FINITE GROUPS WITH SHORT NONNORMAL CHAINS

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This note is a continuation of the author's work [6], describing the structure of a finite group given some information about the distribution of the subnormal subgroups in the lattice of all subgoups. The notation is that of [6], briefly as follows:

DEFINITION. An *upper chain of length n* in the finite group G is a sequence of subgroups of G; $G = G_0 > G_1 > \cdots > G_n$, such that for each i, G_i is a maximal subgroup of G_{i-1}. Let $h(G) = n$ if every upper chain in G of length n contains a proper ($\neq G$) subnormal entry, and there is at least one upper chain in G of length $(n-1)$ which contains no proper subnormal entry.

Let $k(G)$ denote the derived length of G, $|G|$ denote the order of G, $\pi(G)$ denote the number of distinct prime divisors of $|G|$, $l(G)$ denote the Fitting length of G, and $\omega(G)$ denote the length of the longest upper chain in G. Note that if G is solvable, $\omega(G)$ is simply the number of prime factors of $|G|$.

We obtain the following theorem.

THEOREM. *If G is a finite solvable, non-nilpotent group, then*

$$(1) \quad k(G) \leqq h(G)$$

$$(2) \quad l(G) \leqq h(G) - \pi(G) + 2.$$

For reference we list a few lemmas, proven in [6], concerning the function h. All groups under consideration are assumed to be finite.

LEMMA 1. *If N is a normal subgroup of G, then $h(G/N) \leqq h(G)$.*

LEMMA 2. *If H is a nonnormal maximal subgroup of G, then $h(H) < h(G)$.*

LEMMA 3. *If $G = H \times K$, where H is not nilpotent, then $h(G) \geqq h(H) + \omega(K)$.*

LEMMA 4. *If N is a proper normal subgroup of G and $h(G/N) = h(G)$, then N is cyclic of prime power order.*

In [6, Theorem 1] the author showed that for a solvable group G, $l(G) \leqq h(G)$. We now prove a stronger result relating $h(G)$ and $k(G)$.

THEOREM 1. *If G is a finite solvable group such that $2 \leqq h(G) \leqq n$, then $k(G) \leqq n$.*

PROOF. The proof is by induction on n. For $n = 2$, the theorem follows from [6, Theorem 5], so assume the theorem is true for all groups K satisfying $2 \leqq h(K) \leqq (n-1)$, and is false for some group K satisfying $h(K) = n$. Among such groups, let G be one of minimal order. We show that such a group G does not exist. For such a group G we have:

(1) G has a proper non-nilpotent homomorphic image.

Suppose not and let N denote a minimal normal subgroup of G, and let L denote a nonnormal maximal subgroup of G. Since $h(G) \geqq 2$, such an L exists. Since G/N is nilpotent, and L is nonnormal, $N \nleqq L$. Therefore $LN = G$, and $L \cap N = \{1\}$. L is core free and nilpotent, so L does not contain a non trivial subgroup subnormal in G. Thus $h(G) \geqq 1 + \omega(L)$. But then

$$1 + \omega(L) \leqq h(G) < k(G) \leqq 1 + k(L) \leqq 1 + \omega(L),$$

which is a contradiction.

(2) If A is an abelian normal subroup of G and G/A is not nilpotent, then $h(G/A) = h(G)$.

In any case $h(G/A) \leqq h(G)$. If $h(G/A) < h(G)$, we have by induction that $k(G/A) \leqq h(G/A)$, so that $k(G) \leqq h(G)$. This contradicts the choice of G, so is impossible.

(3) $k(G) = n + 1$

Let A denote an abelian normal subgroup of G such that G/A is not nilpotent. By (1), such an A exists. By (2), $h(G/A) = n$, so

$$n = h(G) < k(G) \leqq k(G/A) + 1 = h(G/A) + 1 \leqq n + 1.$$

(4) G has a unique minimal normal subgroup.

If N is a minimal normal subgroup of G, then either G/N is nilpotent or $k(G/N) \leqq n$, so clearly there are at most two minimal normal subgroups. Suppose there are two, say A and B such that G/A is not nilpotent, and G/B is nilpotent. Let L be a nonnormal maximal subgroup of G. Now $B \nleqq L$, so $G = BL$, and $B \cap L = \{1\}$. Also $h(G/A) = n$, so $k(G/A) = n$, and hence $k(G/B) = k(L) = n + 1$. Now $G' \geqq B$, and $G^{(n)} = A \ngeqq B$, so let r be minimal with respect to $G^{(r)} \geqq B$, and $G^{(r+1)} \ngeqq B$. $G/G^{(r+1)}$ is not nilpotent, and each nonnormal maximal subgroup of G is a complement to B so without loss of generality we may assume that $G^{(r+1)} \leqq L$. But since $G^{(r+1)}B = G^{(r+1)} \times B = G^{(r)}$, we have $k(G^{(r)}) = k(G^{(r+1)})$, which is a contradiction. Thus there is ust one minimal normal

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subgroup. Note that by the minimality of G, and (2) and (3), the unique minimal normal subgroup is actually of prime order and is actually $G^{(n)}$.

(5) G does not contain a normal Sylow subgroup.

This follows from (2) and (4), and the fact that the next to last entry in an h-chain for G is cyclic, primary, and not subnormal. Here h-chain refers to any upper chain in G of length $h(G)$ with only its terminal entry subnormal in G.

(6) For each r, $h(G/G^{(r)}) = r$ or 1.

This is certainly true for $r = 1$ and $r = n$, so suppose $h(G/G^{(r+1)}) = r + 1$, and consider $G/G^{(r)}$. If $h(G/G^{(r)}) \neq 1$, then since $k(G/G^{(r)}) = r$, $h(G/G^{(r)}) \geqq 1$. To show equality, we suppose that $h(G/G^{(r)}) > r$, and let $G = H_0 > H_1 > \cdots H_r > G^{(r)}$ be an h-chain for $G/G^{(r)}$. Now H_r is not subnormal in G, and $h(G/G^{(r+1)}) = r + 1$, so $H_r/G^{(r+1)}$ is cyclic of prime power order. But then $G^{(r)}/G^{(r+1)}$ is cyclic, which implies that $G^{(r)}$ is cyclic, hence $r = n$. Thus (6) follows.

(7) $G^{(3)} \leqq \Phi(G)$.

Certainly $G^{(3)}$ is in each normal maximal subgroup, so let L denote a non-normal maximal subgroup. By lemma 1, $h(L) \leqq (n-1)$, so by induction $k(L) \leqq (n-1)$. By (4), $G^{(n)}$ is a minimal normal subgroup of G, so since $k(G)$ $k(G) = (n+1)$, $L \geqq G^{(n)}$. Now $L \ngeqq G'$ so let s be minimal with respect to $G^{(s)} \nleqq L$. We show that $s = 2$. Certainly $G^{(s)}L = G$, and $G^{(s)} \cap L \geqq G^{(s+1)}$, and by (6), $h(G/G^{(s+1)}) = s + 1$. Let C denote the core of L in G. $C \cap G^{(s)} = L \cap G^{(s)}$, and $k(G/C) = s + 1$, so $h(G/C) = s + 1$. Let

$$L = L_1 > L_2 > \cdots > L_s > \cdots > C$$

be a composition series for L thru C. In the chain:

$$G = G_0 > L_1 > L_2 \cdots > L_s$$

no proper entry is subnormal in G. However $h(G/G^{(s+1)}) = s + 1$, so $L_s/G^{(s+1)}$ is cyclic, and moreover L_s/C is of prime order. Let $D = C \cap G^{(s)}$. Then C/D and $G^{(s)}/D$ are abelian, so $CG^{(s)}/D$ is abelian. But since $k(G/D) = s + 1$, $k(G/CG^{(s)}) = s$. Hence $k(L/C) = s$, but $\omega(L/C) \leqq s$, and so $s = 2$.

(i) G has a Sylow tower.

$G/G^{(3)}$ is not nilpotent so by (6), $h(G/G^{(3)}) = 3$. By [6, Theorem 2, Theorem 3], $G/G^{(3)}$ has a Sylow tower. Hence by (7), $G/\phi(G)$ has a Sylow tower. Thus G has a Sylow tower, which contradicts (5), and hence G does not exist, and the theorem is proved.

Mann [5, Theorem 8] described the structure of a group G satisfying $\pi(G) = n$ and having each nth maximal subgroup subnormal. Some of the same structure was noted by the author [6, Theorem 4] under the weaker hypothesis, $h(G) = n$. To see that this is truly a weaker hypothesis, consider the group G given by $G = S_3 \wr Z_x$. $|G| = 72$, and the Sylow-3-subgroup is a minimal normal subgroup.

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An h-chain for G must begin with a Sylow-2-subgroup, and thus $h(G) = 4$. However there are subgroups of order 2 in G which are fourth maximal and not subnormal. In particular if G_1 denotes the copy of $S_3 \times S_3$ in G then a subgroup of order 2 in G_1 is not subnormal in G_1. Notice also that $h(G_1) = 4$, showing in general that the h function is not strictly decreasing on subgroups. We now show that the rest of Theorem 8 in [5] follows from the hypothesis $h(G) = n$.

THEOREM 2. *If G is a finite solvable group such that $h(G) = \pi(G) \geqq 2$, then $G = NH$ where N is a normal nilpotent Hall subgroup with elementary abelian Sylow subgroups, H is a complement to N, H is cyclic, and if $\pi(H) \geqq 2$, then* $|H|$ is *square free.*

PROOF. Let N be the product of all normal Sylow subgroups of G and let H be a complement to N. By [6, Theorem 4], N has the required structure, and if $\pi(H) \geqq 2$, then $|H|$ is square free. All that remains to be shown is that in the case $\pi(H) \geqq 2$, H is cyclic. Let Q denote a nonnormal Sylow subgroup of G, and let P denote a normal Sylow subgroup of G, Then Q either centralizes P or acts in a fixed point free manner on P. To see this, consider an upper chain from G thru $N(Q)$ to Q. Since this chain has at least $\pi(G) - 1$ entries, none of which is subnormal in G, and $h(G) = \pi(G)$, each entry is a Sylow complement in its predecessor. Let $S > T$ be the link in this chain such that $[S:T] = |P|$. If $S \leqq N(Q)$ then P and Q commute elementwise. If $S \nleqq N(Q)$ then $N(Q) \cap P = \{1\}$, and so Q acts in a fixed point free manner on P since P is a minimal normal subgroup of S. Moreover if Q centralizes P, then P is cyclic of prime order. We see this by looking at a chain thru $N(Q)$ and PQ to Q. From [6, Theorem 4] it follows that if $\pi(H) \geqq 2$, then $|H|$ is square free, so it remains to show that H is abelian. We consider two cases:

Case 1. N is cyclic. In this case H is isomorphic to a subgroup of $\mathrm{Aut}(N)$ and is thus abelian.

Case 2. N is not cyclic. In this case let P denote a non cyclic Sylow subgroup of N. Let R and Q denote non isomorphic Sylow subgroups of H. Then R and Q each act in a fixed point free manner on P, but by Burnside [1, p. 335] this implies that RQ is cyclic. Since R and Q were arbitrary, H is abelian.

Mann pointed out that groups of this very special structure actually do satisfy the condition that every nth maximal subgroup is subnormal.

In [6, Theorem 1] it was noted that $l(G) \leqq h(G)$, and later in the same paper it was remarked that in case $h(G) - \pi(G) = 0$, $l(G) \leqq 3$. Theorem 2 above shows that in this case $l(G) \leqq 2$. We now give a better bound on $l(G)$.

THEOREM 3. *It G is a finite solvable non-nilpotent group then* $l(G) \leqq h(G) - \pi(G) + 2$.

PROOF. The theorem is true for groups of small order so let G be a counter example of minimal order. We show that G does not exist. Such a group G must satisfy:

(1) $\Phi(G) = 1$

This follows since $l(G/\phi(G)) = l(G)$, $h(G/\Phi(G)) \leqq h(G)$, and $\pi(G/\Phi(G)) = \pi(G)$.

(2) Each minimal normal subgroup of G is a Sylow subgroup.

Let M denote a minimal normal subgroup. If G/M is nilpotent, then $l(G) = 2$ and since $h(G) - \pi(G) \geqq 0$, the theorem follows. So G/M is not nilpotent. Suppose M is not a Sylow subgroup. Certainly $l(G/M) \leqq l(G)$. If $l(G/M) = l(G)$ then

$$l(G) \leqq h(G/M) - \pi(G/M) + 2 \leqq h(G) - \pi(G) + 2,$$

and the theorem is true. So suppose $l(G/M) < l(G)$. In this case let L denote a complement to M in G. By (1) such a complement exists. Then $l(L) = l(G) - 1$ so $L \ntriangleleft G$, and so by Lemma 2, $h(L) < h(G)$ and so by the minimality of G,

$$l(G) = l(L) + 1 \leqq h(L) - \pi(L) + 3 \leqq h(G) - \pi(G) + 2.$$

This contradiction shows that M is a Sylow subgroup.

(3) G has a unique minimal normal subgroup.

If N is any minimal normal subgroup of G and T is a complement to N, then by Lemma 2 or Lemma 4, $h(T) < h(G)$. In either case $l(T) \leqq h(G) - \pi(G) + 2$ If there are two minimal normal subgroups say N_1 and N_x, then $l(G/N_i) \leqq h(G) - \pi(G) + 2$ so

$$l(G) = l(G/N_1 \cap N_2) \leqq h(G) - \pi(G) + 2.$$

This contradicts the choice of G, so there is only one minimal normal subgroup.

For the remainder of the proof let M denote the unique minimal normal subgroup of G, and let L denote a complement to M. Since M is unique, L is core free and so $h(G) \geqq 1 + \omega(L)$.

(4) $h(L) = h(G) - 1 = \omega(L)$.

As in (3), $h(L) \leqq h(G) - 1$. If $h(L) \leqq h(G) - 2$, then

$$l(G) \leqq l(L) + 1 \leqq h(G) - \pi(G) + 2.$$

This is a contradiction hence (4) follows.

(5) $l(L) \geqq 3$

This follows from Theorem 2. If $l(L) = 2$, $h(L) - \pi(L) = 0$ and so $h(G) - \pi(G) = 0$ and from Theorem 2, $l(G) \leqq 2$.

(6) Each normal prime power subgroup of L is either a Sylow subgroup of L (hence of G) or is cyclic.

Let N denote a normal prime power subgroup of L. By (5) L/N is not nilpotent. Since $l(L/N) = l(L) - 1$ we have

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$$h(L) - h(L/N) \leqq \pi(L) - \pi(L/N) + 1.$$

Thus if N is not a Sylow subgroup, $h(L) - h(L/N) \leqq 1$. However from (4), $h(L) = \omega(L)$ so that $\omega(N) = 1$. In fact if N is a Sylow subgroup we still have $h(L) - h(L/N) \leqq 2$ so that $\omega(N) \leqq 2$. If $l(L/N) = l(L)$, then

$$h(L) - \pi(L) + 2 \leqq h(L/N) - \pi(L/N) + 2$$

so that $0 < \pi(L) - \pi(L/N) \leqq 1$, and thus N is a Sylow subgroup.

To recap: Let N denote a prime power normal subgroup of L. If $l(L/N = l(L)$ then N is a cyclic Sylow subgroup, and if $l(L/N) < l(L)$ then N is either a Sylow subgroup or is cyclic, and in any case $\omega(N) \leqq 2$.

(7) The Fitting subgroup of L contains a Sylow subgroup of L.

Let F denote the Fitting subgroup of L. Since $l(L/F) + 1 = l(L)$,

$$h(L) - h(L/F) \leqq \pi(L) - \pi(L/F) + 1.$$

If F does not contain a Sylow subgroup of L we have $h(L) - h(L/F) = 1$. But by (4), $h(L) = \omega(L)$, so that $\omega(F) = 1$. But then $F \lneqq Z(P)$ for some Sylow subgroup P. This is impossible so F contains a Sylow subgroup of L.

(8) Using the same notation as in (7), F is a Hall subgroup of L.

Suppose not and let T be a Sylow subgroup of F such that T is Sylow subgroup of L. By (6), T is cyclic. By (7) $T \neq F$ so let $K \lhd F$ such that K is a Sylow subgroup of L. Again by (7), $l(L) = l(L/T) + 1$, so $l(L/K) = l(L)$ and so K is of prime order. Thus the Sylow subgroups of F are cyclic, and so F is cyclic. But then L/F is abelian, contrary to (5).

(9) Let H be the next to last entry in an h-chain for L, (i.e. H can be joined to L by an upper chain of length $h(L)-2$ with no entry in the chain subnormal in L.) Then H acts irreducibly on M.

The chain $G = LM > L_1M > \cdots > HM$, where the $\{L_i\}$ from an upper chain from L to H, is $h(G) - 2$ entries long and has no entry subnormal in G. If H normalized a subgroup M_1 of M, then since H is not subnormal in L, HM_1 is not subnormal in G and is $(h(G) - 1)$th maximal and is hence cyclic. But M is a Sylow subgroup so this is impossible.

(10) Let H be as in (9). If T is a subgroup of Fitt(L) of prime order such that H normalizes T then H centralizes T.

Consider the group MHT. $MT \lhd MHT$ and if H does not centralize T, H acts in a fixed point free manner on MT. But then MT is nilpotent, which is contrary to the fact that L is core free.

(11) Let H be as in (9), then H centralizes $F = \text{Fitt}(L)$.

From (10) H centralizes the cyclic Sylow subgroups of F. By (6), $|F|$ is a cube free, so let X denote a Sylow subgroup of F of order q^2, q a prime. $X \lhd L$ and since the h-chain for L thru H has each entry of prime index in its predecessor,

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H normalizes a subgroup X_1 of X of prime order. But then by (10) H centralizes X_1, and by [3, Theorem 3.3.2] $X = X_1 \times X_2$ with X_2 H-invariant. Again by (10) H centralizes X_2, hence H centralizes X.

We have shown that H centralizes F, but $C_L(F) = F \ngeqq H$. This contradiction shows that G does not exist, and so the theorem follows.

It was noted [6, Theorem 6] that if $h(G) \leqq 3$ then G is solvable, while the simple group A_5 has $h(A_5) = 4$. Janko [4] described the groups with each fourth maximal subgroup normal. We now show that these results follow from the hypothesis $h(G) = 4$.

THEOREM 4. *If G is a finite non-solvable group with $h(G) = 4$, then G is isomorphic $SL(2, 5)$ or $LF(2, p)$ where $p = 5$ or p is a prime such that $(p - 1)$ and $(p + 1)$ are products of at most 3 primes and $p \equiv \pm 3$ or $\equiv 13 \pmod{40}$.*

PROOF. If G is simple and $h(G) = 4$ then each fourth maximal subgroup is trivial and this is just Janko's theorem. So suppose G is non-solvable and non-simple group with $h(G) = 4$. Then G must satisfy the following:

(1) Each non-normal maximal subgroup of G is solvable. This follows from Lemma 2 and [6, Theorem 6].

(2) If $N \lhd G$ then either N or G/N is solvable, and in particular if G/N is not solvable, N is cyclic of prime power order.

This follows from Lemma 4 and [6, Theorem 6].

(3) If S is a solvable normal subgroup of G then $S \trianglelefteq \phi(G)$.

Suppose not and let L be a maximal subgroup of G such that $L \ngeqq S$. G/S is not solvable so L is not solvable hence by (1) $L \lhd G$. Now $L/L \cap S$ is not solvable hence $h(L/L \cap S) \geqq 4$. Then by Lemma 3, $h(G/L \cap S) \geqq 4 + \omega(S/(L \cap S)) > 4$ which is impossible.

(4) $h(G/\Phi(G)) = 4$, and $\phi(G)$ is a cyclic p-group. This follows from [6, Theorem 6] and Lemma 4.

(5) $G/\Phi(G)$ has a cube free order.

Let $S/\Phi(G)$ denote a Sylow subgroup of $G/\Phi(G)$. Consider an upper chain $G > G_1 \geqq \cdots N(S) \geqq \cdots S > \cdots \Phi(G)$. Since all subnormal solvable subgroups of G lie in $\Phi(G)$ no entry in this chain properly containing $\Phi(G)$ is subnormal in G. Also since $h(G_1) \leqq 3$, G_1 is solvable, and so $\omega(G_1/\Phi(G)) \leqq 3$. If $G_1 = S$ then $\omega(G_1/\Phi(G)) = 3$ so that $G_1/\Phi(G)$ is nilpotent of class $\leqq 2$, and by a theorem of Deskins [2, Theorem 1] $G/\Phi(G)$ is solvable. This is impossible, thus $S \nleqq G_1$ and so $\omega(S) \leqq 2$.

(6) $G/\Phi(G)$ is simple.

Let $N/\Phi(G)$ denote a proper minimal normal subgroup of $G/\Phi(G)$. By (3) $N/\Phi(G)$ is not solvable, so since $|G/\Phi(G)|$ is cube free, $N/\Phi(G)$ is simple. Notice that $4 \mid |N/\Phi G|$ and $|G/N|$ is odd so G/N is solvable. Let $T/\Phi(G)$ denote a Sylow

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2-subgroup of $N/\Phi(G)$. Consider a chain

$$G > N(T) > N(T) \cap N \geqq T > T_1 > \Phi(G).$$

Since T_1 is not subnormal in G, this chain does not contain a subnormal entry properly containing $\phi(G)$, so T is second maximal in this chain. i.e. $T = N(T) \cap N$. But then by Burnside's theorem N is solvable. This contradiction shows N does not exist.

Since $G/\phi(G)$ is simple G does not have a normal maximal subgroup, and so by (1) all maximal subgroups of G are solvable. Since each upper chain of length 4 in G contains in a solvable subgroup, each fourth maximal subgroup of G is normal. Janko's theorem [4, Theorem 3] yields the desired result.

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