INDEFINITE QUADRATIC POLYNOMIALS

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1. Introduction. Let

$$Q(\mathbf{x})=\sum_{i=1}^{n}\sum_{j=1}^{n}\alpha_{ij}x_ix_j \qquad (\alpha_{ij}=\alpha_{ji}) \tag{1}$$

be an indefinite quadratic form with real coefficients. A well-known result, due to Birch, Davenport and Ridout [**1**], [**5**] and [**6**], states that if $n\geq 21$ then for any $\varepsilon>0$ there is an integer vector $\mathbf{x}\neq\mathbf{0}$ such that

$$|Q(\mathbf{x})|<\varepsilon. \tag{2}$$

Recently [**3**] we have quantified this result, obtaining a function $g(n)$ such that $g(n)\to\frac{1}{2}$ as $n\to\infty$ and such that for any $\eta>0$ and all large enough X there is an integer vector $\mathbf{x}$ satisfying

$$0<|\mathbf{x}|\leq X \quad \text{and} \quad |Q(\mathbf{x})|\ll X^{-g(n)+\eta}, \tag{3}$$

where $|\mathbf{x}|=\max|x_i|$ and the implicit constant in Vinogradov's $\ll$-notation is independent of X.

Suppose that when Q is expressed as a sum of squares of real linear forms, with positive and negative signs, there are r positive signs and $n-r$ negative signs, then we may say that Q is of type $(r, n-r)$. We shall call a quadratic polynomial

$$F(\mathbf{x})=Q(\mathbf{x})+L(\mathbf{x})+C \tag{4}$$

indefinite if the quadratic part $Q(\mathbf{x})$ is indefinite. It is not possible to obtain a complete analogue of (3) with Q replaced by a general quadratic polynomial. For example, if Q and L have integer coefficients and $C=\frac{1}{2}$ then clearly it is not possible to obtain a result like (3). So we shall suppose that $C=0$.

THEOREM. *Let $F(\mathbf{x})=Q(\mathbf{x})+L(\mathbf{x})$ be a quadratic polynomial in n variables and having no constant term. Suppose that Q is indefinite of type $(r, n-r)$, where*

$$\min(r, n-r)\geq 4. \tag{5}$$

Then there exists an absolute constant A such that for

$$f(n)=-\tfrac{1}{3}+A/n \tag{6}$$

and any $\eta>0$ and all large enough X there is an integer vector $\mathbf{x}$ satisfying

$$0<|\mathbf{x}|\leq X \quad \textit{and} \quad |F(\mathbf{x})|\ll X^{-f(n)+\eta}. \tag{7}$$

The proof of the theorem shows that $A<33$ and no doubt this could be improved; the major interest of the result is that $f(n)\to\frac{1}{3}$ as $n\to\infty$.

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The methods used here are capable of providing a non-trivial result when $\min(r, n-r)=2$ or 3, but the exponents obtained

$$-\tfrac{1}{5}+O(n^{-1}) \quad \text{and} \quad -\tfrac{2}{7}+O(n^{-1}),$$

respectively, are weaker. The detailed calculations in these cases are left to the reader.

2. Preliminary lemmas. Our first lemma, which is Lemma 1 of [**3**], is essentially a reformulation of the result of Birch and Davenport [**2**]. It shows that indefinite diagonal quadratic forms take small values.

LEMMA 1. *For any $\tau>0$ there exists $C(\tau)$ with the following property. For any real $\lambda_1,\ldots,\lambda_5$, not all of the same sign, and real numbers $X_1,\ldots,X_5$, Y, all at least 1, satisfying*

$$Y(Y^5\Pi)^\tau < C(\tau)X_i^{1/2}\,|\lambda_i\Pi^{-1}|^{\frac{1}{4}} \quad \text{for} \quad 1\le i\le 5, \tag{8}$$

where $\Pi=|\lambda_1\cdots\lambda_5|$, there exist integers $x_1,\ldots,x_5$, not all zero, such that

$$0\le x_i\le X_i \quad \text{for} \quad i=1,\ldots,5 \tag{9}$$

and

$$|\lambda_1x_1^2+\ldots+\lambda_5x_5^2|<Y^{-1}. \tag{10}$$

In order to replace the indefinite quadratic polynomial F with another polynomial that is almost a diagonal quadratic form we make use of the following lemma of Birch and Davenport [**1**], it is essentially a sophisticated version of Dirichlet's pigeon-hole principle.

LEMMA 2. *Suppose that $m<n$ and let $L_1(\mathbf{x}),\ldots,L_m(\mathbf{x})$ be m real linear forms in n variables $x_1,\ldots,x_n$, say*

$$L_i(\mathbf{x})=\sum_{j=1}^{n}\gamma_{ij}x_j \quad \text{for} \quad i=1,\ldots,m. \tag{11}$$

Then, for any $P\ge 2$, there exists a non-zero integer vector $\mathbf{x}$ satisfying

$$|\mathbf{x}|\le P^m \quad \text{and} \quad \max_i|L_i(\mathbf{x})|\le C_0P^{m-n}\sum_{j=1}^{n}|\gamma_{ij}|, \tag{12}$$

where C_0 is an absolute constant.

Our next lemma is the crucial result and its proof takes up the remaining sections of this paper.

LEMMA 3. *Let $F(\mathbf{x})=Q(\mathbf{x})+L(\mathbf{x})$ be a quadratic polynomial in n variables, having no constant term. Let $Q(\mathbf{x})$ be indefinite of type $(4, n-4)$. Then for any $\eta>0$ and all sufficiently large X there is an integer vector $\mathbf{x}$ satisfying*

$$0<|\mathbf{x}|\le X \quad \text{and} \quad |F(\mathbf{x})|\ll X^{-\frac{1}{3}+(49/3n)+\eta}, \tag{13}$$

provided that n is large.

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We now deduce the main theorem from Lemma 3. Replacing Q by $-Q$, if necessary, we may suppose that $\min(r, n-r) = r$. Using an appropriate integral unimodular transformation $\mathbf{x} = U\mathbf{y}$ and then completing the square, we can express Q in the form

$$-\alpha_1\xi_1^2 - \ldots - \alpha_{n-r}\xi_{n-r}^2 + \alpha_{n-r+1}\xi_{n-r+1}^2 + \ldots + \alpha_n\xi_n^2, \tag{14}$$

where $\alpha_1, \ldots, \alpha_n$ are positive and $\xi_1, \ldots, \xi_n$ are linear forms with real coefficients having a triangular matrix

$$\xi_j = \beta_{jj}x_j + \ldots + \beta_{jn}x_n. \tag{15}$$

Taking $x_i = 0$ for $i = n-r+5, \ldots, n$, we see that Q represents a form Q_1 of type $(4, n-r)$ in $n-r+4$ variables. Now $r \leq \frac{1}{2}n$ and so $n-r+4 \geq \frac{1}{2}n+4$. Thus, for any $\eta > 0$ and large enough Ξ, there exist $x_1, \ldots, x_{n-r+4}$ such that

$$0 < \max|x_i| \leq \Xi \quad \text{and} \quad |F_1(\mathbf{x})| < \Xi^{-\frac{1}{3}+(A/n)+\eta}, \tag{16}$$

for some absolute constant A. Since Q is of type $(r, n-r)$ the forms ξ_j given by (15) are independent and so $\beta_{jj} \neq 0$ for $j = 1, \ldots, n$. Inverting transformation U we find that there is a number $B = B(Q)$, independent of X and Ξ, such that

$$|\mathbf{y}| \leq B\,|\mathbf{x}|. \tag{17}$$

Taking $\Xi = X/B$ in (16), we then obtain a suitable solution of the inequalities (7) since $\mathbf{y} = \mathbf{0}$ only if $\mathbf{x} = \mathbf{0}$.

3. A suitable diagonalization. We may now suppose that $Q(\mathbf{x})$ is an indefinite quadratic form of type $(4, n-4)$. As in Davenport [**4**], there is a non-singular linear transformation $\mathbf{y} = T\mathbf{x}$ which takes $Q(\mathbf{x})$ into a quadratic form $Q'(\mathbf{y})$ satisfying

$$Q'(y_1, y_2, y_3, y_4, 0, \ldots, 0) > 0 \tag{18}$$

if $y_1, \ldots, y_4$ are not all zero, and

$$Q'(0, 0, 0, 0, y_5, \ldots, y_n) < 0 \tag{19}$$

provided that $y_5, \ldots, y_n$ are not all zero. Since T is non-singular, $|\mathbf{x}|$ is bounded by constant multiples of $|\mathbf{y}|$ and vice-versa. Therefore there is no loss of generality in proving Lemma 3 under the additional hypothesis that Q satisfies (18) and (19). Then

$$\alpha_{11} = Q(1, 0, \ldots, 0) > 0. \tag{20}$$

With $Q(\mathbf{x})$ we associated the bilinear form

$$B(\mathbf{x}, \mathbf{y}) = \sum_{i=1}^{n} \sum_{j=1}^{n} \alpha_{ij}x_i y_j \tag{21}$$

and use a suitably chosen linear transformation

$$\mathbf{x} = u_1\mathbf{z}^1 + \ldots + u_5\mathbf{z}^5, \tag{22}$$

where $\mathbf{z}^1, \ldots, \mathbf{z}^5$ are non-zero vectors in $\mathbb{Z}^n$, to show that $F(\mathbf{x})$ represents a quadratic polynomial that is close to a diagonal quadratic form in 5 variables.

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Let

$$L(\mathbf{x}) = \sum_{j=1}^{n} \lambda_j x_j. \tag{23}$$

We choose $\mathbf{z}^1$ by applying Lemma 2 with $m=1$, $n=4$,

$$L_1(\mathbf{x}) = \lambda_1 x_1 + \lambda_2 x_2 + \lambda_3 x_3 + \lambda_4 x_4 \tag{24}$$

and P replaced by Z. We obtain a non-zero integer vector

$$\mathbf{z}^1 = (\zeta_1, \ldots, \zeta_4, 0, \ldots, 0) \tag{25}$$

satisfying

$$|\mathbf{z}^1| \leq Z \quad \text{and} \quad |L(\mathbf{z}^1)| \ll Z^{-3}. \tag{26}$$

Since Q satisfies (18), we have

$$Q(\mathbf{z}^1) = B(\mathbf{z}^1, \mathbf{z}^1) > 0. \tag{27}$$

Having chosen $\mathbf{z}^1, \ldots, \mathbf{z}^{j-1}$, we choose $\mathbf{z}^j$ by applying Lemma 2 with $m=j$, $L_i(\mathbf{x}) = B(\mathbf{z}^i, \mathbf{x})$ for $i=1, \ldots, j-1$ and $L_j(\mathbf{x}) = L(\mathbf{x})$. We obtain a non-zero integer vector $\mathbf{z}^j$ such that

$$|\mathbf{z}^j| \leq P^j \quad \text{for} \quad 2 \leq j \leq 5, \tag{28}$$

$$|B(\mathbf{z}^1, \mathbf{z}^j)| \ll Z P^{j-n} \quad \text{for} \quad 2 \leq j \leq 5, \tag{29}$$

$$|B(\mathbf{z}^i, \mathbf{z}^j)| \ll P^{i+j-n} \quad \text{for} \quad 2 \leq i, j \leq 5, i \neq j, \tag{30}$$

and

$$|L(\mathbf{z}^j)| \ll P^{j-n} \quad \text{for} \quad 2 \leq j \leq 5. \tag{31}$$

Since the exponents of P are negative, the effect of the transformation (22) is to take Q into a polynomial that is almost a diagonal form.

4. Proof of theorem. Under the linear transformation (22),

$$Q(\mathbf{x}) = \Phi(u_1, \ldots, u_5) = \sum_{i=1}^{5} \sum_{j=1}^{5} \beta_{ij} u_i u_j, \tag{32}$$

where $\beta_{ij} = B(\mathbf{z}^i, \mathbf{z}^j)$, and

$$L(\mathbf{x}) = \Lambda(u_1, \ldots, u_5) = \sum_{j=1}^{5} \mu_j u_j, \tag{33}$$

where $\mu_j = L(\mathbf{z}^j)$. Thus

$$F(\mathbf{x}) = \Phi_0(\mathbf{u}) + \Phi_1(\mathbf{u}) + \Lambda(\mathbf{u}) = \Psi(\mathbf{u}), \tag{34}$$

say, where

$$\Phi_0(\mathbf{u}) = \beta_{11} u_1^2 + \ldots + \beta_{55} u_5^2 \tag{35}$$

and $\Phi_1(\mathbf{u}) = \Phi(\mathbf{u}) - \Phi_0(\mathbf{u})$.

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We consider the values taken by $\Psi(\mathbf{u})$, where

$$|u_1| \leq XZ^{-1}/5 \quad \text{and} \quad |u_i| \leq XP^{-i}/5 \quad \text{for} \quad 2 \leq i \leq 5 \tag{36}$$

so that $|\mathbf{x}| \leq X$. Now

$$|\beta_{11}| \ll Z^2 \quad \text{and} \quad |\beta_{ii}| \ll P^{2i} \quad \text{for} \quad 2 \leq i \leq 5 \tag{37}$$

so that

$$\Pi = |\beta_{11} \ldots \beta_{55}| \ll Z^2 P^{28}. \tag{38}$$

On taking

$$Y = X^{\frac{1}{2}-\varepsilon} Z^{-\frac{1}{2}} P^{-7} \tag{39}$$

for some fixed $\varepsilon > 0$ and choosing $\tau > 0$ sufficiently small, we have

$$(XZ^{-1})^{\frac{1}{2}} |\beta_{11} \Pi^{-1}|^{\frac{1}{4}} \gg Y(Y^5 \Pi)^{\tau} \tag{40}$$

and, for $i = 2, \ldots, 5$,

$$(XP^{-i})^{\frac{1}{2}} |\beta_{ii} \Pi^{-1}|^{\frac{1}{4}} \gg Y(Y^5 \Pi)^{\tau}. \tag{41}$$

Further, let n be large,

$$P = X^{7/3n} \quad \text{and} \quad Z = X^{\frac{1}{3}} \tag{42}$$

so that

$$X^2 P^{-n} = XZ^{-4} = X^{-\frac{1}{3}}. \tag{43}$$

If any $|\beta_{ii}| < Y^{-1}$ then, taking $\mathbf{x} = \mathbf{z}^i \neq 0$, we have

$$\begin{aligned} |F(\mathbf{x})| &\leq |\beta_{ii}| + |L(\mathbf{z}^i)| \\ &\ll Y^{-1} + Z^{-1} + P^{5-n} \ll Y^{-1}, \end{aligned} \tag{44}$$

from (26) and (31), so that $\mathbf{z}^i$ is a suitable solution of the inequality (13). Now we may suppose that each $|\beta_{ii}| \geq Y^{-1}$ and, from (29) and (30), we see that the off-diagonal coefficients of Φ are $o(Y^{-1})$, provided that n is large enough. Therefore Φ is nearly diagonal and is non-singular, so that if $\mathbf{u} \neq \mathbf{0}$ then $\mathbf{x} \neq \mathbf{0}$. Since Q represents Φ, Φ is of type $(r, 5-r)$ where $r \leq 4$.

Since $\beta_{11} > 0$, it now follows that $\beta_{11}, \ldots, \beta_{55}$ are not all of the same sign; so we may apply Lemma 1 to the diagonal form Φ_0. We obtain integers $u_1, \ldots, u_5$, not all zero, satisfying (36) and

$$|\beta_{11} u_1^2 + \ldots + \beta_{55} u_5^2| < Y^{-1}. \tag{45}$$

Now, from (29), (30) and (36),

$$\begin{aligned} |\Phi_1(\mathbf{u})| = \left| \sum_{i \neq j}\sum \beta_{ij} u_i u_j \right| &\ll \sum_j ZP^{j-n} XZ^{-1} XP^{-j} \\ &+ \sum_{i \neq j}\sum P^{i+j-n} XP^{-i} XP^{-j} \ll X^2 P^{-n} = X^{-\frac{1}{3}}. \end{aligned} \tag{46}$$

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From (26), (31) and (36), we have

$$\begin{aligned}|\Lambda(\mathbf{u})| &\le \sum_{j=1}^{5} |L(\mathbf{z}^j)|\,|u_j| \\ &\ll XZ^{-4} + XP^{-n} \\ &\ll X^{-\frac{1}{3}}.\end{aligned} \tag{47}$$

Thus there exist $u_1, \ldots, u_5$, not all zero, satisfying (36) and

$$\begin{aligned}|\Psi(\mathbf{u})| &\le |\Phi_0(\mathbf{u})| + |\Phi_1(\mathbf{u})| + |\Lambda(\mathbf{u})| \\ &\ll Y^{-1} + X^{-\frac{1}{3}} \\ &\ll X^{-\frac{1}{3}+(49/3n)+\varepsilon}\end{aligned}$$

and then $|\mathbf{x}| \le X$, $\mathbf{x} \ne \mathbf{0}$ since $\mathbf{u} \ne \mathbf{0}$, and $F(\mathbf{x}) = \Psi(\mathbf{u})$, which completes the proof.

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