Note on a formula connected with Fourier Series.

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An expression for the sum of a series of terms which takes account of the different rates of oscillation of the functions involved is of importance for series such as occur in the problems of diffraction, and a relation of this kind has been given by Poisson.* This relation can be written in the form

$$S_m = \tfrac{1}{2}(u_n - u_{n+m}) + \int_0^m u_{n+t}\,dt + 2\sum_{k=1}^{k=\infty}\int_0^m u_{n+t}\cos 2k\pi t\,dt,$$

where S_m denotes the sum of m terms of the series Σu_n beginning with the term u_n. This formula can be deduced from Dirichlet's integral, for

$$\int_a^b \frac{\sin(2k+1)\pi t}{\sin \pi t} f(t)\,dt \rightarrow \tfrac{1}{2}\left[f(n_1 -) + f(n_1 +) + f(n_1 + 1 -) + \ldots + f(n_2 +)\right],$$

when $k \rightarrow \infty$, where n_1, n_2 are the two integers which satisfy the relations

$$n_1 > a > n_1 - 1,\ n_2 + 1 > b > n_2,$$

and therefore

$$\tfrac{1}{2}\left[f(n_1 -) + f(n_1 +) + \ldots + f(n_2 +)\right] = \int_a^b f(t)\,dt + 2\int_a^b \sum_{k=1}^{k=\infty} f(t)\cos 2k\pi t\,dt,$$

where $f(t)$ is a function of t satisfying the Dirichlet conditions. It follows that

$$\tfrac{1}{2}\left[f(n_1 -) + f(n_1 +) + \ldots + f(n_2 +)\right] = \int_a^b f(t)\,dt + 2\sum_{k=1}^{k=\infty}\int^b f(t)\cos 2k\pi t\,dt \quad \ldots\ldots\ldots\ldots\ldots(1),$$

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* *Mem. de l'Acad. des Sciences*, t. VI, pp. 5–78, 5–91.

when the series on the right hand side converges, or

$$\sum_{n=n_1}^{n=n_2} u_n \int_a^b u_t\,dt + 2\sum_{k=1}^{k=\infty}\int_a^b u_t \cos 2k\pi t\,dt \ldots\ldots\ \ldots\ldots(2),$$

when a is an integer the first term on the left hand side of relation (1) has to be omitted, and when b is an integer the last term on the left hand side of relation (1) has to be omitted; therefore

$$\sum_{n=n_1}^{n=n_2} u_n = \tfrac{1}{2} u_{n_1 -} + \int_1^b u_t\,dt + 2\sum_{k=1}^{k=\infty}\int_{n_1}^b u_t \cos 2k\pi t\,dt \quad \ldots\ldots(3),$$

$$\sum_{n=n_1}^{n=n_2} u_n = \tfrac{1}{2} u_{n_2 +} + \int_a^{n_2} u_t\,dt + 2\sum_{k=1}^{k=\infty}\int_a^{n_2} u_t \cos 2k\pi t\,dt \ldots\ldots\ldots(4),$$

$$\sum_{n=n_1}^{n=n_2} u_n = \tfrac{1}{2}(u_{n_1 -} + u_{n_2 +})$$

$$+\int_{n_1}^{n_2} u_t\,dt + 2\sum_{k=1}^{k=\infty}\int_{n_1}^{n_2} u_t \cos 2k\pi t\,dt,$$

or

$$\sum_{n=n_1}^{n=n_2} u_n = \tfrac{1}{2}(u_{n_1 -} - u_{n_2+1 -})$$

$$+\int_{n_1}^{n_2+1} u_t\,dt + 2\sum_{k=1}^{k=\infty}\int_{n_1}^{n_2+1} u_t \cos 2k\pi t\,dt \ldots\ldots\ldots(5),$$

which is the same as the relation

$$S_m = \tfrac{1}{2}(u_n - u_{n+m}) + \int_0^m u_{n+t}\,dt$$

$$+ 2\sum_{k=1}^{k=\infty}\int_0^m u_{n+t} \cos 2k\pi t\,dt \ldots\ldots\ldots(5'),$$

when S_m is the sum of m terms of the series beginning with the term u_n.

When u_x oscillates slowly as x varies, the first integral on the right hand side is more important than any of the others; the formula obtained for the approximate value of the sum by neglecting all the subsequent integrals is effectively the same as that given by Maclaurin.*

If, however, the rate of oscillation of u_x is in the neighbourhood of the rate of oscillation of $\cos 2p\pi x$, where p is an integer, the

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* *Treatise on Fluxions*, Bk. II, Ch. IV.

most important term on the right hand side is the integral $\int_0^m u_{n+t} \cos 2p\pi t\, dt$, and, if the rate of oscillation of v_x is in the neighbourhood of the rate of oscillation of $\cos(2p+1)\pi x$, the consecutive integrals on the right hand side $\int_0^m u_{n+t} \cos 2p\pi t dt$ and $\int_0^m u_{n+t} \cos 2(p+1)\pi t\, dt$ are of equal importance and contribute the most important part of the sum. Again, it may happen that there is a group or groups of terms of a series whose sum constitutes the most important part of the sum of the series. In this case, if the terms of one of the groups lie between u_{n_1} and u_{n_2}, the sum of the terms of this group is by the preceding

$$\int_a^b u_t\, dt + 2\sum_{k=1}^{k=\infty} \int_a^b u_t \cos 2k\pi t\, dt;$$

and, in general, the origin of t being chosen conveniently, it will be possible to write from the range from a to b

$$u_t = v_t + w_t,$$

where

$$\int_{-\infty}^{\infty} v_t \cos 2k\pi t\, dt - \int_a^b v_t \cos 2k\pi t\, dt + \int_a^b w_t \cos 2k\pi t\, dt$$

can be neglected for all values of k, and the principal part of the sum due to this group is then

$$\int_{-\infty}^{\infty} v_t\, dt + 2\sum_{k=1}^{k=\infty} \int_{-\infty}^{\infty} v_t \cos 2k\pi t\, dt,$$

the most important term in this expression depending on the rate of oscillation of v_t. An example of this is afforded by the series that occur in connection with the effect of a spherical obstacle on a train of waves.

The formula can also be used to obtain certain analytical relations, and some examples of this are given.

(a) Writing in the relation (5′) $u_n = n^2$, the result is

$$\sum_{n=1}^{n=m} n^2 = \tfrac{1}{2}\{1-(m+1)^2\} + \int_0^m (t+1)^2\, dt + 2\sum_{k=1}^{k=\infty} \int_0^m (t+1)^2 \cos 2k\pi t\, dt,$$

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whence, integrating the terms on the right hand side by parts,

$$\tfrac{1}{6} m(m+1)(2m+1) + \tfrac{1}{2}\{(m+1)^2-1\} - \tfrac{1}{3}\{(m+1)^3-1\} = m \sum_{k=1}^{k=\infty} \frac{1}{k^2\pi^2},$$

that is $$\tfrac{1}{6} = \sum_{k=1}^{k=\infty} \frac{1}{k^2\pi^2} \text{ or } \sum_{k=1}^{k=\infty} \frac{1}{k^2} = \frac{\pi^2}{6}.$$

(*b*) Writing $u_n = \cos n z$, the result is

$$\sum_{r=0}^{r=m-1} \cos(n+r)z = \tfrac{1}{2}\{\cos n z - \cos(n+m)z\}$$
$$+ \int_0^m \cos(n+t)z\,dt + 2\sum_{k=1}^{k=\infty} \int_0^m \cos(n+t)z \cos 2k\pi t\,dt,$$

that is

$$\{\sin(n+m-\tfrac{1}{2})z - \sin(n-\tfrac{1}{2})z\}/2\sin\tfrac{1}{2}z = \tfrac{1}{2}\{\cos n z - \cos(n+m)z\}$$
$$+ \{\sin(n+m)z - \sin n z\}\left\{\frac{1}{z} + \sum_{k=1}^{k=\infty}\left(\frac{1}{z+2k\pi} + \frac{1}{z-2k\pi}\right)\right\},$$

whence

$$\tfrac{1}{2}\{\sin(n+m)z - \sin n z\}\cot\tfrac{1}{2}z$$
$$= \{\sin(n+m)z - \sin n z\}\left\{\frac{1}{z} + \sum_{k=1}^{k=\infty}\left(\frac{1}{z+2k\pi} + \frac{1}{z-2k\pi}\right)\right\},$$

that is

$$\tfrac{1}{2}\cot\tfrac{1}{2}z = \frac{1}{z} + \sum_{k=1}^{k=\infty}\left(\frac{1}{z+2k\pi} + \frac{1}{z-2k\pi}\right).$$

(*c*) If relation (5′) is applied to the terms on the right hand side of the identity

$$\sinh\eta / \cosh\eta - \cos\xi) = 1 + 2\sum_{n=1}^{n=\infty} e^{-n\eta}\cos n\xi,$$

the result is

$$\sinh\eta / (\cosh\eta - \cos\xi) = 1 - 1 + 2\int_0^\infty e^{-t\eta}\cos\xi t\,dt$$
$$+ 4\sum_{k=1}^{k=\infty}\int_0^\infty e^{-t\eta}\cos\xi t\cos 2k\pi t\,dt,$$

that is

$$\sinh\eta / (\cosh\eta - \cos\xi) = 2\eta / (\eta^2+\xi^2)$$
$$+ \sum_{k=1}^{k=\infty}\left[2\eta / \{\eta^2 + (\xi+2k\pi)^2\} + 2\eta / \{\eta^2 + (\xi-2k\pi)^2\}\right].$$

(*d*) The relation

$$S = 1/a + 2\sum_{m=1}^{m=\infty} a\cos nz/(a^2+n^2)$$

can be written

$$S = \sum_{n=-\infty}^{n=\infty} a\exp(inz)/(a^2+n^2),$$

and applying the formula this becomes

$$S = \int_{-\infty}^{\infty} a e^{izt}/(a^2+t^2)\,dt + 2\sum_{k=1}^{k=\infty}\int_{-\infty}^{\infty} e^{izt}\cos 2k\pi t/(a^2+t^2)\,dt,$$

that is, when $2\pi > z > 0$,

$$S = \pi e^{-az} + \sum_{k=1}^{k=\infty}\pi e^{-a(z+2k\pi)} + \sum_{k=1}^{k=\infty}\pi e^{-a(2k\pi-z)},$$

or $$S = \pi e^{-az}/(1-e^{-2a\pi}) + \pi e^{-a(2\pi-z)}/(1-e^{-2a\pi})$$
$$= \pi\cosh a(\pi-z)/\sinh a\pi\,;$$

when $0 > z > -2\pi$,

$$S = \pi e^{az} + \sum_{k=1}^{k=\infty}\pi e^{-a(z+2k\pi)} + \sum_{k=1}^{k=\infty} e^{-a(2k\pi-z)},$$

or $$S = \pi\cosh a(\pi+z)/\sinh a\pi.$$

(*e*) Applying the formula to the expression

$$f(a, b) = \sum_{n=-\infty}^{n=\infty} e^{-an^2-2bn},$$

where the real part of a is positive,

$$f(a, b) = \int_{-\infty}^{\infty} e^{-at^2-2bt}\,dt + 2\sum_{k=1}^{k=\infty}\int_{-\infty}^{\infty} e^{-at^2-2bt}\cos 2k\pi t\,dt,$$

whence

$$f(a, b) = (\pi/a)^{\frac{1}{2}}\left[\exp(b^2/a) + \sum_{k=1}^{k=\infty}\exp\{(b^2-k^2\pi^2)/a - 2kb\pi i/a\}\right.$$
$$\left. + \sum_{k=1}^{k=\infty}\exp\{(b^2-k^2\pi^2)/a + 2kb\pi i/a\}\right]$$

or

$$f(a, b) = (\pi/a)^{\frac{1}{2}} e^{b^2/a}\sum_{k=-\infty}^{k=\infty}\exp(-k^2\pi^2/a - 2kb\pi i/a)\,;$$

that is, writing

$$a_1 = \pi^2/a,\ \ b_1 = \pi b i/a,$$

the relation is

$$a^{\frac{1}{2}}\, e^{-b^2/2a} f(a, b) = a_1^{\frac{1}{2}}\, e^{-b_1^2/2a_1} f(a_1, b_1).$$

The formula can also be applied to a single term, for example.

$$\cos mz = \int_{m-t}^{m+t'} \cos zt\,dt + 2\sum_{k=1}^{k=\infty}\int_{m-t}^{m+t'} \cos zt \cos 2k\pi t\,dt,$$

where $1 > t > 0$, $1 > t' > 0$, and m is a positive integer, whence

$$\cos mz = \cos mz\,\Big[(\sin t'z + \sin tz)/z$$
$$+ \sum_{k=1}^{k=\infty} \{\sin t'(z+2k\pi) + \sin t(z+2k\pi)\}/(2k\pi+2)$$
$$+ \sum_{k=l}^{k=\infty} \{\sin t'(z-2k\pi) + \sin t(z-2k\pi)\}/(z-2k\pi)\Big]$$
$$+ \sin mz\,\Big[(\cos t'z - \cos tz)/z$$
$$+ \sum_{k=1}^{k=\infty} \{\cos t'(z+2k\pi) - \cos t(z+2k\pi)\}/(z+2k\pi)$$
$$+ \sum_{k=1}^{k=\infty} \{\cos t'(z-2k\pi) - \cos t(z-2k\pi)\}/(z-2k\pi)\Big],$$

writing $t' = t$, this relation becomes

$$1 = 2\sin tz/z + 2\sum_{k=1}^{k=\infty} \sin t(z+2k\pi)/(z+2k\pi)$$
$$+ 2\sum_{k=1}^{k=\infty} \sin t(z-2k\pi)/(z-2k\pi),$$

or $$\sum_{k=-\infty}^{k=\infty} \sin t(z+2k\pi)/(z+2k\pi) = \tfrac{1}{2}, \text{ when } 1 > t > 0.$$

Substituting this result above, it follows that

$$\sum_{k=-\infty}^{k=\infty} \cos t(z+2k\pi)/(z+2k\pi) = \sum_{k=-\infty}^{k=\infty} \cos t'(z+2k\pi)/(z+2k\pi),$$

whence, writing $t' = \frac{1}{2}$,

$$\sum_{k=-\infty}^{k=\infty} \cos t(z+2k\pi)/(z+2k\pi)$$
$$= \cos\tfrac{1}{2}z \sum_{k=-\infty}^{k=\infty} \cos k\pi/(z+2k\pi) = \tfrac{1}{2}\cot\tfrac{1}{2}z.$$

Therefore combining these results,

$$\sum_{k=-\infty}^{k=\infty} \cos 2k\pi t/(z+2k\pi) = \tfrac{1}{2}\cos(\tfrac{1}{2}-t)z/\sin\tfrac{1}{2}z,$$
$$\sum_{k=-\infty}^{k=\infty} \sin 2k\pi t/(z+2k\pi) = \tfrac{1}{2}\sin(\tfrac{1}{2}-t)z/\sin\tfrac{1}{2}z.$$