1, B_1, \ldots, A_g, B_g, C_1, \ldots, C_n) \in \mathrm{GL}(r, \mathbb{C})^{2g+n} \;\middle|\; (\prod_{i=1}^{g} A_i^{-1} B_i^{-1} A_i B_i) C_1 \cdots C_n = I_r \right\}.
\end{aligned}$$

Note that the connection ∇_L on the line bundle L induces a one-dimensional representation ρ_{∇_L} of $\pi_1(X \setminus D, *)$. Define a tuple $(b^{(i)}_j)$ by $b^{(i)}_j := e^{-2\pi\sqrt{-1}\nu^{(i)}_j}$, and consider the closed subvariety

$$Y = \left\{ \begin{array}{l} ((A_k, B_k), (C_i)) \in \\ \mathrm{Hom}(\pi_1(X \setminus D, *), \mathrm{GL}(r, \mathbb{C})) \end{array} \;\middle|\; \begin{array}{l} \rho_{\nabla_L}(\alpha_k) = \det(A_k) \text{ and } \rho_{\nabla_L}(\beta_k) = \det(B_k) \\ \text{for } 1 \leqslant \text{ i} \leqslant \text{ g and} \\ \det(TI_r - C_i) = \prod_{j=0}^{r-1}(T - b^{(i)}_j) \text{ for} 1 \leqslant i \leqslant n \end{array} \right\}$$

of $\mathrm{Hom}(\pi_1(X \setminus D, *), \mathrm{GL}(r, \mathbb{C}))$. There is a canonical action of $\mathrm{GL}(r, \mathbb{C})$ on Y given by the adjoint action of $\mathrm{GL}(r, \mathbb{C})$ on itself, and we can take the corresponding categorical quotient

$$\mathrm{Ch}_{X \setminus D, (b^{(i)}_j)} := Y /\!\!/ \mathrm{GL}(r, \mathbb{C}) = \mathrm{Spec}\, \Gamma(Y, \mathcal{O}_Y)^{\mathrm{GL}(r, \mathbb{C})}. \tag{5.24}$$

Under the genericity assumption in (5.19) of the eigenvalues of the residues, this quotient is in fact a geometric quotient, and we have a Riemann–Hilbert morphism

$$\mathrm{RH} : \mathcal{M}^{\boldsymbol{\alpha}}_{\mathrm{PC}}(\boldsymbol{\nu}, \nabla_L) \longrightarrow \mathrm{Ch}_{X \setminus D, (b^{(i)}_j)}.$$

By [Ina], the above Riemann–Hilbert morphism RH is a proper and surjective holomorphic map, which is generically an isomorphism. So $\mathcal{M}^{\boldsymbol{\alpha}}_{\mathrm{PC}}(\boldsymbol{\nu}, \nabla_L)$ gives an analytic resolution of singularities of $\mathrm{Ch}_{X \setminus D, (b^{(i)}_j)}$. Since the character variety $\mathrm{Ch}_{X \setminus D, (b^{(i)}_j)}$ is affine by its definition, it is evident that

$$\mathrm{tr.deg}_{\mathbb{C}} \Gamma(\mathrm{Ch}_{X \setminus D, (b^{(i)}_j)}, \mathcal{O}_{\mathrm{Ch}_{X \setminus D, (b^{(i)}_j)}}) = \dim \mathrm{Ch}_{X \setminus D, (b^{(i)}_j)} = 2(r^2 - 1)(g - 1) + r(r - 1)n. \tag{5.25}$$

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By Theorem 5.22 and (5.25) (or by Corollary 5.24), we have the following.

Corollary 5.25. *The Riemann–Hilbert morphism* $\mathrm{RH} : \mathcal{M}^{\boldsymbol{\alpha}}_{\mathrm{PC}}(\boldsymbol{\nu}, \nabla_L) \longrightarrow \mathrm{Ch}_{X\setminus D,(b^{(i)}_j)}$ *is not an algebraic morphism.*

Appendix

Let k be an algebraically closed field of arbitrary characteristic. We will prove a lemma on the upper semi-continuity of the dimension of global sections of vector bundles on an algebraic space containing a projective variety over k.

Recall that an algebraic space $\mathcal{X}$ of finite type over $\operatorname{Spec} k$ is said to be locally separated over $\operatorname{Spec} k$ if there is a scheme U of finite type over $\operatorname{Spec} k$ together with an étale surjective morphism $U \longrightarrow \mathcal{X}$ such that $U \times_{\mathcal{X}} U$ is a locally closed subscheme of $U \times_{\operatorname{Spec} \mathbb{C}} U$. A locally separated algebraic space $\mathcal{X}$ of finite type over $\operatorname{Spec} k$ is irreducible if the underlying topological space $|\mathcal{X}|$ is irreducible. In other words, any two non-empty open subspaces $U_1,\, U_2 \subset \mathcal{X}$ intersect: $U_1 \cap U_2 \neq \emptyset$.

Lemma A.1. *Let $\mathcal{X}$ be a locally separated, smooth, irreducible algebraic space of finite type over $\operatorname{Spec} k$. Assume that $\overline{X}$ is an open subspace of $\mathcal{X}$ such that $\overline{X}$ is isomorphic to a smooth projective variety over k. Let T be an affine variety, and let $\mathcal{F}$ be a locally free sheaf of finite rank on $\mathcal{X} \times T$. For each point $t \in T$, denote by $\Gamma(\mathcal{X} \times \{t\},\, \mathcal{F}|_{\mathcal{X}\times\{t\}})$ the space of global sections of the restriction $\mathcal{F}|_{\mathcal{X}\times\{t\}}$. Then the function*

$$T \longrightarrow \mathbb{Z}_{\geqslant 0}, \quad t \longmapsto \dim \Gamma(\mathcal{X} \times \{t\},\, \mathcal{F}|_{\mathcal{X}\times\{t\}})$$

is upper semi-continuous.

Proof. Since the upper semi-continuity is a local property on T, we may replace T with a neighborhood at any point of T. Take a finite number of smooth affine varieties $\{U_i\}_{i=1}^n$ and an étale surjective morphism

$$f \,:\, \overline{X} \sqcup \coprod_{i=1}^{n} U_i \longrightarrow \mathcal{X},$$

whose restriction to $\overline{X}$ coincides with the given inclusion map $f|_{\overline{X}} : \overline{X} \hookrightarrow \mathcal{X}$. After shrinking U_i and T, we may assume that $\mathcal{F}|_{U_i\times T} \cong \mathcal{O}^{\oplus r}_{U_i\times T}$ for every i. Since $\mathcal{X}$ is irreducible, we have $\overline{X} \cap (\bigcap_{i=1}^n f(U_i)) \neq \emptyset$. So there is a non-empty affine open subset $V \subset \overline{X} \cap \bigcap_{i=1}^n f(U_i)$. Take a non-empty smooth affine variety $\widetilde{V}$ with étale morphisms $\widetilde{V} \longrightarrow V$ and $\widetilde{f}_i : \widetilde{V} \longrightarrow U_i$ for $1 \leqslant i \leqslant n$ such that the diagram

$$\begin{array}{ccc} \widetilde{V} & \xrightarrow{\widetilde{f}_i} & U_i \\ \big\downarrow & & \big\downarrow{\scriptstyle f} \\ V & \longrightarrow & \mathcal{X} \end{array}$$

is commutative for every $1 \leqslant i \leqslant n$.

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Let $\widetilde{X}$ be the normalization of $\overline{X}$ in the field $K(\widetilde{V})$ of rational functions on $\widetilde{V}$. Then $\widetilde{X}$ is a projective variety with the following canonical commutative diagram.

$$\begin{array}{ccc} \widetilde{V} & \longrightarrow & \widetilde{X} \\ \big\downarrow & & \big\downarrow \\ V & \longrightarrow & \overline{X} \end{array}$$

After shrinking $\widetilde{V}$ if necessary, $\widetilde{V} \longrightarrow \widetilde{X}$ is an open immersion. We can take a very ample divisor $D \subset \overline{X}$ such that $\overline{X} \setminus V \subset D$. Choose a very ample divisor $\widetilde{D}$ on $\widetilde{X}$ such that the inclusion $\widetilde{X} \setminus \widetilde{V} \subset \widetilde{D}$ holds set theoretically and that $D \times_{\overline{X}} \widetilde{X} \subset \widetilde{D}$.

We can construct a projective variety P_i with a very ample divisor $D_i \subset P_i$ such that $P_i \setminus D_i$ is isomorphic to U_i. We can also take a very ample divisor $D'_i \subset P_i$ such that $P_i \setminus \widetilde{f}_i(\widetilde{V}) \subset D'_i$ holds set theoretically and that $D'_i = D_i + B_i$ holds for a divisor B_i without any common component with D_i.

For $i < j$, the fiber product $U_i \times_{\mathcal{X}} U_j$ is a smooth quasi-affine scheme over $\operatorname{Spec} k$. So we can construct a projective scheme P_{ij} over $\operatorname{Spec} k$ that contains $U_i \times_{\mathcal{X}} U_j$ as a Zariski open subscheme. Choose a very ample divisor $D_{ij} \subset P_{ij}$ such that $P_{ij} \setminus (U_i \times_{\mathcal{X}} U_j) \subset D_{ij}$.

Since $\overline{X}$ is projective and D is very ample, we can take a sufficiently large integer l such that $H^p(\overline{X} \times \{t\}, \mathcal{F}|_{\overline{X}\times\{t\}}(lD)) = 0$ for all $p \geqslant 1$ and $t \in T$. After shrinking T, the space of sections $\Gamma(\mathcal{F}|_{\overline{X}\times T}(lD))$ is a free $\Gamma(\mathcal{O}_T)$–module of finite rank and the map $\Gamma(\mathcal{F}|_{\overline{X}\times T}(lD)) \otimes k(t) \longrightarrow \Gamma(\mathcal{F}|_{\overline{X}\times\{t\}}(lD))$ is bijective for any $t \in T$, where $k(t)$ is the residue field of $\mathcal{O}_{T,t}$.

Choose generators $s_1, \cdots, s_N$ of $\Gamma(\mathcal{F}|_{\overline{X}\times T}(lD))$. Consider the pullbacks of these sections by the morphism $P_i \setminus D'_i \hookrightarrow \widetilde{f}_i(\widetilde{V}) \xrightarrow{(f|_{U_i})|_{\widetilde{f}_i(\widetilde{V})}} V \hookrightarrow \overline{X}$ and denote them by

$$s_1|_{P_i\setminus D'_i}, \ldots, s_N|_{P_i\setminus D'_i} \in \Gamma(\mathcal{F}|_{(U_i\setminus(U_i\cap D'_i))\times T}) \cong \Gamma(\mathcal{O}^{\oplus r}_{(P_i\setminus D'_i)\times T}).$$

There is a sufficiently large integer l_i such that each $s_1|_{P_i\setminus D'_i}, \cdots, s_N|_{P_i\setminus D'_i}$ can be lifted to a section of $\Gamma(\mathcal{O}_{P_i\times T}(l_i D'_i))$.

After shrinking T, the space of sections $\Gamma(\mathcal{O}_{P_i\times T}(l_i D_i))$ is a free $\Gamma(\mathcal{O}_T)$–module of finite rank. Fix a basis $t^{(i)}_1, \cdots, t^{(i)}_{N_i}$ of it. Let $t^{(i)}_\ell|_{\widetilde{X}\setminus\widetilde{D}}$ be the pullback of $t^{(i)}_\ell$ by the composition of the maps

$$\widetilde{X} \setminus \widetilde{D} \hookrightarrow \widetilde{V} \longrightarrow \widetilde{f}_i(\widetilde{V}) \hookrightarrow U_i = P_i \setminus D_i \hookrightarrow P_i.$$

Then there is an integer $\widetilde{l} \geqslant l$ such that all $t^{(i)}_1|_{\widetilde{X}\setminus\widetilde{D}}, \cdots, t^{(i)}_{N_i}|_{\widetilde{X}\setminus\widetilde{D}}$ can be lifted to sections $\widetilde{t}^{(i)}_1, \cdots, \widetilde{t}^{(i)}_{N_i}$ of $\Gamma(\mathcal{F}_{\widetilde{X}\times T}(\widetilde{l}\,\widetilde{D}))$ for $1 \leqslant i \leqslant n$.

Consider the pullback $t^{(i)}_\gamma|_{P_{ij}\setminus D_{ij}}$ of $t^{(i)}_\gamma$ by the composition of maps

$$P_{ij} \setminus D_{ij} \hookrightarrow U_i \times_N U_j \longrightarrow U_i \hookrightarrow P_i.$$

If we choose l_{ij} sufficiently large, all $t^{(i)}_1|_{P_{ij}\setminus D_{ij}}, \cdots, t^{(i)}_{N_i}|_{P_{ij}\setminus D_{ij}}$ can be lifted to sections

$$t^{(i)}_{j,1}, \cdots, t^{(i)}_{j,N_i}$$

of $\Gamma(\mathcal{O}_{P_i\times T}(l_{ij}D_{ij})^{\oplus r})$. We may also assume that all $t^{(j)}_1|_{P_{ij}\setminus D_{ij}}, \cdots, t^{(j)}_{N_j}|_{P_{ij}\setminus D_{ij}}$ can be lifted to sections $t^{(j)}_{i,1}, \cdots, t^{(j)}_{i,N_j}$ of $\Gamma(\mathcal{O}_{P_i\times T}(l_{ij}D_{ij})^{\oplus r})$.

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Take a resolution

$$\mathcal{L}_1 \xrightarrow{\partial_{\mathcal{L}_\bullet}} \mathcal{L}_0 \xrightarrow{\psi} \mathcal{F}|^{\vee}_{\overline{X}\times T} \longrightarrow 0,$$

where $\mathcal{L}_i = \mathcal{O}_{\overline{X}\times T}(-m_i)^{\oplus R_i}$ for $i = 1, 2$ and $m_i \gg 1$. After shrinking T, both $\Gamma(\mathcal{L}_i^{\vee})$ and $\Gamma(\mathcal{L}_i^{\vee}(lD))$ are free $\Gamma(\mathcal{O}_T)$–modules for $i = 1, 2$. Let $\mathcal{F}|_{\widetilde{X}\times T}$ and $\mathcal{L}_i|_{\widetilde{X}\times T}$ respectively be the pullbacks of $\mathcal{F}|_{\overline{X}\times T}$ and $\mathcal{L}_i$ by the morphism $\widetilde{X}\times T \longrightarrow \overline{X}\times T$. Then there is the following commutative diagram with exact rows.

$$\begin{array}{ccccccc} 0 \longrightarrow & \Gamma(\mathcal{F}|_{\overline{X}\times T}) & \longrightarrow & \Gamma(\mathcal{L}_0^{\vee}) & \xrightarrow{\Gamma(\partial^{\vee}_{\mathcal{L}_\bullet})} & \Gamma(\mathcal{L}_1^{\vee}) \\ & \downarrow & & \downarrow & & \downarrow \\ 0 \longrightarrow & \Gamma(\mathcal{F}|_{\widetilde{X}\times T}(\widetilde{l}\,\widetilde{D})) & \longrightarrow & \Gamma(\mathcal{L}_0^{\vee}|_{\widetilde{X}\times T}(\widetilde{l}\,\widetilde{D})) & \longrightarrow & \Gamma(\mathcal{L}_1^{\vee}|_{\widetilde{X}\times T}(\widetilde{l}\,\widetilde{D})) \end{array}$$

Consider the homomorphism

$$\Phi \;:\; \Gamma(\mathcal{L}_0^{\vee}) \oplus \bigoplus_{i=1}^{n} \Gamma(\mathcal{O}_{P_i\times T}(l_iD_i)^{\oplus r}) \;\longrightarrow\; \Gamma(\mathcal{L}_1^{\vee}) \oplus \Gamma(\widetilde{\mathcal{L}}_0^{\vee}(\widetilde{l}\widetilde{D}))^{\oplus n} \oplus \bigoplus_{i<j} \Gamma(\mathcal{O}_{P_{ij}\times T}(l_{ij}D_{ij})^{\oplus r})$$

defined by

$$\Big(\alpha,\, \Big(\sum_{\gamma=1}^{N_i} c_\gamma^{(i)} t_\gamma^{(i)}\Big)\Big) \longmapsto \Big(\Gamma(\partial^{\vee}_{\mathcal{L}_\bullet})(\alpha),\, \Big(\iota(\alpha) - \sum_{\gamma=1}^{N_i} c_\gamma^{(i)} \Gamma(\psi^{\vee})(\widetilde{t}_\gamma^{(i)})\Big)_i,\, \Big(\sum_{\gamma=1}^{N_i} c_\gamma^{(i)} t_{j,\gamma}^{(i)} - \sum_{\gamma=1}^{N_j} c_\gamma^{(j)} t_{i,\gamma}^{(j)}\Big)_{i<j}\Big),$$

where $\iota :\, \Gamma(\mathcal{L}_0^{\vee}) \longrightarrow \Gamma(\mathcal{L}_0^{\vee}(\widetilde{l}\,\widetilde{D}))$ is the canonical inclusion map and

$$\Gamma(\psi^{\vee}) :\, \Gamma(\mathcal{F}|_{\widetilde{X}\times T}(\widetilde{l}\,\widetilde{D})) \longrightarrow \Gamma(\mathcal{L}_0^{\vee}|_{\widetilde{X}\times T}(\widetilde{l}\,\widetilde{D}))$$

is the map induced by ψ.

Claim. $\Gamma(\mathcal{X}\times\{t\},\, \mathcal{F}|_{\mathcal{X}\times\{t\}}) \;=\; \ker(\Phi\otimes k(t))$ for any $t \in T$.

Proof of Claim. Take a section $s \in \Gamma(\mathcal{X}\times\{t\},\, \mathcal{F}|_{\mathcal{X}\times\{t\}})$. Its restriction $s|_{\overline{X}\times\{t\}}$ is a section of $\Gamma(\overline{X},\, \mathcal{F}|_{\overline{X}\times\{t\}}) \subset \Gamma(\overline{X}, \mathcal{F}|_{\overline{X}\times\{t\}}(lD))$. From the choice of l_i, the pullback $(f|_{U_i\cap\widetilde{f}_i(\widetilde{V})})^*(s|_{\overline{X}\times\{t\}})$ can be lifted to a section σ_i of $\Gamma(\mathcal{O}_{P_i\times\{t\}}(l_iD'_i))$. On the other hand, we have $(s|_{U_i})|_{\widetilde{f}_i(\widetilde{V})} = (f|_{U_i\cap\widetilde{f}_i(\widetilde{V})})^*(s|_{\overline{X}\times\{t\}})$. Since $s|_{U_i}$ does not have pole along B_i, it follows that σ_i belongs to $\Gamma(\mathcal{O}_{P_i\times\{t\}}(l_iD_i))$. So we get an element $(\psi^{\vee}(s|_{\overline{X}\times\{t\}}),\, (\sigma_i)_i)$ of

$$\Big(\Gamma(\mathcal{L}_0^{\vee}) \oplus \bigoplus_{i=1}^{n} \Gamma(\mathcal{O}_{P_i\times T}(l_iD_i)^{\oplus r})\Big) \otimes k(t).$$

By the construction, we have $\Phi(\psi^{\vee}(s|_{\overline{X}\times\{t\}}),\, (\sigma_i)_i) = 0$. So we get the inclusion map $\Gamma(\mathcal{X}\times\{t\},\, \mathcal{F}|_{\mathcal{X}\times\{t\}}) \subset \ker(\Phi\otimes k(t))$.

To prove the reverse direction, take a section $(\alpha,\, (s_i)) \in \ker(\Phi\otimes k(t))$. Since $\Gamma(\partial_{\mathcal{L}_\bullet})(\alpha) = 0$, there is a section $s \in \Gamma(\mathcal{F}|_{\overline{X}\times\{t\}})$ such that $\psi^{\vee}(s) = \alpha$. Considering the middle component of $\Phi(\alpha,\, (s_i)) = 0$, we obtain the equality $s|_{\overline{X}\times_{\mathcal{X}}U_i} = s_i|_{\overline{X}\times_{\mathcal{X}}U_i}$, because the maps $\Gamma(\mathcal{F}|_{\widetilde{X}\times T}(\widetilde{l}\widetilde{D})) \longrightarrow \Gamma(\mathcal{L}_0^{\vee}|_{\widetilde{X}\times T}(\widetilde{l}\widetilde{D}))$ and $\Gamma(\mathcal{F}|_{(\overline{X}\times_{\mathcal{X}}U_i)\times\{t\}}) \longrightarrow \Gamma(\mathcal{F}|_{(\widetilde{X}\setminus\widetilde{D})\times\{t\}})$ are injective. So

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$(s,\ (s_i))$ is in the kernel of

$$\Gamma(\mathcal{F}|_{(\overline{X}\sqcup\coprod_{i=1}^{n} U_i)\times\{t\}}) \longrightarrow \Gamma(\mathcal{F}|_{(\overline{X}\sqcup\coprod_{i=1}^{n} U_i)\times_{\mathcal{X}}(\overline{X}\sqcup\coprod_{i=1}^{n} U_i)\times\{t\}}),$$

which is in fact $\Gamma(\mathcal{X}\times\{t\},\ \mathcal{F}|_{\mathcal{X}\times\{t\}})$. So we also have the inclusion

$$\ker(\Phi\otimes k(t)) \subset \Gamma(\mathcal{X}\times\{t\}, \mathcal{F}|_{\mathcal{X}\times\{t\}}).$$

This proves the claim.

Since the claim holds, it suffices to show that

$$\{t\in T \mid \dim\ker(\Phi\otimes k(t)) \geqslant d\}$$

is Zariski closed for any $d\in\mathbb{Z}_{\geqslant 0}$. Note that

$$\dim\ker(\Phi\otimes k(t)) = \operatorname{rank}_{\Gamma(\mathcal{O}_T)}\Big(\Gamma(\mathcal{L}_0^{\vee})\oplus\bigoplus_{i=1}^{n}\Gamma(\mathcal{O}_{P_i\times T}(l_iD_i)^{\oplus r})\Big) - \operatorname{rank}(\Phi\otimes k(t)).$$

Since the subset of T given by locus of all points satisfying the condition

$$\operatorname{rank}(\Phi\otimes k(t)) \leqslant \operatorname{rank}_{\Gamma(\mathcal{O}_T)}\Big(\Gamma(\mathcal{L}_0^{\vee})\oplus\bigoplus_{i=1}^{n}\Gamma(\mathcal{O}_{P_i\times T}(l_iD_i)^{\oplus r})\Big) - d$$

is Zariski closed, the proof of the lemma is complete. □

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Conflicts of Interest

None.

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