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Higher Order Scattering on Asymptotically Euclidean Manifolds

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Abstract. We develop a scattering theory for perturbations of powers of the Laplacian on asymptotically Euclidean manifolds. The (absolute) scattering matrix is shown to be a Fourier integral operator associated to the geodesic flow at time π on the boundary. Furthermore, it is shown that on $\mathbb{R}^n$ the asymptotics of certain short-range perturbations of Δ^k can be recovered from the scattering matrix at a finite number of energies.

1 Introduction

In this paper, we develop a scattering theory for powers of the Laplacian on a class of manifolds which includes perturbations of Euclidean space and apply this theory to obtain new inverse scattering results on Euclidean space. This theory is a natural extension of the work of Melrose [14], who developed a theory of scattering for the Laplacian on asymptotically Euclidean manifolds. We show that the higher order scattering matrix has very similar properties in this case. In particular, we show that the Melrose-Zworski calculus of Legendrian distributions, [15], can be applied to construct the Poisson operator for the scattering problem and thus deduce that the scattering matrix is a Fourier integral operator associated to geodesic flow at time π. For higher order operators, this result appears to be new for the class of perturbations we consider even for $\mathbb{R}^n$. This theory is then also applied to extend the inverse results of Joshi and Sá Barreto on recovering the asymptotics of perturbations, [8], [9], [10], [11], [12], to this higher order case.

An asymptotically Euclidean manifold is a smooth manifold with boundary $(X, \partial X)$ which is equipped with a scattering metric. A scattering metric is a smooth Riemannian metric, g, on the interior of X which blows up in a prescribed way at the boundary: there exists a product decomposition close to the boundary, $p \mapsto (x, y) \in [0, \epsilon) \times \partial X$, such that g takes the form

$$g = \frac{dx^2}{x^4} + \frac{h(x, y, dy)}{x^2}, \tag{1.1}$$

with h smooth on the closed space and $h_{|x=0}$ a non-degenerate metric on ∂X. This is slightly different from Melrose's definition in [14] but was shown to be equivalent in [11]. The co-tensor, $h_{|x=0}$, is independent of the decomposition chosen and thus we have a natural metric on ∂X. We shall assume throughout that a product decomposition close to the boundary has been chosen and fixed.

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It is important to realize that $\mathbb{R}^n$ with the Euclidean metric is a special case of such a manifold. To see this, put $x = |z|^{-1}$ and $\omega = z|z|^{-1}$; we then have

$$dz^2 = \frac{dx^2}{x^4} + \frac{d\omega^2}{x^2}, \tag{1.2}$$

which also shows that the induced metric on the sphere at infinity is the Euclidean metric on the unit sphere.

Melrose showed that given a smooth function, f, on ∂X, and $\lambda \in \mathbb{R} \setminus \{0\}$ that there is a unique function u, smooth on the interior of X, of the form

$$e^{i\lambda/x} x^{\frac{n-1}{2}} f_+ + e^{-i\lambda/x} x^{\frac{n-1}{2}} f_-, \tag{1.3}$$

with $f_\pm$ smooth functions on $(X, \partial X)$ and f_- restricted to the boundary equal to f such that $(\Delta - \lambda^2)u = 0$. The scattering matrix is then defined to be the map, $S(\lambda)$, on $C^\infty(\partial X)$, defined by

$$S(\lambda)\colon f \mapsto f_{+|\partial X}. \tag{1.4}$$

It was shown in [14] that $S(\lambda)$ extends to a unitary operator on $L^2(\partial X)$ with the density induced by h. Melrose and Zworski [15] studied the micro-local structure of this operator and showed that it is a zeroth order, classical Fourier integral operator associated to geodesic flow at time π. In the special case that $h(x, y, dy) = h(0, y, dy) + \mathcal{O}(x^\infty)$, Christiansen [4] and Parnovski [16] showed that for $\lambda < 0$, modulo smoothing,

$$S(\lambda) = ie^{i\pi\sqrt{\Delta_{\partial X} + \frac{(n-2)^2}{4}}}. \tag{1.5}$$

Here we develop analogous results for perturbations of powers of the Laplacian. For $k \in \mathbb{N}$, we define a short range perturbation of Δ^k to be a symmetric differential operator of order $2k-1$ which close to the boundary can be written in local coordinates in the form $x^2 P(x, y, x^2 D_x, xD_y)$ where $P(x, y, \tau, \eta)$ is smooth in (x, y) and is a polynomial of order $2k-1$ in (τ, η). In particular, a real-valued, smooth function vanishing to second order at ∂X defines a short range perturbation. In Section 2 only, we allow certain pseudodifferential perturbations as well.

We prove

Theorem 1.1 *Let V be a short range perturbation of Δ^k and let $\lambda \in \mathbb{R} \setminus \{0\}$. Then, given $f \in C^\infty(\partial X)$, there exists a smooth function, u, on X^0 such that $(\Delta^k + V - \lambda^{2k})u = 0$ and u is of the form*

$$e^{i\lambda/x} x^{\frac{n-1}{2}} f_+ + e^{-i\lambda/x} x^{\frac{n-1}{2}} f_-, \tag{1.6}$$

with $f_\pm$ smooth functions on X such that f_- restricted to the boundary is equal to f. The function u is unique modulo smooth functions vanishing to infinite order at ∂X.

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The non-uniqueness here corresponds to the possibility of embedded discrete spectrum. The scattering matrix $S(\lambda)$ can then be defined precisely as before as the indeterminacy will not affect the lead term at the boundary, and we show that it has a unitary extension.

We also prove

Theorem 1.2 *Let V be a short range perturbation of Δ^k and let $\lambda \in \mathbb{R} \setminus \{0\}$. Then $S(\lambda)$ is a zeroth order classical Fourier integral operator associated to geodesic flow at time π on ∂X.*

It is interesting to compare the scattering matrix for different powers and the following is an immediate corollary to our construction.

Corollary 1.1 *Let $0 < k_1 \leq k_2 \in \mathbb{N}$, and suppose V_j is a short range perturbation of Δ^{k_j}, of the form $x^l p_j(x, y, x^2 D_x, xD_y)$, with $p_j(x, y, \xi, \eta)$ a polynomial in (ξ, η). Let $S_j(\lambda)$ be the scattering matrix associated to $\Delta^{k_j} + V_j - \lambda^{2k_j}$. Then $S_1(\lambda) - S_2(\lambda)$ is a Fourier integral operator of order $1 - l$.*

It is therefore immediate that (1.5) holds also for the higher order scattering matrix when $V = 0$.

In the special case of $\mathbb{R}^n$ we study the problem of recovering asymptotics of a perturbation from scattering data. As in [12], we need an *aradiality* condition to recover the perturbation. In fact, in [12] it was observed that recovery is not possible without it. We shall say a perturbation is aradial modulo Schwartz functions if it is asymptotically equal to a sum

$$\sum_{l=2}^{\infty} \sum_{\alpha} f_{\alpha,-l} D_z^{\alpha}, \tag{1.7}$$

with each term $\sum_{\alpha} f_{\alpha,-l} D_z^{\alpha}$ of the form $|z|^{-l}$ times a composition of vector fields tangent to the sphere. The aradiality conditions allows any zeroth order perturbation.

Theorem 1.3 *Let V_1, V_2 be short range perturbations of Δ^k on $(\mathbb{R}^n, dz^2)$, $n \geq 3$, such that $V_1 - V_2$ is an aradial differential operator of order l. Let $S_j(\lambda)$ be the scattering matrix associated to $\Delta^k + V_j - \lambda^{2k}$, and suppose that for $l + 1$ values of $\lambda > 0$, $S_1(\lambda) - S_2(\lambda)$ is smoothing. Then the coefficients of V_1 and V_2 agree modulo Schwartz functions.*

Note that we assume neither that V_1, V_2 are of order l nor that they are aradial.

Our approach to this higher order scattering problem is highly influenced by that of Melrose, [14], and that of Melrose-Zworski, [15]. In particular, we use the scattering calculus developed by Melrose and used for the case $k = 1$ to study the general case and use techniques similar to those of [14] to establish the existence of the scattering matrix. To establish the micro-local structure of the scattering matrix we proceed as in [15] to construct the Poisson operator for the scattering problem as a Legendrian distribution associated to a pair of intersecting Legendrian submanifolds using the calculus developed there.

To prove the inverse result, we choose to follow the approach of [9] rather than [10] in order to increase the readability of the paper for non-experts. In particular, we establish our results for the case of $\mathbb{R}^n$ without explicitly using the Melrose-Zworski Legendrian calculus. The reader may regard these proofs as a warm-up for the construction of the Poisson operator in the general case. As in [8], [9], [10], [11], [12], the proof proceeds by establishing that the principal symbol of the difference of the scattering matrices determines and is determined by a weighted integral of the lead term of the difference of the perturbations over geodesics of length π. The injectivity of this transformation is then deduced by using some elementary calculus and some deep results of Bailey and Eastwood, [3], on the integral geometry of tensor fields on projective space.

The main reference for higher order scattering on $\mathbb{R}^n$ is Chapter 14 of [7] where a scattering theory for perturbations of a much more general class of constant coefficient operators on $\mathbb{R}^n$ is developed—this extends ideas developed by Agmon and Hörmander in [2] and Agmon in [1]. The complementary problem of studying the recovery of compactly supported perturbations of higher order operators on $\mathbb{R}^n$ was studied by Liu in [13].

The results of Bailey and Eastwood [3] which we use in Section 4 to prove the inverse results are restricted to projective spaces and spheres. Since in addition, we believe the principal interest in this problem is in Euclidean space, we restrict ourselves to studying that important special case as the analysis is much more accessible. The problem of recovering the asymptotics of metrics for the $k = 1$ case has been studied in [11] and it is likely that a similar result could be proven in the higher order case.

The question of recovering the entire perturbation from the scattering matrix at fixed energy is an interesting one but is still open even in the case with $k = 1$.

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2 Basic Scattering Theory of $\Delta^k + V$

In this section, we construct solutions of $\Delta^k + V - \lambda^{2k}$ having specified behaviour at the boundary, leading to the definition of the scattering matrix.

Throughout, Δ denotes the Laplacian associated to a scattering metric on a compact manifold $(X, \partial X)$, k is a positive integer, and $\lambda \in \mathbb{R} \setminus \{0\}$. We assume a product decomposition with boundary defining function x has been fixed. We call the smooth functions vanishing to infinite order at the boundary the Schwartz functions on X. We use the product decomposition to extend functions on the boundary smoothly into X by making them constant in the normal direction and cutting off.

The results of this section are very closely related to the results of [14]. We have tried to state the results in a manner accessible to those unfamiliar with that paper, but in order to avoid repetition we omit proofs which follow essentially as in [14], only giving some indication of how they need to be modified to work in this setting. We recall some of the definitions and results of [14] but refer the reader to the original for full details.

In this section only, we will allow a somewhat larger class of perturbations of Δ^k. In order to describe them, we recall some notation.

Let $\mathcal{V}_{\mathrm{sc}}(X)$ be the space of all smooth vector fields of finite length with respect to a scattering metric. Near a point on the boundary, $x^2 \frac{\partial}{\partial x}$ and $x \frac{\partial}{\partial y_i}$, $i = 1, \ldots, n-1$, form a basis for $\mathcal{V}_{\mathrm{sc}}(X)$, where y_i are coordinates on ∂X. The set of scattering differential operators of order m is

$$\operatorname{Diff}^m_{sc}(X) = \operatorname{span}_{0\le j\le m}\big(\mathcal{V}_{sc}(X)\big)^j.$$

The short range perturbations which we are allowing in the remainder of this paper are elements of $x^2 \operatorname{Diff}^{2k-1}_{sc}(X)$.

Most of the results of this section hold for a wider class of perturbations: elements of $\Psi^{2k-1,2}_{sc}(X)$, part of the (small) calculus of scattering pseudodifferential operators. We refer the reader to [14, Sections 4 and 5] for the full definition and some properties of $\Psi^{m,l}_{sc}(X)$. However, we remark that $x^j \operatorname{Diff}^m_{sc}(X) \subset \Psi^{m,j}_{sc}(X)$ and $(\Delta - z)^{-1} \in \Psi^{-2,0}_{sc}(X)$ when $z \notin [0,\infty)$ [14, Theorem 1]. Moreover, if we consider the manifold $(\mathbb{R}^n, dz)$ (which by an earlier discussion under radial compactification becomes a compact manifold $(\mathbb{R}^n)_{rc}$ with scattering metric), the operators corresponding to $\Psi^{m,l}_{sc}\big((\mathbb{R}^n)_{rc}\big)$ have Schwartz kernels of the form

$$A(z,z') = \frac{1}{(2\pi)^n}\int e^{i(z-z')\cdot\zeta} a_L(z,\zeta)\, d\zeta$$

with a_L satisfying

$$|D^\alpha_z D^\beta_\zeta a_L(z,\zeta)| \le C_{\alpha\beta}(1+|z|)^{-l-|\alpha|}(1+|\zeta|)^{m-|\beta|}$$

[14, (4.1), (4.2)].

Definition 2.1 We shall say a differential operator V is a short range perturbation of Δ^k if it is symmetric and if $V \in x^2 \operatorname{Diff}^{2k-1}_{sc}(X)$. We call a pseudodifferential operator a pseudodifferential short range perturbation of Δ^k if it is symmetric and it is an element of $\Psi^{2k-1,2}_{sc}(X)$.

Throughout this section V is a pseudodifferential short range perturbation of Δ^k.

We prove

Proposition 2.1 *Let V be a pseudodifferential short range perturbation of Δ^k and $\lambda \in \mathbb{R}\setminus\{0\}$. Given $f \in C^\infty(\partial X)$, there exists $u \in C^\infty(X)$, unique modulo Schwartz functions, such that $u_{|\partial X} = f$ and $(\Delta^k + V - \lambda^{2k})(e^{-i\lambda/x}x^{\frac{n-1}{2}}u)$ is Schwartz.*

Note we make no assumptions on the sign of λ.

This will follow easily once we have proven

Lemma 2.1 *Let V be a pseudodifferential short range perturbation of Δ^k and $\lambda \in \mathbb{R}\setminus\{0\}$. If $f \in C^\infty(\partial X)$, then*

$$(\Delta^k + V - \lambda^{2k})(e^{-i\lambda/x}x^{\frac{n-1}{2}+\alpha} f) = e^{-i\lambda/x}(k\lambda^{2k-1}C_\alpha x^{\frac{n+1}{2}+\alpha} f + x^{\frac{n+3}{2}+\alpha} g)$$

with $g \in C^\infty(X)$, where $C_0 = 0$ and $C_\alpha \neq 0$ for $\alpha \neq 0$.

Proof For $k = 1$, this follows as in [14], using [15, Lemma 8] for the mapping properties of pseudodifferential short range perturbations. For general k, one simply iterates. ∎

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Now to prove Proposition 2.1, we simply choose the lead term to be f and then repeatedly iterate to compute all the terms of the Taylor series. The result follows from Borel's lemma. The uniqueness follows from the fact that $C_\alpha \neq 0$ for $\alpha \neq 0$.

We will repeatedly use the operator

$$Q = Q(\lambda) = \sum_{j=0}^{k-1} \lambda^{2j} \Delta^{k-j-1} \tag{2.1}$$

since $\Delta^k - \lambda^{2k} = Q(\Delta - \lambda^2) = (\Delta - \lambda^2)Q$. Using the same techniques as Theorem 1 of [14], one can show that for $\lambda \in \mathbb{R} \setminus \{0\}$, $Q(\lambda)^{-1} \in \Psi_{\mathrm{sc}}^{-2k+2,0}(X)$.

A key idea of many of the proofs is that

$$\Delta^k - \lambda^{2k} + V = Q(\Delta - \lambda^2 + V'), \tag{2.2}$$

where $V' = Q^{-1}V \in \Psi_{\mathrm{sc}}^{1,2}(X)$. The importance of this is that the symbol of V' vanishes to second order at the boundary. As noted in Remark 3 of [14], many of the results there hold if Δ is replaced by $\Delta + W$, when $W \in x^2 C^\infty(X)$, because they depend on the properties of the principal symbol and the boundary symbol, which are unchanged by the addition of such a W. What we are doing here is allowing a somewhat more general perturbation, but with the same kind of decay at the boundary. Thus, the results of Melrose's Propositions 9–11 hold if Δ is replaced by $\Delta + W$, with $W \in \Psi_{\mathrm{sc}}^{1,2}(X)$.

The following proposition is closely related to Proposition 11 of [14] and follows from the modifications of Propositions 9–11 indicated above and from (2.2).

Proposition 2.2 *If $\lambda \in \mathbb{R} \setminus \{0\}$, $u \in L^2_{\mathrm{sc}}(X)$, V is a pseudodifferential short range perturbation of Δ^k and $(\Delta^k + V - \lambda^{2k})u = 0$, then u is Schwartz.*

If $k = 1$ and V is a short range (differential) perturbation, then $\Delta + V$ has no positive eigenvalues. However, for (differential) short range perturbations of Δ^k, $k > 1$, the situation can be different.

Proposition 2.3 *There are (differential) short-range perturbations V of Δ^k, $k > 1$, even, such that $\Delta^k + V$ has positive eigenvalues.*

Proof We give two ways of constructing such examples. Let $j \in \mathbb{N}$.

Using the min-max principle, for any asymptotically Euclidean manifold X one can find $\tilde{V} \in x^2 C^\infty(X)$ so that $\Delta + \tilde{V}$ has a negative eigenvalue $-\sigma$. Then $(\Delta + \tilde{V})^{2j}$ has a positive eigenvalue σ^{2j}. It is possible to choose $\tilde{V}$ to be compactly supported and such that $\Delta + \tilde{V}$ has many negative eigenvalues.

To construct a Schwartz potential V such that $\Delta^{2j} + V$ has an embedded eigenvalue, choose $\tau > 0$ and let x be a globally defined boundary-defining function, $x \in C^\infty(X)$. Then

$$(\Delta^{2j} - \tau^{4j})e^{-\tau/x} x^{(n-1)/2} = \mathcal{O}(x^{(n+3)/2} e^{-\tau/x}).$$

Just as in Proposition 2.1 we can use Lemma 2.1 to successively solve away the errors, resulting in a u which satisfies

$$(\Delta^{2j} - \tau^{4j})u = g = \mathcal{O}(x^\infty e^{-\tau/x})$$

with $u = e^{-\tau/x}x^{(n-1)/2}\big(1+\mathcal{O}(x)\big)$. In fact, by doing the asymptotic summation judiciously, we can ensure that $x^{-(n-1)/2}e^{\tau/x}u$ is nowhere vanishing. Set $V = -u^{-1}g$. Then $\Delta^{2j} + V$ has eigenvalue τ^{4j} with eigenfunction u. ∎

We continue with the results needed to define the scattering matrix.

Theorem 2.1 *If $\lambda \in \mathbb{R} \setminus \{0\}$, $V \in \Psi_{sc}^{2k-1,2}(X)$ is symmetric, and f is a Schwartz function on X, orthogonal to the L^2 null space of $\Delta^k + V - \lambda^{2k}$ if there is any, then there exists u with $x^{-(n-1)/2}e^{-i\lambda/x}u \in C^\infty(X)$ such that*

$$(\Delta^k + V - \lambda^{2k})u = f.$$

This is an analogue of [14, Propositions 12 and 14].

Proof As in Proposition 14 of [14],

$$u = \lim_{t\downarrow 0}\big(\Delta^k + V - (\lambda + it)^{2k}\big)^{-1} f$$

where the limit exists in $x^{-1/2-\delta}H_{sc}^{\infty}(X)$ for any $\delta > 0$. To show that the limit exists, let $u_t = \big(\Delta^k + V - (\lambda + it)^{2k}\big)^{-1} f$. Then, since f is orthogonal to the null space of $\Delta^k + V - \lambda^{2k}$, so is u_t for $t > 0$. Having made this observation, the proof follows as in [14, Proposition 14], giving us the same additional microlocal regularity as well.

To finish the proof, one uses the analog of Proposition 12 of [14]. That is, if $\tilde{u} \in C^{-\infty}(X)$, $WF_{sc}^{*,-1/2}(\tilde{u}) \cap R_+(\lambda) = \varnothing$, and $(\Delta + Q^{-1}V - \lambda^2)\tilde{u} \in \dot{C}^\infty(X)$, then $e^{-i\lambda/x}x^{-(n-1)/2}\tilde{u} \in C^\infty(X)$. The proof follows just as the proof of [14, Proposition 12], first noting that $[\tilde{\Delta}_0, Q^{-1}V] \in \Psi_{sc}^{2,1}(X)$ and thus

$$[\tilde{\Delta}_0, Q^{-1}V]\colon x^s H_{sc}^{\infty}(X) \to x^{s+1}H_{sc}^{\infty}(X).$$

The second observation that allows the proof to proceed as in [14, Proposition 12] is that if $\mathrm{Diff}_c^{l}(X)u \subset H_{sc}^{\infty,m}(X)$, then $\mathrm{Diff}_c^{l-l'}(X)A\,\mathrm{Diff}_c^{l'}(X)u \subset H_{sc}^{\infty,m+s}(X)$ when $l' \leq l$ is a nonnegative integer and $A \in \Psi_{sc}^{*,s}(X)$. This then allows the inductive step in the proof of [14, Proposition 12] to proceed just as it does there. ∎

We shall also use the following boundary pairing result, the analogue of [14, Proposition 13].

Proposition 2.4 *Suppose $V \in \Psi_{sc}^{2k-1,2}(X)$ is symmetric, $\lambda \in \mathbb{R} \setminus \{0\}$, u_i, $i = 1, 2$ satisfies $(\Delta^k + V - \lambda^{2k})u_i = f_i$, f_i is Schwartz, and $u_i = x^{(n-1)/2}(e^{i\lambda/x}a_i^+ + e^{-i\lambda/x}a_i^-)$, with $a_i^\pm \in C^\infty(X)$. Let $b_i^\pm = (a_i^\pm)_{|\partial X}$. Then*

$$2ik\lambda^{2k-1}\int_{\partial X}(b_1^+\overline{b_2^+} - b_1^-\overline{b_2^-})\,dh = \int_X u_1\overline{f_2} - f_1\overline{u_2}\,dg.$$

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Proof Using $Q = Q(\lambda) = \sum_{j=0}^{k-1} \lambda^{2j}\Delta^{k-j-1}$, we have

$$\begin{aligned}\int_X u_1\overline{f_2} - f_1\overline{u_2}\,dg &= \int_X u_1\overline{(\Delta - \lambda^2 + VQ^{-1})Qu_2} - Q(\Delta - \lambda^2 + Q^{-1}V)u_1\overline{u_2}\,dg \\ &= \int_X u_1\overline{(\Delta - \lambda^2 + VQ^{-1})Qu_2} - (\Delta - \lambda^2 + Q^{-1}V)u_1\overline{Qu_2}\,dg\end{aligned} \tag{2.3}$$

as Q is self-adjoint and $(\Delta - \lambda^2 + Q^{-1}V)u_1 \in \dot{C}^\infty(X)$. Recall that V is self-adjoint as well. Choose a function $\phi \in C^\infty(\mathbb{R})$ such that $\phi(t) = 0$ if $t < 1$ and $\phi(t) = 1$ if $t > 2$. Then (2.3) is equal to

$$\begin{aligned}\lim_{\epsilon\downarrow 0}\int_X \phi(x/\epsilon)\big(u_1\overline{(\Delta - \lambda^2 + VQ^{-1})Qu_2} - (\Delta - \lambda^2 + Q^{-1}V)u_1\overline{Qu_2}\big)\,dg \\ = \lim_{\epsilon\downarrow 0}\int_X [\Delta + Q^{-1}V, \phi(x/\epsilon)]u_1\overline{Qu_2}\,dg.\end{aligned}$$

Since V is short-range, $\lim_{\epsilon\downarrow 0}\int_X [Q^{-1}V, \phi(x/\epsilon)]u_1\overline{Qu_2}\,dg = 0$. To compute the remainder, we just use the corresponding results from [14, Proposition 13] and the fact that at the boundary of X,

$$Qu_2 = k\lambda^{2k-2}x^{(n-1)/2}e^{i\lambda/x}b_2^+ + k\lambda^{2k-2}x^{(n-1)/2}e^{-i\lambda/x}b_2^- + \mathcal{O}(x^{(n+1)/2}). \qquad \blacksquare$$

We will need

Theorem 2.2 *If $\lambda \in \mathbb{R}\setminus\{0\}$, V is a pseudodifferential short range perturbation of Δ^k, and $f \in C^\infty(\partial X)$, then there exists $f_\pm \in C^\infty(X)$ such that $u = x^{\frac{n-1}{2}}(e^{i\lambda/x}f_+ + e^{-i\lambda/x}f_-)$ satisfies*

$$(\Delta^k + V - \lambda^{2k})u = 0$$

and the restriction of f_- to ∂X is f. The function u is unique if λ^{2k} is not an eigenvalue of $\Delta^k + V$ and is unique up to the addition of a Schwartz function if λ^{2k} is an eigenvalue.

Proof Most of the proof follows from Proposition 2.1 and Theorem 2.1. We use Theorem 2.1 to solve away the error obtained by constructing the formal expansion as in Proposition 2.1. We note that since the error we wish to solve away is of the type $(\Delta^k + V - \lambda^2)g$, where $g \in x^{-1/2-\epsilon}L^2_{\mathrm{sc}}(X)$, and since eigenfunctions are Schwartz we may still integrate by parts to obtain that the error is orthogonal to the eigenspace. Finally, suppose that there are two such u satisfying the conditions of the theorem. Then their difference v satisfies $(\Delta^k + V - \lambda^{2k})v = 0$ and $v = x^{(n-1)/2}e^{i\lambda/x}g_+ + x^{(n+1)/2}e^{-i\lambda/x}g_-$ with $g_\pm \in C^\infty(X)$. Then the boundary pairing result of Proposition 2.4 gives us that $(g_+)_{|\partial X} = 0$, and thus v is in $L^2_{\mathrm{sc}}(X)$. Consequently, v is an eigenfunction and is thus Schwartz. $\blacksquare$

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We can therefore define the scattering matrix as in the usual case:

Definition 2.2 The scattering matrix is a map on $C^\infty(\partial X)$ taking a function f to f_+ restricted to ∂X where f, f_+ are as in Theorem 2.2.

Using Proposition 2.4 and the uniqueness it implies, it is easy to check that the scattering matrix can be extended to a unitary operator on $L^2(\partial X)$ which satisfies $S(\lambda)^{-1} = S(-\lambda)$.

We remark that if $V \in \Psi_{\mathrm{sc}}^{2k-1,2}(X)$ is symmetric, then $\Delta^k + V$, initially considered as an operator on the Schwartz functions, has a unique self-adjoint extension to a domain in $L^2_{\mathrm{sc}}(X)$. We end this section by describing the continuous part of its spectral measure. Let P be the Poisson operator whose existence is given by Theorem 2.2:

$$P(\lambda)\colon C^\infty(\partial X) \ni f \mapsto u \in x^{(n-1)/2} e^{i\lambda/x} C^\infty(X) + x^{(n-1)/2} e^{-i\lambda/x} C^\infty(X)$$

where u is the function in Theorem 2.2. Then the continuous part of the spectral measure of $\Delta^k + V$ is given by

$$dE_c(\lambda) = \frac{1}{2\pi} P(\lambda) P^*(\lambda) d\lambda$$

where λ^{2k} is the spectral variable and $\lambda \in [0, \infty)$. This can be seen by first noting that if $V = 0$ this follows from [4, Lemma 2.2] (see also [5, Lemma 5.2]) and the general case then follows as in [17, Appendix to XI.6]. We note that $\Delta^k + V$ has no singular continuous spectrum, though, as previously noted, there may be discrete spectrum.

3 The Poisson Operator for Euclidean Space

As an introduction to the general case, we use the techniques of [9] to construct the Poisson operator in the Euclidean case, where the construction can be done more explicitly. Following [9] we look for a Poisson operator in the form $e^{i\lambda z.\omega} a(z, \omega)$ with a a polyhomogeneous symbol in z and ω is a smooth parameter.

Let $\tilde{a}(z, \omega)$ be any polyhomogeneous symbol in z, with ω a parameter. Then

$$\Delta(e^{i\lambda z.\omega} \tilde{a}) = \Big(\lambda^2 \tilde{a} - 2i\lambda\omega . \frac{\partial \tilde{a}}{\partial z} + \Delta \tilde{a}\Big) e^{i\lambda z.\omega}, \tag{3.1}$$

so we conclude that

$$\Delta^k(e^{i\lambda z.\omega} \tilde{a}) = \Big(\lambda^{2k} \tilde{a} - 2ki\lambda^{2k-1}\omega . \frac{\partial \tilde{a}}{\partial x} + b\Big) e^{i\lambda z.\omega}, \tag{3.2}$$

with b a symbol two orders lower than $\tilde{a}$.

Now V is a short range perturbation of Δ^k, which for $\mathbb{R}^n$ means that

$$V = \sum_{|\alpha| \le 2k-1} f_\alpha D_x^\alpha, \tag{3.3}$$

with $f_\alpha \in S_{cl}^{-2}(\mathbb{R}^n)$. We conclude that it maps $e^{i\lambda z.\omega} S^m$ to $e^{i\lambda z.\omega} S^{m-2}$ and hence that if $\tilde{a} \in S^m_{phg}$ we have,

$$(\Delta^k + V - \lambda^{2k})(e^{i\lambda z.\omega} \tilde{a}) = \Big(-2ki\lambda^{2k-1}\omega . \frac{\partial \tilde{a}}{\partial z} + c\Big) e^{i\lambda z.\omega}, \tag{3.4}$$

with $c \in S^{m-2}_{phg}$.

Returning to the Poisson operator, by taking the lead term of a to be the constant function 1 we have an error in S^{-2}_{phg}. To solve away this and subsequent error terms, we proceed precisely as in [9, Section 2], using a term in S^{-j+1}_{phg} to solve away an error term in S^{-j}_{phg} (modulo terms in S^{-j-1}_{phg}). We solve the transport equations,

$$-2ki\lambda^{2k-1}\omega.\frac{\partial b}{\partial z} = d,$$

where d is the error, along the geodesics on the unit sphere from ω to $-\omega$. The only difference from the $k = 1$ case of [9] is a factor of $\lambda^{2(k-1)}$. As before, the transport equation degenerates on approach to $-\omega$ and in particular blows up like $(\pi - s)^{-r}$ as s, the geodesic distance from ω, tends to π, when solving for the term of homogeneity of order $-r$.

So as before we need a different ansatz close to $-\omega$. We can locally in ω rotate our coordinate system so that ω is the north pole. So close to the south pole, we look for the Poisson operator in the form

$$\int_0^\infty \int \Big(\frac{1}{S|z|}\Big)^{\gamma} S^{\alpha} e^{i\lambda(Sz'.\mu - \sqrt{1+S^2}|z|)} a\Big(\frac{1}{S|z|}, S, \mu\Big)\, dS\, d\mu, \tag{3.5}$$

with $a(t, S, \mu)$ a smooth function compactly supported on $[0, \epsilon) \times [0, \epsilon) \times S^{n-2}$. We denote the class of functions that can be written in this form plus a Schwartz error by $I^{\gamma,\alpha}$. We recall from [9] that this class is asymptotically complete in γ. It follows from a stationary phase computation carried out in [9] that away from the south pole this is equivalent to the previous ansatz with a symbol of order $-\gamma + \frac{n-1}{2}$.

We recall from [9],

Proposition 3.1 *If $u(z, \omega) \in I^{\gamma,\alpha}$ and $f \in C^\infty(\partial X \times \partial X)$ then*

$$e^{i\lambda|z|} \int u(|z|\theta, \omega) f(\theta, \omega)\, d\theta\, d\omega$$

is a smooth symbolic function in $|z|$ of order $-1 - \alpha$ and its lead coefficient is $|z|^{-\alpha-1}\langle K, f\rangle$ where K is the pull-back of the Schwartz kernel of a pseudo-differential operator of order $\alpha - \gamma - (n-2)$ by the map $\theta \mapsto -\theta$. The principal symbol of K determines and is determined by the lead term of the symbol, $a(t, S, \mu)$, of u as $S \to 0+$.

We also have as a special case of Proposition 15 from [15],

Proposition 3.2 *If $u \in I^{\gamma,\alpha}$, then*

$$(\Delta^k - \lambda^{2k})u \in I^{\gamma+1,\alpha+1},$$

and

$$Vu \in I^{\gamma+2,\alpha+2}.$$

It is also important to note that,

Lemma 3.1 *If $u \in I^{\gamma,\frac{n-3}{2}}$ and $(\Delta^k+V-\lambda^{2k})u \in I^{\gamma+j,\frac{n-1}{2}}$, then $(\Delta^k+V-\lambda^{2k})u \in I^{\gamma+j,\frac{n+1}{2}}$.*

This follows from a slight extension of the argument in the proof of Lemma 3.2 in [9]. We also recall from [9] that

Proposition 3.3 *If $u \in I^{\infty,\alpha} = \bigcap_\gamma I^{\gamma,\alpha}$, then $u = e^{-i\lambda|z|}f(z)$, with f a classical symbol of order $-\alpha-1$.*

To carry out our construction we first use the original ansatz, obtaining a symbol which blows up on approach to the south pole. Near the south pole, we use the second ansatz, (3.5), taking $\gamma = -\frac{n-1}{2}$, $\alpha = \frac{n-3}{2}$. The argument is then identical to the one in [9]. At the end of the construction, we obtain an approximate Poisson operator, $\tilde{P}$, such that $(\Delta^k + V - \lambda^{2k})\tilde{P}(\lambda)$ is Schwartz. This error will in fact be orthogonal to the eigenspace at energy λ (if there is one) as the eigenfunctions are necessarily Schwartz and the orthogonality follows from self-adjointness and integration by parts. We can therefore apply the resolvent (Theorem 2.1) and gain a term of the form $e^{i\lambda|z|}f$ with f smooth, solving away the error.

4 The Inverse Problem

In this section we prove Theorem 1.3. Suppose we have two short range perturbations, V_1, V_2, of Δ^k and we want to compare their scattering matrices. In particular, suppose

$$V_1 - V_2 = \sum_{|\alpha|\le 2k-1} a_\alpha(z) D_z^\alpha, \tag{4.1}$$

with $a_\alpha \in S_{phg}^{-1-r}(\mathbb{R}^n)$. If we carry out our construction for each V_j, then the first r terms of the construction with the first ansatz will be the same but the term of homogeneity $-r$ will be different. In particular, the forcing terms in the transport equations will differ by

$$W_{-r} = e^{-i\lambda z.\omega}(V_1 - V_2)e^{i\lambda z.\omega} = \sum_{|\alpha|\le 2k-1} \omega^\alpha a_{\alpha,-r-1}\lambda^{|\alpha|}, \tag{4.2}$$

where $a_{\alpha,-r-1}$ is the lead term of a_α. We therefore conclude that the terms of homogeneity $-r$ will differ at $\big(\gamma(s),\omega\big)$ by

$$\frac{i|z|^{-r}}{2k\lambda^{2k-1}(\sin s)^r}\int_0^s W_{-r}\big(\gamma(s'),\omega\big)(\sin s')^{r-1}\,ds', \tag{4.3}$$

where γ is a geodesic on the sphere running from ω to $-\omega$. As we have shown that the lead singularity of the first ansatz as $s \to \pi-$ is essentially the principal symbol of the scattering matrix, we conclude by the same argument that the difference of the scattering matrices will be of order $-r$ and that the principal symbol of the difference will determine and be determined by

$$\frac{1}{\lambda^{2k-1}}\int_0^\pi W_{-r}\big(\gamma(s),\omega\big)(\sin s')^{r-1}\,ds, \tag{4.4}$$

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for all geodesics γ with ω equal to $\gamma(0)$.

Note that as W depends polynomially on λ, so does the principal symbol. In particular, if we know the scattering matrix for $2k$ different values of $\lambda > 0$, then we can separate out the parts coming from the differing orders of $|\alpha|$ in the forcing term. More generally, if we have a perturbation of order $l \leq 2k-1$, then we determine the asymptotics from $l+1$ values. So in the sequel, assuming the perturbation is of order $l \leq 2k-1$, we only consider forcing terms of the form

$$\sum_{|\alpha|=l} a_\alpha(z) D_z^\alpha, \tag{4.5}$$

where $l \leq 2k-1$. To solve the inverse problem, then, we must show that $a_\alpha(z)$, $|\alpha| = l$, can be recovered from knowledge of the transform (4.4), where we replace the sums in (4.1) and (4.2) by sums only over the terms with $|\alpha| = l$. That is, we need to show that the transform (4.4), a weighted integral along geodesics of length π, is invertible. The injectivity of the transform was proven in [10] for the case $l = 0$. For higher order perturbations ($l > 0$) the question is more subtle and requires the notion of *aradiality*. We say a perturbation is *aradial* if it has no radial component, that is, if the perturbation is a span of vector fields tangent to the sphere. Clearly, all zeroth order perturbations are aradial. The injectivity of the transform (4.4) for aradial first order perturbations was shown in [12] and for second order aradial perturbations in [11]. The impossibility of recovering first order perturbations that are not aradial was also shown in [12]. We now look at the general case.

First we re-express the forcing term more invariantly. We regard the perturbation as an l-form, $\mu = \sum_{|\alpha|=l} a_{\alpha,-r-1}(z) dz^\alpha$. Since dz^α is symmetric, μ is symmetric. The aradiality means that μ makes sense as a form on the sphere and is determined by its values as a map from the tangent space of the sphere to $\mathbb{R}$.

To re-express the forcing term we rotate so that $\omega = (0, \ldots, 0, 1)$ and $\gamma(s) = \big(0, \ldots, 0, \sin(s), \cos(s)\big)$. We then have $\frac{d\gamma}{ds}(s) = \big(0, \ldots, 0, \cos(s), -\sin(s)\big)$. The transform is then just $\int_0^\pi a_{(0,\ldots,0,l)}\big(\gamma(s)\big)(\sin s)^{r-1}\, ds$. We show that this is equal to

$$(-1)^l \int_0^\pi (\sin s)^{l+r-1} \Big\langle \mu, \frac{d\gamma}{ds}(s) \Big\rangle\, ds$$

where $\langle \mu, \frac{d\gamma}{ds}(s)\rangle$ is the pairing of μ with $\frac{d\gamma}{ds} \otimes \cdots \otimes \frac{d\gamma}{ds}$ (l copies). To prove this, observe that as γ lies in a plane this is really a two-dimensional question, so without loss of generality it's enough to take $n = 2$. Now μ is a symmetric l-form so

$$\langle \mu, \nu \rangle = \sum_{j=0}^{l} a_{(j,l-j)} \nu^{(j,l-j)}.$$

Thus

$$\Big\langle \mu, \frac{d\gamma}{ds}(s) \Big\rangle = \sum_{j=0}^{l} a_{(j,l-j)}\big(\gamma(s)\big)\big(-\sin(s)\big)^{l-j}\big(\cos(s)\big)^j.$$

It follows from aradiality and the fact that the perturbation is of order l that

$$a_{(j,l-j)} = \binom{l}{j} \Big(\frac{-\cos(s)}{\sin(s)}\Big)^j a_{(0,l)},$$

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and thus we have

$$\big(-\sin(s)\big)^l\Big\langle \mu, \frac{d\gamma}{ds}(s)\Big\rangle = a_{(0,l)}\sum_{j=0}^{l}\binom{l}{j}\big(\cos(s)\big)^{2j}\big(\sin(s)\big)^{2l-2j},$$

which is of course equal to $a_{(0,l)}$. This establishes the equality.

We now want to show that if μ is a symmetric l-form on the sphere and

$$\int_0^\pi \Big\langle \mu\big(\gamma(s)\big), \frac{d\gamma}{ds}(s)\Big\rangle \big(\sin(s)\big)^{l+r-1}\,ds = 0, \tag{4.6}$$

for all geodesics γ on the sphere, then $\mu = 0$. Since we know this is true for any geodesic γ, we have

$$I_{l+r-1,\gamma,\alpha} = \int_0^\pi \Big\langle \mu\big(\gamma(s+\alpha)\big), \frac{d\gamma}{ds}(s+\alpha)\Big\rangle \big(\sin(s)\big)^{l+r-1}\,ds = 0$$

for any α. Differentiating with respect to α (see [10]), we deduce that $I_{j,\gamma,\alpha} = 0$ implies that $I_{j-2,\gamma,\alpha} = 0$ and thus, differentiating repeatedly, if $l+r-1$ is even and $I_{l+r-1,\gamma,\alpha} = 0$ then μ is even. Also, if $l+r-1$ is odd and $I_{l+r-1,\gamma,\alpha} = 0$, then μ is odd.

Since we can always reduce by two, it is enough to consider the cases where r is 1 or 2. Considering $r = 1$, we have for any geodesic γ that

$$\int_0^\pi \big(\sin(s)\big)^l\Big\langle \mu\big(\gamma(s)\big), \frac{d\gamma}{ds}(s)\Big\rangle\,ds = 0.$$

If we take a geodesic starting at $z_n = 0$, then we deduce that

$$\int_0^\pi \Big\langle (z_n^l\mu)\big(\gamma(s)\big), \frac{d\gamma}{ds}(s)\Big\rangle\,ds = 0.$$

By rotational invariance we have that

$$\int_0^\pi \Big\langle (p\mu)\big(\gamma(s)\big), \frac{d\gamma}{ds}(s)\Big\rangle\,ds = 0,$$

for all homogeneous polynomials, p, of order l. As $p\mu$ is even, we can regard it as a symmetric tensor on projective space which is in the kernel of the generalized x-ray transform on symmetric l-tensors. Fortunately, the kernel of this operator has been identified by Bailey and Eastwood, [3]. They showed that the kernel is precisely the symmetrized covariant derivatives of symmetric $(l-1)$-tensors. Therefore, to show that $\mu = 0$ we need to show that if $p\mu$ is a symmetrized covariant derivative for all homogeneous polynomials p of order l, then $\mu = 0$.

We show this by showing that μ has to vanish at the north pole. By rotational invariance it will then follow that μ vanishes everywhere. For the case $r = 2$, we apply the same arguments to $z_n\mu$ and the result will follow from the case $r = 1$.

We first prove

Lemma 4.1 *Let μ be a symmetric co-tensor of order l on $\mathbb{R}^{n-1}$ such that $p\mu$ is a symmetrized covariant derivative for every homogeneous polynomial of order l. Then μ vanishes at the origin.*

Proof To prove this lemma, we introduce some new notation. If $\alpha = (\alpha_1, \dots, \alpha_r) \in \{1, \dots, n-1\}^r$ then

$$\partial_\alpha = \frac{\partial}{\partial x_{\alpha_1}} \cdots \frac{\partial}{\partial x_{\alpha_r}}. \tag{4.7}$$

Note that $\partial_\alpha \neq \partial^\alpha$ even when both sides make sense. We also put

$$dx_\alpha = dx_{\alpha_1} \cdots dx_{\alpha_r}. \tag{4.8}$$

Let $\tilde{\alpha}_t = (\alpha_1, \dots, \alpha_{t-1}, \alpha_{t+1}, \dots, \alpha_t)$.

The symmetrized covariant derivative, $\nabla_s \eta$, of a tensor η is obtained by taking the usual covariant derivative and then averaging over the symmetric group to make it symmetric. On $\mathbb{R}^{n-1}$, if the symmetric $l-1$ tensor $\eta = \sum \phi_\gamma dx_\gamma$, and $\nabla_s \eta = \sum \psi_\alpha dx_\alpha$, then

$$\psi_\alpha = \frac{1}{l} \sum_{j=1}^{l} \partial_{\alpha_j} \phi_{\tilde{\alpha}_j}. \tag{4.9}$$

Our proof of the lemma follows from the observation that the symmetric $l-1$ tensors must satisfy certain PDEs. If $\alpha, \beta \in \{1, \dots, n-1\}^r$, we define an exchange to be a map swapping certain places in α with ones in β. We include the case where no swaps take place. The order of the exchange is the number of swaps. The sign of the exchange will be -1 to the power of the order. If the exchange is e, we denote the sign of e by $\operatorname{sgn}(e)$ and the new value of α by $e(\alpha, \beta)^{(1)}$ and of β by $e(\alpha, \beta)^{(2)}$. For example, if e exchanges α_1 with β_1 then

$$e(\alpha, \beta)^{(1)} = (\beta_1, \alpha_2, \dots, \alpha_r)$$
$$e(\alpha, \beta)^{(2)} = (\alpha_1, \beta_2, \dots, \beta_r)$$
$$\operatorname{sgn}(e) = -1.$$

If $\alpha, \beta \in \{1, \dots, n-1\}^l$, and $\mu = \sum \psi_\gamma dx_\gamma$ is a covariant symmetrized derivative then

$$\sum_e \operatorname{sgn}(e) \partial_{e(\alpha,\beta)^{(2)}} \psi_{e(\alpha,\beta)^{(1)}} = 0, \tag{4.10}$$

where the sum is taken over all exchanges. To prove this one substitutes the expression for a covariant derivative and observes that the terms with no swaps will cancel with some of the terms from swaps of order one but that the swaps of order one will then have remainder terms which are canceled by swaps of order two and so on.

Suppose we have that every homogeneous polynomial of order l times μ is a symmetrized covariant derivative. We show that each term ψ_α vanishes at the origin. We consider two cases. The first case is that α does not contain all possible values from $\{1, \ldots, n-1\}$ (which will always happen when $l < n-1$). Let r be the value not taken. Let $\beta = (r, \ldots, r)$. We have that

$$\sum_e \operatorname{sgn}(e)\partial_{e(\alpha,\beta)^{(2)}}(x_\beta \psi_{e(\alpha,\beta)^{(1)}}) = 0 \tag{4.11}$$

where $x_\beta = x_{\beta_1}x_{\beta_2}\cdots x_{\beta_l}$. Upon evaluating at $x = 0$, all terms will vanish except $\partial_\beta(x_\beta\psi_\alpha)$, which will equal ψ_α, which must therefore be zero.

In the second case, α takes all values from $\{1, \ldots, n-1\}$. We let β_j equal 2 if α_j equals 1 and 1 otherwise. As before, (4.11) is satisfied. Upon evaluation at zero, the only terms that will not vanish are those for which $e(\alpha, \beta) = (\alpha, \beta)$ (up to the order of α_j and the order of β_j) and such e will have positive sign as an even number of swaps will be involved. We therefore have that a positive multiple of ψ_α is zero and thus that ψ_α is zero. ■

This completes the proof for $\mathbb{R}^{n-1}$ with the Euclidean metric. We, however, wish to obtain a similar result for the sphere. Projecting the upper hemi-sphere onto the plane, we obtain coordinates on the sphere and the metric agrees with the Euclidean one at the north pole (which corresponds to the origin). The symmetrized covariant derivative will then agree with the Euclidean one up to terms vanishing there. As we are applying the result to polynomials of order l these additional terms do not affect the argument and so we conclude that the result will also hold for the sphere.

So we have shown that if the principal symbol vanishes then the lead term of an *aradial* perturbation vanishes also and thus by induction the asymptotics of aradial perturbations are recoverable from fixed energy scattering data.

5 Review of Legendrian Distributions

In this section we review and rephrase the material we need from [14] and [15]. Here X is a compact manifold with boundary ∂X and g is a scattering metric on X and we have chosen a product decomposition of the form (1.1). Our account is necessarily brief and we refer the reader to [14] and [15] for more details.

There is a natural bundle over X called the scattering cotangent bundle which is denoted ${}^{\mathrm{sc}}T^*(X)$. This is the dual to the bundle of smooth vector fields of bounded length with respect to some (and hence all) scattering metrics on X. The restriction of ${}^{\mathrm{sc}}T^*(X)$ to ∂X is denoted ${}^{\mathrm{sc}}T^*(X)_{|\partial X}$ and carries a natural contact structure. If y are local coordinates on ∂X and μ are the corresponding dual coordinates, then (y, μ, τ) form local coordinates on ${}^{\mathrm{sc}}T^*(X)_{|\partial X}$, where τ is the coefficient of $\frac{dx}{x^2}$.

We recall from Section 2 that a differential operator $P(x, y, xD_y, x^2D_x)$ will be in $\Psi^{m,k}_{\mathrm{sc}}(X, {}^{\mathrm{sc}}\Omega^{1/2})$ if it is of order m and the total symbol as an operator in xD_y, x^2D_x vanishes to k-th order at the boundary. It then has a well-defined symbol at the boundary

$$j(P) = x^k p_k + x^{k+1} p_{k+1}, \quad p_k, p_{k+1} \in C^\infty(\mathbb{R} \times T^*\partial X).$$

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Definition 5.1 An intersecting pair with conic points is a subset, $\widetilde{W}$, of ${}^{\mathrm{sc}}T^*(X)_{|\partial X}$ which is a union of the closure of a smooth Legendrian submanifold, W, and $W^\#$, which is a finite union of global sections of the form $W^\#(\lambda_j) = \{(y, 0, \lambda_j)\}$, and contains $\overline{W} \setminus W$. We also require $\overline{W}$ to have an at most conic singularity at $\mu = 0$; that is, it is smooth if polar coordinates are introduced along $\overline{W} \setminus W$.

The process of introducing polar coordinates along $\overline{W} \setminus W$ can be given an invariant meaning and is then called blow-up. We denote the blown-up manifold by $\widehat{W}$.

The metric g induces a metric h on the boundary as nearby it is of the form

$$\tau^2 + h'(y, \mu) + xg'$$

as a function on ${}^{\mathrm{sc}}T^*X$, we obtain h' from h via the isomorphism

$$\mu.\frac{dy}{x} \longmapsto \mu.dy.$$

Example 5.1 For each $y' \in \partial X$ and $0 \neq \lambda \in \mathbb{R}$, let $G_{y'}(\lambda)$ be equal to the set of (τ, y, μ), such that $\tau^2 + |\mu|^2 = \lambda^2$, $\mu \neq 0$, and putting $\mu = |\mu|\hat{\mu}$, such that

$$\begin{aligned} \tau &= |\lambda| \cos(s) \\ |\mu| &= |\lambda| \sin(s) \qquad (5.1) \\ (y, \hat{\mu}) &= \exp(sH_{\frac{1}{2}h})(y', \hat{\mu}') \end{aligned}$$

where $s \in (0, \pi)$, $(y', \hat{\mu}') \in T^*\partial X$, and $h(y', \hat{\mu}') = 1$. Then $G_{y'}(\lambda) \cup \{(\lambda, y, 0)\}$ is an intersecting pair with conic points. We denote this pair $\widetilde{G}(\lambda)$. This is the pair in which we are interested here. The set $G^\sharp(\lambda) = \{(-\lambda, y, 0, y', 0)\}$ is also important in our construction. Note that $G^\sharp(\lambda)$ is the initial or incoming surface and that $G^\sharp(-\lambda)$ is the outgoing surface. Note that in the coordinates defined by (5.1), $G^\sharp(\lambda)$ is $s = \pi$ and $G^\sharp(-\lambda)$ is $s = 0$, when λ is positive.

Associated with these intersecting pairs at each conic point is a unique homogeneous Lagrangian submanifold $\Lambda(\widetilde{W}, \lambda_i)$ of $T^*(\partial X)$. For the pair $\widetilde{G}(\lambda)$ we are interested in, this is precisely the relation of being π apart along a lifted geodesic. (See Proposition 4 of [15].) For simplicity, we shall henceforth take λ to be positive. The λ negative case is similar, or could be deduced from the positive case.

Melrose and Zworski associated to any such intersecting pair a class of smooth functions whose asymptotics on approach to the boundary are determined by symbols on the Legendrians. A symbol bundle over the smooth Legendrian $W(\lambda)$ in the pair $\widetilde{W}$ can be defined and is denoted $\hat{E}^{m,p}$. The sections of this bundle are of the form

$$aS^{p-m}|dx|^{m-n/4}$$

where a is a smooth section of $C^\infty(\hat{W}; \Omega_b^{\frac{1}{2}} \otimes M_{\hat{H}})$, S is a defining function of the boundary of W, M is the Maslov bundle, and $\Omega_b^{\frac{1}{2}}$ is the b-half density bundle. For G above, one could take $S = \sin s$. Melrose and Zworski remove this singularity at the endpoints by rescaling but for us it will be easier not to do so.

Proposition 5.1 *If $\widetilde{W}(\lambda)$ is an intersecting pair with conic points then there is a class of smooth half-densities on X^o, denoted $I^{m,p}_{\mathrm{sc}}(X, \widetilde{W})$, such that $\bigcap_{m,p} I^{m,p}_{\mathrm{sc}}(X, \widetilde{W})$ is equal to the class of half-densities vanishing to infinite order at the boundary. There exists a symbol map*

$$\hat{\sigma}_{sc,m,p} \colon I^{m,p}_{\mathrm{sc}}(X, \widetilde{W}, {}^{\mathrm{sc}}\Omega^{1/2}) \to C^\infty(\hat{W}; \hat{E}^{m,p})$$

which gives a short exact sequence

$$0 \to I^{m+1,p}_{\mathrm{sc}}(X, \widetilde{W}, {}^{\mathrm{sc}}\Omega^{1/2}) \to I^{m,p}_{\mathrm{sc}}(X, \widetilde{W}, {}^{\mathrm{sc}}\Omega^{1/2}) \to C^\infty(\hat{W}; \hat{E}^{m,p}) \to 0.$$

This is Proposition 12 from [15]. The Legendrian half-densities of order m are given locally away from the conic points by oscillatory integrals

$$u = (2\pi)^{-\frac{n}{4}-\frac{k}{2}} \int e^{i\phi(y,u)} a(x, y, u) x^{m-k/2+n/4}\, du, \tag{5.2}$$

with a smooth on $[0, \epsilon) \times U \times U'$ with U, U' open and ϕ parameterizes W, that is

$$W = \{\big(y, -\phi(y, u), d_y\phi\big) : d_u\phi = 0\}. \tag{5.3}$$

Near the conic singularity a more general form is required and we refer the reader to [15]. The order m here is adjusted by $-k/2 + n/4$ and so is inconsistent with Section 3. We keep this inconsistency as the second definition provides for better invariance properties but less clarity.

An important related fact we need to know is how Legendrian distributions map under scattering pseudo-differential operators. We recall Proposition 13 from [15].

Proposition 5.2 *Suppose $P \in \Psi^{l,k}_{\mathrm{sc}}(X, {}^{\mathrm{sc}}\Omega^{1/2})$ has symbol $x^k p_k + x^{k+1} p_{k+1}$ with respect to a product decomposition of X near ∂X, and suppose that*

$$W \subset {}^{\mathrm{sc}}T^*_{\partial X}(X)$$

is a smooth Legendre submanifold. Then for any $m \in \mathbb{R}$,

$$P \colon I^m_{\mathrm{sc}}(X, W; {}^{\mathrm{sc}}\Omega^{1/2}) \to I^{m+k}_{\mathrm{sc}}(X, W; {}^{\mathrm{sc}}\Omega^{1/2}) \tag{5.4}$$

$$\sigma_{sc,m+k}(Pu) = (p_{k|G})\sigma_{sc,m}(u) \otimes |dx|^k. \tag{5.5}$$

Furthermore, if p_k vanishes identically on W then

$$P \colon I^m_{\mathrm{sc}}(X, W; {}^{\mathrm{sc}}\Omega^{1/2}) \to I^{m+k+1}_{\mathrm{sc}}(X, W; {}^{\mathrm{sc}}\Omega^{1/2})$$

and

$$\sigma_{sc,m+k+1}(Pu) = \left(\frac{1}{i}\left(L_V + \left(\frac{1}{2}(k+1) + m - \frac{n}{4}\right)\frac{\partial p_k}{\partial \tau}\right) + p_{k+1|W}\right) a \otimes |dx|^{m+k+1-\frac{n}{4}}$$

where $\sigma_{sc,m}(u) = a \otimes |dx|^{m-\frac{n}{4}}$ and V is the rescaled Hamiltonian vector field associated to p_k.

We omit the definition of the rescaled Hamiltonian vector field but recall that for Δ on the pair G we are studying it is equal to

$$2\lambda \sin s \frac{\partial}{\partial s}$$

in the semi-global coordinates given by (5.1).

We also need two push-forward theorems, Propositions 16 and 17, from [15]. They relate the singularities of the scattering matrix to the asymptotics in small x of the Poisson operator. Given a product decomposition near the boundary, there is a natural pairing

$$B\colon C^{-\infty}(X, {}^{\mathrm{sc}}\Omega^{1/2}) \times C^{\infty}(\partial X; {}^{\mathrm{sc}}\Omega^{1/2}) \to C^{-\infty}\big([0,\epsilon), {}^{\mathrm{sc}}\Omega^{1/2}\big) \tag{5.6}$$

$$B(u,f) = x^{\frac{n-1}{2}} \int_{\partial X} u(x,y) f(y). \tag{5.7}$$

Proposition 5.3 *For any intersecting pair of Legendre submanifolds with conic points, W, the partial pairing* (5.7) *gives a map*

$$B\colon I^{m,p}_{\mathrm{sc}}(X, \widetilde{W}; {}^{\mathrm{sc}}\Omega^{1/2}) \times C^{\infty}(\partial X; {}^{\mathrm{sc}}\Omega^{1/2}) \mapsto \sum_j I^{p+\frac{n-1}{4}}\big([0,\epsilon), W'(\bar{\tau}_j; {}^{\mathrm{sc}}\Omega^{1/2})\big)$$

where the $W'(\bar{\tau}_j) = \{(0, -\tau_j dx/x^2)\}$ are the Legendre submanifolds corresponding to the components of $W^{\#}$ and

$$B(u,f) = \sum_j e^{-i\bar{\tau}/x} x^{p+n/4} Q^0_{\bar{\tau}_j}(u,f) \Big|\frac{dx}{x^2}\Big|^{\frac{1}{2}} + O(x^{p+n/4+1})$$

with

$$Q^0_{\bar{\tau}}(u) \in I^{p-m-\frac{n-1}{4}}_{phg}\big(\partial X, \Lambda(\widetilde{W}, \bar{\tau})\big),$$

and the principal symbol of $Q^0_{\bar{\tau}}(u)$ determines and is determined by the lead singularity of the principal symbol of u on W on approach to $W'(\bar{\tau}_j)$.

When the Legendrian distribution is actually associated to a smooth Legendrian submanifold the push-forward becomes much simpler and this simplifies the construction of the Poisson operator.

Proposition 5.4 *If G is a smooth Legendre variety and $u \in I^m_{\mathrm{sc}}(X, G', {}^{\mathrm{sc}}\Omega^{1/2})$ near $\tau = \bar{\tau}$, then the distribution $Q^0_{\bar{\tau}}$ is a Dirac delta distribution.*

6 The Poisson Operator in the General Case

In this section, we apply the calculus reviewed in Section 5 to construct the Poisson operator and prove that the higher order scattering matrix is indeed a zeroth order, classical, Fourier integral operator, proving Theorem 1.2 and Corollary 1.1. We shall refer heavily to [15, Section 15] as our construction is a modification of the one there.

We assume a product decomposition of X close to the boundary of the form (1.1) has been chosen and is fixed throughout this section. We then have as in [15] that Δ, the intrinsic Laplacian acting on scattering half-densities on X, induces an operator

$$\Delta_X \in \mathrm{Diff}^2_{\mathrm{sc}}\big(X \times \partial X, {}^{\mathrm{sc}}\Omega^{1/2}(X \times \partial X)\big)$$

by

$$\Delta_X\left(u\left|\frac{dx}{x^2}\right|^{\frac{1}{2}}\left|\frac{dy}{x^{n-1}}\right|^{\frac{1}{2}}\left|\frac{dy'}{x^{n-1}}\right|^{\frac{1}{2}}\right) = \Delta\left(u(.,y')\left|\frac{dx}{x^2}\right|^{\frac{1}{2}}\left|\frac{dy}{x^{n-1}}\right|^{\frac{1}{2}}\right)\left|\frac{dy'}{x^{n-1}}\right|^{\frac{1}{2}},$$

where (x, y, y') is a point in $X \times \partial X$. Throughout this section V will be a short range higher order perturbation, that is a scattering differential operator of order $2k-1$ as a differential operator and order 2 at the boundary.

We recall from [10], using the notation of (1.1) that

Lemma 6.1 *The symbol at the boundary of Δ_X is $p = p_0 + xp_1$ with $p_0 = \tau^2 + h(0, y, \mu)$ and with p_1 equal to $-i(n-1)\tau + c$ where $c = \frac{\partial}{\partial x} h(x, y, \mu)_{|x=0}$ is quadratic in μ.*

Our choice of product decomposition ensures that there is no τ term in c. We need to compute the symbols at the boundary of $\Delta_X^k - \lambda^{2k} + V$. The short range assumption on V ensures that it does not contribute—indeed this is one reason why this definition of short range is appropriate. Now we can decompose

$$(\Delta^k - \lambda^{2k}) = Q(\lambda)(\Delta_X - \lambda^2),$$

with

$$Q(\lambda) = \sum_{j=0}^{k-1} \lambda^{2j} \Delta_X^{k-1-j}.$$

We deduce that the lead symbol of $(\Delta_X^k - \lambda^{2k})$ is $p_0^k - \lambda^{2k}$ and the second symbol (sub-principal term) is equal to $k\lambda^{2(k-1)}$ times p_1 on the zero set of the principal symbol. It also follows that the Hamiltonian on the boundary of the lead term is just $k\lambda^{2(k-1)}$ times that of p_0 on the zero set of the principal symbol. The fact that both these terms have been changed simply by multiplication by $k\lambda^{2(k-1)}$ ensures the simplicity of extending results from the $k = 1$ case to the higher order case.

We also note the following lemma which is important in our construction to show that the transport equations are solvable.

Lemma 6.2 *If $L \in I^{m,-\frac{1}{4}}\big(X \times \partial X, \tilde{G}(\lambda), {}^{\mathrm{sc}}\Omega^{1/2}\big)$ is such that*

$$(\Delta_X^k - \lambda^{2k} + V)L \in I^{m+j,\frac{3}{4}}\big(X \times \partial X, \tilde{G}(\lambda), {}^{\mathrm{sc}}\Omega^{1/2}\big),$$

then

$$(\Delta_X^k - \lambda^{2k} + V)L \in I^{m+j,\frac{7}{4}}\big(X \times \partial X, \tilde{G}(\lambda), {}^{\mathrm{sc}}\Omega^{1/2}\big).$$

This is a modification of Lemma 15 from [15] and in fact, the $k = 1$ case, though not explicitly stated, is essential to the construction there also.

Proof The proof is no different from that of the special case in $\mathbb{R}^n$ (Lemma 3.1). ∎

Proposition 6.1 *For any* $0 \neq \lambda \in \mathbb{R}$ *there exists*

$$K \in I^{m,p}\big(X \times \partial X, \tilde{G}(\lambda); {}^{\mathrm{sc}}\Omega^{1/2}\big)$$

such that

$$(\Delta_X^k - \lambda^{2k} + V)K \in \mathcal{C}^\infty(X \times \partial X; {}^{\mathrm{sc}}\Omega^{1/2}) \text{ and}$$
$$Q_\lambda^0(K) = \mathrm{Id},$$

with

$$m = -\frac{2n-1}{4}, \quad p = -\frac{1}{4},$$

and the principal symbol of K on G is

$$C \sin(s)^{\frac{n-1}{2}} \frac{|ds|^{\frac{1}{2}}\,|dy|^{\frac{1}{2}}\,|d\hat{\mu}|^{\frac{1}{2}}}{(\sin s)^{\frac{1}{2}}} |dx|^{m-\frac{2n-1}{4}},$$

where $C(y, \hat{\mu})$ *is a non-zero smooth function.*

Proof As in [15], we first construct $K^b \in I^{m,p}\big(X \times \partial X, \tilde{G}(\lambda); {}^{\mathrm{sc}}\Omega^{1/2}\big)$ such that

$$(\Delta_X^k - \lambda^{2k} + V)K^b \in I_{\mathrm{sc}}^{\frac{3}{4}}\big(X \times \partial X, G^\sharp(-\lambda)\big) \text{ and} \tag{6.1}$$

$$Q_\lambda^0(K^b) = \mathrm{Id}\,. \tag{6.2}$$

We construct K^b as an asymptotic sum of

$$K_j \in I_{\mathrm{sc}}^{-\frac{2n-1}{4}+j,-\frac{1}{4}}\big(X \times \partial X, \tilde{G}(\lambda); {}^{\mathrm{sc}}\Omega^{1/2}\big).$$

We wish

$$(\Delta_X^k - \lambda^{2k} + V)K_0 \in I_{\mathrm{sc}}^{-\frac{2n-1}{4}+2,\frac{3}{4}}\big(X \times \partial X, \tilde{G}(\lambda); {}^{\mathrm{sc}}\Omega^{1/2}\big) \text{ and} \tag{6.3}$$

$$\sigma_0\big(Q_\lambda^0(K_0)\big) = \sigma_0(\mathrm{Id}) \tag{6.4}$$

and then it is automatically in

$$I_{\mathrm{sc}}^{-\frac{2n-1}{4}+2,\frac{7}{4}}\big(X \times \partial X, \tilde{G}(\lambda); {}^{\mathrm{sc}}\Omega^{1/2}\big)$$

by Lemma 6.2. We also want

$$(\Delta_X^k - \lambda^{2k} + V)\Big(\sum_{l=0}^{j-1} K_l\Big) \in I_{\mathrm{sc}}^{-\frac{2n-1}{4}+j+2,\frac{3}{4}}\big(X \times \partial X, \tilde{G}(\lambda); {}^{\mathrm{sc}}\Omega^{1/2}\big)$$

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and this, of course, implies that it will also be an element of

$$I_{\mathrm{sc}}^{-\frac{2n-1}{4}+j+2,\frac{7}{4}}\big(X\times\partial X,\tilde{G}(\lambda);{}^{\mathrm{sc}}\Omega^{1/2}\big).$$

Now near $G\cap G^{\sharp}(\lambda)$, where G is smooth, we can as in [15] give an explicit construction and it is then only necessary to have that the principal symbol of $Q^0_\lambda(K_0)$ is equal to 1 to ensure that $Q^0_\lambda(K^b)=\mathrm{Id}$. We look for K_j of the form

$$x^j e^{i\lambda\phi(y,y')/x}a_j(x,y,y',\lambda)\nu,\, a_j\in C^\infty(X\times\partial X),$$

with ν a fixed scattering half-density and ϕ the cosine of the Riemannian distance from y to y'. Let a_j' be the restriction of a_j to $x=0$. Taking geodesic normal coordinates, y, about each y' the transport equation for a_j' is of the form

$$(y\cdot\partial_y+j)a_j'+b_ja_j'=c_j\in C^\infty(X\times\partial X)$$

near $y=0$ where c_0 is identically zero and b_j vanishes quadratically at $y=0$. So as in [15], the terms K_j exist sc-microlocally close to $G^{\sharp}(\lambda)$.

We now need to continue each K_j up to $G^{\sharp}(-\lambda)$. We do so by solving transport equations for the principal symbols and iteratively solving away the error.

The principal symbol of K_0, $\sigma_m(K_0)$, is of the form

$$b\frac{|ds|^{\frac{1}{2}}\,|dy|^{\frac{1}{2}}\,|d\hat{\mu}|^{\frac{1}{2}}}{(\sin s)^{\frac{1}{2}}}|dx|^{m-\frac{2n-1}{4}}.$$

On the lifted geodesic $\beta(s)$ the sub-principal term $c\big(\beta(s)\big)=2k\lambda^{2k-1}\sin(s)d\big(\beta(s)\big)$ for some smooth d. From Proposition 5.2, the transport equation for b is

$$\frac{2k\lambda^{2k-1}}{i}\left(\sin(s)\frac{d}{ds}+\Big(\frac{1-n}{2}\Big)\cos(s)+i\sin(s)d\big(\beta(s)\big)\right)b=0.$$

Writing $\tilde{b}=(e^{i\int d(\beta(s'))\,ds'}\sin(s)^{\frac{1-n}{2}})b$, we thus have

$$\frac{d\tilde{b}}{ds}=0.$$

This means that

$$b=C\sin(s)^{\frac{n-1}{2}}e^{-i\int d(\beta(s'))\,ds'}. \tag{6.5}$$

As $s\to\pi-$, that is near $G^{\sharp}(\lambda)$, this has a singularity of the form $(\pi-s)^{\frac{n-1}{2}}$ and as $s\to 0+$, of the form $s^{\frac{n-1}{2}}$.

As the order on $G^{\sharp}(\pm\lambda)$ is $-\frac{1}{4}$, the order $p-m$ is equal to the order of singularity and thus the solution of the transport equation is a legitimate symbol and we can construct K_0.

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Now by Lemma 6.2,

$$(\Delta_X^k + V - \lambda^{2k})(K_0) \in I^{-\frac{2n-1}{4}+2,-\frac{1}{4}+2}(X \times \partial X, \tilde{G}; {}^{\mathrm{sc}}\Omega^{1/2}),$$

and we look for

$$K_1 \in I^{-\frac{2n-1}{4}+1,-\frac{1}{4}}(X \times \partial X, \tilde{G}; {}^{\mathrm{sc}}\Omega^{1/2}),$$

such that

$$(\Delta_X^k + V - \lambda^{2k})(K_0 + K_1) \in I^{-\frac{2n-1}{4}+3,-\frac{1}{4}+1}(X \times \partial X, \tilde{G}; {}^{\mathrm{sc}}\Omega^{1/2}),$$

and thus by Lemma 6.2 is in $I^{-\frac{2n-1}{4}+3,-\frac{1}{4}+2}(X \times \partial X, \tilde{G}; {}^{\mathrm{sc}}\Omega^{1/2})$. Letting the principal symbol of K_1 be $b_1|dx|^{-\frac{2n-1}{4}+1-\frac{2n-1}{4}}$ times the trivializing density above, we obtain a transport equation; arguing as above, it becomes

$$\sin(s)^{\frac{n-1}{2}} e^{-i\int d(\beta(s'))\,ds'} \frac{d}{ds} e^{i\int d(\beta(s'))\,ds'} \left(\sin(s)^{\frac{1-n}{2}+1} b_1\right) = g(s) e^{-i\int d(\beta(s'))\,ds'} \sin(s)^{\frac{n-1}{2}},$$

with $g(s)$ a smooth function on $[0,\pi]$ (and smoothly depending on the suppressed parameters). Canceling, we obtain that

$$\frac{d}{ds}\left(e^{i\int d(\beta(s')\,ds'} \sin(s)^{\frac{1-n}{2}+1} b_1\right) = g(s),$$

which has a solution in the appropriate symbol class. The same argument, after appropriately shifting indices, constructs all the terms K_j.

Asymptotically summing, we obtain K^b such that

$$(\Delta_X^k + V - \lambda^{2k})K^b \in I^{7/4}\left(G^\sharp(-\lambda)\right).$$

These errors can now be removed by an iterative construction of their Taylor series, *cf.* Lemma 16 of [15], and we obtain K as desired. ∎

We have therefore constructed the Poisson operator modulo smooth terms vanishing to infinite order at the boundary. We wish to remove this error by applying the resolvent. As before, this is possible even if there is embedded discrete spectrum, as the error is orthogonal to the eigenspace at that energy. This is seen by a simple integration by parts, since the elements of the eigenspace are Schwartz they are orthogonal to the image of smooth functions of tempered growth. We have thus constructed $P(\lambda)$, the Poisson operator for the problem, as a paired Legendrian distribution. The remainder of the proof that $S(\lambda)$ is a Fourier integral operator now follows as in [15, Proposition 19].

To prove Corollary 1.1, one observes that if $P_1(\lambda)$ is the Poisson operator for $\Delta - \lambda^2$, then

$$(\Delta^k + V - \lambda^{2k})P_1(\lambda) = Q(\lambda)(\Delta - \lambda^2)P_1(\lambda) + VP_1(\lambda) = VP_1(\lambda). \tag{6.6}$$

This means that the Poisson operator for $\Delta^k + V - \lambda^{2k}$ can be constructed as a perturbation of $P_1(\lambda)$ and as $VP_1(\lambda) \in I^{-\frac{2n-1}{4}+l,-\frac{1}{4}+l}$, the scattering matrices will agree to order $1-l$, and the corollary follows. Note, however, that even if one fixes the perturbation, the scattering matrices will not agree to order more than $1-l$ as the solutions of the transport equations will differ.

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