On Integral Relations connected with the Confluent Hypergeometric Function.

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§ 1. *Introductory.*

In the *Bulletin of the American Mathematical Society** Whittaker defines the confluent hypergeometric function $W_{k,m}(x)$ by the equation

$$W_{k,m}(x)=\frac{\Gamma(k+\frac{1}{2}-m)}{2\pi}e^{-\frac{1}{2}x+\frac{1}{2}i\pi}x^k\int(-t)^{-k-\frac{1}{2}+m}\left(1+\frac{t}{x}\right)^{k-\frac{1}{2}+m}e^{-t}\,dt \tag{1}$$

where the path of integration begins at $t=+\infty$, and after encircling the point $t=0$ in the counter-clockwise direction, returns to $t=+\infty$ again.

In the same memoir it is shown that this function satisfies the differential equation

$$\frac{d^2W}{dx^2}+\left\{-\frac{1}{4}+\frac{k}{x}+\frac{\frac{1}{4}-m^2}{x^2}\right\}W=0, \tag{2}$$

and that its asymptotic expansion is

$$e^{-\frac{1}{2}x}x^kI(x)$$

where

$$I(x)=1+\frac{m^2-(k-\frac{1}{2})^2}{1\,!\,x}+\frac{\{m^2-(k-\frac{1}{2})^2\}\{m^2-(k-\frac{3}{2})^2\}}{2\,!\,x^2}+\ldots, \tag{3}$$

a series which cannot terminate unless $k-\frac{1}{2}\pm m$ is a positive integer.

It is evident that (2) is unaltered if m is changed into $-m$, or if k and x are replaced by $-k$ and $-x$ simultaneously. The four functions

$$W_{k,\pm m}(x),\quad W_{-k,\pm m}(-x)$$

are therefore solutions of the differential equation (2).

* 2nd Series, Vol. X., p. 125.

The object of the present paper is to show that any W-function for which (3) is a non-terminating series can be expressed in terms of any other W-function for which (3) is a terminating series; in §4 we shall see that the relation (1) is a particular case of this result. The last section is devoted to a discussion of certain special cases.

§2. *Solution of Equation (2) as a Definite Integral.*

Transforming (2) by the substitution

$$y = e^{-px} x^{-r} W_{-k,m}(-x),$$

we obtain the differential equation

$$x^2 y'' + 2x(px+r)y' + \{(p^2 - \tfrac{1}{4})x^2 + (2pr+k)x + (r^2 - r + \tfrac{1}{4} - m^2)\}y = 0 \tag{4}$$

Now assume that (4) can be satisfied by the definite integral

$$y = \int_c v(s)(x-s)^q \, ds \tag{5}$$

where c is some contour to be afterwards determined. Substituting from (5) in (4) we have

$$\begin{aligned} 0 &= \int_c v(s)\big[q(q-1)(x-s)^{q-2}x^2 + 2q(x-s)^{q-1}(px+r)x + (x-s)^q \\ &\qquad \{(p^2-\tfrac{1}{4})x^2 + (2pr+k)x + (r^2 - r + \tfrac{1}{4} - m^2)\}\big]\,ds \\ &= \int_c v(s)(x-s)^{q-2}\big[(x-s)^4(p^2-\tfrac{1}{4}) + (x-s)^3 \\ &\qquad \{2pq + 2(p^2-\tfrac{1}{4})s + 2pr + k\} \\ &\qquad + (x-s)^2\{q(q-1) + 2q(2ps+r) + (p^2-\tfrac{1}{4})s \\ &\qquad + (2pr+k)s + (r^2 - r + \tfrac{1}{4} - m^2)\} \\ &\qquad + (x-s)\{2q(q-1)s + 2qs(ps+r)\} + q(q-1)s^2\big]\,ds. \end{aligned} \tag{6}$$

Now let

$$p^2 - \tfrac{1}{4} = 0, \quad 2pq + 2(p^2-\tfrac{1}{4})s + 2pr + k = 0,$$

i.e.

$$\left.\begin{array}{r} p = \tfrac{1}{2} \\ q + r + k = 0 \end{array}\right\} \qquad \left.\begin{array}{r} p = -\tfrac{1}{2} \\ -q - r + k = 0 \end{array}\right\}. \tag{7}$$

Equation (6) then becomes

$$\begin{aligned} 0 = \int_c v(s)(x-s)^{q-2}\big[(x-s)^2\{q(q-1) + 2q(2ps+r) \\ + (2pr+k)s + (r^2 - r + \tfrac{1}{4} - m^2)\} \\ + (x-s)\{2q(q-1)s + 2qs(ps+r)\} + q(q-1)s^2\big]\,ds \end{aligned}$$

where p, q, r, k are subject to conditions (7).

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Raising each term to the q^{th} degree in $(x-s)$, we have

$$0=\int_c (x-s)^q \left[\{q(q-1)+2q(2ps+r)+(2pr+k)s + (r^2-r+\tfrac{1}{4}-m^2)\}v(s) + \frac{1}{q}\frac{d}{ds}\{v(s)(2q\overline{q-1}\,s+2q\overline{ps^2+rs})\} + \frac{1}{q(q-1)}\frac{d^2}{ds^2}\{v(s)\,q(q-1)s^2\}\right]ds$$
$$-\left[2s(x-s)^q(q-1+ps+r)v(s)+qs^2(x-s)^{q-1}v(s) + s\{2v(s)+sv'(s)\}(x-s)^q\right]_c$$
$$=\int_c (x-s)^q\left[s^2v''+\{2(q+r+1)s+2ps^2\}v'+\{(4pq+2pr+4p+k)s + (q^2+2qr+r^2+q+r+\tfrac{1}{4}-m^2)\}v\right]ds$$
$$-\left[s(x-s)^{q-1}\{2(x-s)(q+ps+r)+qs\}v+s^2(x-s)^q v'\right]_c$$

where $v(s)$ have been replaced by v.

The contour c will be so chosen that the function

$$s(x-s)^{q-1}\{2(x-s)(q+ps+r)+qs\}v+s^2(x-s)^q v' \qquad (8)$$

will vanish when taken round the contour. The function $v(s)$ will then have to satisfy the differential equation

$$s^2v''+\{2(q+r+1)s+2ps^2\}v'+\{(4pq+2pr+4p+k)s + (q^2+2qr+r^2+q+r+\tfrac{1}{4}-m^2)\}v=0 \qquad (9)$$

where p, q, r, k satisfy conditions (7).

Case 1.

Let $\qquad p=\tfrac{1}{2} \qquad q+r+k=0.$

Equation (9) becomes

$$s^2v''+\{2(1-k)s+s^2\}v'+\{(q+2)s+(k^2-k+\tfrac{1}{4}-m^2)\}v=0. \qquad (10)$$

If this has a solution of the form

$$v=e^{-\alpha s}s^{-\beta}W_{\kappa,\mu}(s),$$

then it must be the same as

$$s^2v''+2s(\alpha s+\beta)v'+\{(\alpha^2-\tfrac{1}{4})s^2+(2\alpha\beta+\kappa)s+(\beta^2-\beta+\tfrac{1}{4}-\mu^2)\}v=0,$$

and hence we must have

$$(\alpha^2-\tfrac{1}{4})s^2+(2\alpha\beta+\kappa)s+(\beta^2-\beta+\tfrac{1}{4}-\mu^2)\equiv(q+2)s+(k^2-k+\tfrac{1}{4}-m^2)$$

$$2\alpha s^2+2\beta s\equiv s^2+2(1-k)s$$

i.e. $\alpha=\tfrac{1}{2},\quad \beta=1-k,\quad \mu=\pm m,\quad \kappa=1-r.$

The solution of (10) is thus

$$v(s)=e^{-\frac{1}{2}s}s^{k-1}W_{1-r,\pm m}(s).$$

Case 2.

Let $\qquad p=-\tfrac{1}{2}\qquad -q-r+k=0.$
Then by a similar method to the above it may be shown that the corresponding solution is

$$v(s)=e^{\frac{1}{2}s}\ s^{-k-1}W_{r-1,\pm m}(s).$$

Hence in case (i) we have

$$e^{-\frac{1}{2}x}x^{-r}W_{-k,m}(-x)=A\int_c e^{-\frac{1}{2}s}s^{k-1}W_{1-r,\pm m}(s)(x-s)^q\,ds$$

and in case (ii)

$$e^{\frac{1}{2}x}x^{-r}W_{k,m}(x)=B\int_c e^{\frac{1}{2}s}s^{-k-1}W_{r-1,\pm m}(s)(x-s)^q\,ds$$

where q, r, k are subject to conditions (7), and A, B are constants.

Changing the signs of x and k and writing $r=1-k'$ in the former and $r=1+k'$ in the latter, these became

$$e^{\frac{1}{2}x}x^{k'-1}W_{k,m}(x)=A\int_c e^{-\frac{1}{2}s}s^{-k-1}W_{k',\pm m}(s)(x+s)^{k+k'-1}\,ds \qquad (11)$$

$$e^{\frac{1}{2}x}x^{-k'-1}W_{k,m}(x)=B\int_c e^{\frac{1}{2}s}s^{-k-1}W_{k',\pm m}(s)(x-s)^{k-k'-1}\,ds \qquad (12)$$

§ 3. *Determination of the Constant A.*

Taking the positive value of m in (11) we have

$$e^{\frac{1}{2}x}x^{k'-1}W_{k,m}(x)=A\int_c e^{-\frac{1}{2}s}s^{-k-1}W_{k',m}(s)(x+s)^{k+k'-1}\,ds$$

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whence

$$I(x)=A\int_c e^{-s}s^{k'-k-1}$$

$$\left\{1+\frac{m^2-(k'-\frac{1}{2})^2}{1\,!\,s}+\frac{\{m^2-(k'-\frac{1}{2})^2\}\{m^2-(k'-\frac{3}{2})^2\}}{2\,!\,s^2}+\,..\right\}$$

$$\times\left\{1+\frac{(k+k'-1)}{1\,!}\,\frac{s}{x}+\frac{(k+k'-1)(k+k'-2)}{2\,!}\,\frac{s^2}{x^2}+\ldots\right\}ds$$

$$=A\int_c e^{-s}s^{\lambda}\left\{1+\frac{\alpha\beta}{1\,!\,s}+\frac{\alpha(\alpha+1)\beta(\beta-1)}{2\,!\,s^2}+\ldots\right\}$$

$$\left\{1+\frac{\gamma}{1\,!}\,\frac{s}{x}+\frac{\gamma(\gamma-1)}{2\,!}\,\frac{s^2}{x^2}+\ldots\right\}ds$$

where

$$\lambda=k'-k-1,\quad \alpha=m-k'+\tfrac{1}{2},\quad \beta=m+k'-\tfrac{1}{2},\quad \gamma=k+k'-1$$

$$=\sum_{r=0}^{\infty}A\,\frac{\gamma(\gamma-1)\ldots(\gamma-r+1)}{r\,!\,x^r}(-1)^{\lambda+r}\int_c e^{-s}(-s)^{\lambda+r}$$

$$\left\{1+\frac{\alpha\beta}{1\,!\,s}+\frac{\alpha(\alpha+1)\beta(\beta-1)}{2\,!\,s^2}+\ldots\right\}ds$$

Now choose for c a contour beginning at $s=+\infty$, and, after encircling the origin $s=0$ in the counter-clockwise direction, returning to $s=+\infty$. The expression (8) will vanish when taken round this contour, and hence the path of integration is a valid one. We then have

$$I(x)=\sum_{r=0}^{\infty}A\,\frac{\gamma(\gamma-1)\ldots(\gamma-r+1)}{r\,!\,x^r}(-1)^{\lambda+r}\,\frac{2\pi}{i}$$

$$\left[\frac{1}{\Gamma(-\lambda-r)}-\frac{\alpha\beta}{1\,!\,\Gamma(-\lambda-r+1)}+\frac{\alpha(\alpha+1)\beta(\beta-1)}{2\,!\,\Gamma(-\lambda-r+2)}-\ldots\right]$$

provided λ is not zero or an integer

$$=\sum_{r=0}^{\infty}A\,\frac{\gamma(\gamma-1)\ldots(\gamma-r+1)}{r\,!\,x^r}(-1)^{\lambda+r}\,\frac{2\pi}{i\,\Gamma(-\lambda-r)}$$

$$\left[1-\frac{\alpha\beta}{1\,!\,(-\lambda-r)}+\frac{\alpha(\alpha+1)\beta(\beta-1)}{2\,!\,(-\lambda-r)(-\lambda-r+1)}-\ldots\right]$$

$$=\sum_{r=0}^{\infty}A\,\frac{\gamma(\gamma-1)\ldots(\gamma-r+1)}{r\,!\,x^r}(-1)^{\lambda+r}\,\frac{2\pi}{i\,\Gamma(-\lambda-r)}$$

$$F(\alpha,\ -\beta,\ -\lambda-r,\ 1). \qquad (13)$$

The hypergeometric series for $F(\alpha, -\beta, -\lambda-r, 1)$ is not in general convergent, since $k+k'-r$ is not always positive. Let us, therefore, choose k' so that the expression $I(s)$ in the asymptotic expansion of $W_{k', m}(s)$ shall terminate. We can then express $F(\alpha, -\beta, -\lambda-r, 1)$ in terms of Γ-functions. This will be the case when $k'=n-\frac{1}{2}\pm m$. Choose the positive value of m. Then

$$I(x)=\sum_{r=0}^{\infty} A \frac{\gamma(\gamma-1)\ldots(\gamma-r+1)}{r!\ x^r}(-1)^{\lambda+r}\frac{2\pi}{i} \frac{\Gamma(-\lambda-r-\alpha+\beta)}{\Gamma(-\lambda-r-\alpha)\,\Gamma(-\lambda-r+\beta)}$$

$$=(-1)^{\lambda}\frac{2\pi A}{i}\frac{\Gamma(-\lambda-\alpha+\beta)}{\Gamma(-\lambda-\alpha)\,\Gamma(-\lambda+\beta)}$$

$$\left\{1+\sum_{r=1}^{\infty}(-1)^r\frac{\gamma(\gamma-1)\ldots(\gamma-r+1)}{r!\ x^r}\right.$$

$$\left.\times\frac{(-\lambda-1-\alpha)\ldots(-\lambda-r-\alpha)(-\lambda-1+\beta)\ldots(-\lambda-r+\beta)}{(-\lambda-1-\alpha+\beta)\ldots(-\lambda-r-\alpha+\beta)}\right\}$$

$$=(-1)^{n-\frac{3}{2}+m-k}\frac{2\pi A}{i}\frac{\Gamma(k+n-\frac{1}{2}+m)}{\Gamma(k-m+\frac{1}{2})\,\Gamma(k+m+\frac{1}{2})}I(x).$$

Hence
$$A=\frac{(-1)^{k-n+\frac{1}{2}-m}\,\Gamma(k+m+\frac{1}{2})\,\Gamma(k-m+\frac{1}{2})}{2\pi i\,\Gamma(k+n+m-\frac{1}{2})}.$$

The corresponding result when m is negative can be immediately obtained. Again, if we replace $W_{k', \pm m}(s)$ in (12) by the other pair of solutions $W_{-k', \pm m}(-s)$, and then change the signs k' and s simultaneously, equation (12) becomes equation (11). It therefore follows that $A=B$.

Hence, finally,

$$e^{\frac{1}{2}x}x^{k'-1}W_{k, m}(x)=\frac{(-1)^{k-k'}\Gamma(k+m+\frac{1}{2})\,\Gamma(k-m+\frac{1}{2})}{2\pi i\,\Gamma(k+k')}$$

$$\int_c e^{-\frac{1}{2}s}s^{-k-1}W_{k', \pm m}(s)(x+s)^{k+k'-1}ds \tag{14}$$

provided (1) $k'=n-\frac{1}{2}\pm m$, where n is a positive integer or zero, (2) $\pm m-k-\frac{1}{2}$ is not an integer or zero, and (3) c is the contour already specified.

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§ 4. *Deduction of Whittaker's definition.*

Put $k' = m + \frac{1}{2}$ in (14) and we obtain the relation

$$e^{\frac{1}{2}x} x^{m-\frac{1}{2}} W_{k,m}(x) = \frac{(-1)^{k-m-\frac{1}{2}} \Gamma(k-m+\frac{1}{2})}{2\pi i}$$

$$\int_c e^{-\frac{1}{2}s} s^{-k-1} W_{m+\frac{1}{2}, \pm m}(s)(x+s)^{k+m-\frac{1}{2}} ds$$

$$= \frac{(-1)^{k-m-\frac{1}{2}} \Gamma(k-m+\frac{1}{2})}{2\pi i}$$

$$\int_c e^{-s} s^{-k+m-\frac{1}{2}} (x+s)^{k+m-\frac{1}{2}} ds$$

whence

$$W_{k,m}(x) = -\frac{\Gamma(k-m+\frac{1}{2})}{2\pi i} e^{-\frac{1}{2}x} x^k$$

$$\int_c e^{-s}(-s)^{m-k-\frac{1}{2}} \left(1+\frac{s}{x}\right)^{k+m-\frac{1}{2}} ds$$

which is equivalent to (1).

§ 5. *Special Cases.*

(1) *Relation connecting the Parabolic Cylinder Function and the Error Function.*

These functions are connected with the W-function by the relations

$$D_{2k-\frac{1}{2}}(\sqrt{2s}) = 2^{k-\frac{1}{4}} s^{-\frac{1}{4}} W_{k,\frac{1}{4}}(s)$$

$$\text{Erfc.}\,(s) = \tfrac{1}{2} s^{-\frac{1}{2}} e^{-\frac{1}{2}s^2} W_{-\frac{1}{4},\frac{1}{4}}(s^2).$$

Putting $k' = -\frac{1}{4}$, $m = \frac{1}{4}$ in (14) and assuming $k \pm \frac{1}{4}$ is not an integer or zero, we have

$$e^{\frac{1}{2}x} x^{-\frac{3}{4}} W_{k,\frac{1}{4}}(x) = (-1)^{k+\frac{3}{4}} \frac{\Gamma(k+\frac{1}{4})\,\Gamma(k+\frac{3}{4})}{2\pi i\,\Gamma(k-\frac{1}{4})}$$

$$\int_c e^{-\frac{1}{2}s} s^{-k-1} W_{-\frac{1}{4},\frac{1}{4}}(s)(x+s)^{k-\frac{5}{4}} ds$$

whence

$$e^{\frac{1}{2}x} x^{-1} D_{2k-\frac{1}{2}}(\sqrt{2x}) = \frac{(-2)^{k-\frac{1}{4}}}{\pi} (k-\tfrac{1}{4})\,\Gamma(k+\tfrac{1}{4})$$

$$\int_c s^{-k-\frac{3}{4}} (x+s)^{k-\frac{5}{4}} \,\text{Erfc.}\,(\sqrt{s})\, ds.$$

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Corollary.

$$e^{\frac{1}{2}x} x^{-1} D_{2k-\frac{1}{2}}(\sqrt{2x}) = -\frac{(-2)^{k-\frac{5}{4}}}{\pi}(k-\tfrac{1}{4})\,\Gamma(k+\tfrac{1}{4})$$
$$\int_c s^{-k-\frac{3}{4}}(x+s)^{k-\frac{5}{4}}\{\pi^{\frac{1}{2}}-\gamma(\tfrac{1}{2},s)\}\,ds$$

where $\gamma(n, x)$ is the incomplete gamma function.

(2) *Relation connecting the Pearson-Cunningham Function and Sonine's Polynomial.*

These functions are expressed in terms of the W-function by the relations*

$$\omega_{\alpha,\beta}(x) = \frac{(-1)^{\alpha-\frac{1}{2}\beta}}{\Gamma(\alpha-\frac{1}{2}\beta+1)}\, x^{-\frac{1}{2}(\beta+1)}\, e^{-\frac{1}{2}x}\, W_{\alpha+\frac{1}{2},\,\frac{1}{2}\beta}(x)$$

$$T^{\gamma}_{\beta}(x) = \frac{1}{\gamma\,!\,(\beta+\gamma)\,!}\, x^{-\frac{1}{2}(\beta+1)}\, e^{\frac{1}{2}x}\, W_{\gamma+\frac{1}{2}\beta+\frac{1}{2},\,\frac{1}{2}\beta}(x)$$

where γ is an integer.

Put $k=\alpha+\frac{1}{2}$, $k'=\gamma+\frac{1}{2}\beta+\frac{1}{2}$, $m=\frac{1}{2}\beta$ in (14). Then if $\pm\frac{1}{2}\beta-\alpha$ is not an integer or zero,

$$e^{\frac{1}{2}x} x^{\gamma+\frac{1}{2}\beta-\frac{1}{2}}\, W_{\alpha+\frac{1}{2},\,\frac{1}{2}\beta}(x) = \frac{(-1)^{\alpha-\gamma-\frac{1}{2}\beta}\,\Gamma(\alpha+\frac{1}{2}\beta+1)\,\Gamma(\alpha-\frac{1}{2}\beta+1)}{2\pi i\,\Gamma(\alpha+\frac{1}{2}\beta+\gamma+1)}$$
$$\int_c e^{-\frac{1}{2}s} s^{-\alpha-\frac{3}{2}}\, W_{\gamma+\frac{1}{2}\beta+\frac{1}{2},\,\frac{1}{2}\beta}(s)\,(x+s)^{\alpha+\frac{1}{2}\beta+\gamma}\,ds,$$

whence

$$e^{x} x^{\beta+\gamma}\,\omega_{\alpha,\beta}(x) = \frac{(-1)^{-\beta-\gamma}\,\gamma\,!\,(\beta+\gamma)\,!\,\Gamma(\alpha+\frac{1}{2}\beta+1)}{2\pi i\,\Gamma(\alpha+\frac{1}{2}\beta+\gamma+1)}$$
$$\int_c e^{-s} s^{-\alpha+\frac{1}{2}\beta-1}(x+s)^{\alpha+\frac{1}{2}\beta+\gamma}\, T^{\gamma}_{\beta}(s)\,ds.$$

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* Whittaker & Watson's *Modern Analysis*, Chap. XVI., Misc. Exs. 8 & 10.

The following relations may be similarly obtained :—

$$(3) \quad D_{2k-\frac{1}{2}}(\sqrt{2x}) = \frac{(-1)^{k-r-\frac{5}{4}}\, 2^{k-\frac{5}{4}}\, r\,!\,(r+\frac{1}{2})\,!\,\Gamma(k+\frac{1}{4})\,\Gamma(k+\frac{3}{4})}{\pi\,\Gamma(k+r+\frac{3}{4})}$$

$$\int_c e^{-s}\, s^{-k-\frac{1}{4}}\,(x+s)^{k+r-\frac{1}{4}}\; T^r_{\frac{1}{2}}(s)\, ds.$$

$$(4) \quad \omega_{n,m}(x) = (-1)^{1-m}\,\frac{n+\frac{1}{2}m}{2\pi i}\; e^{-x}\, x^{1-m}$$

$$\int_c s^{-n-\frac{1}{2}m-1}\,(x+s)^{n+\frac{m}{2}-1}\,[\Gamma(m)-\gamma(m\,.\,s)]\, ds.$$

$$(5) \quad \omega_{r,\frac{1}{2}}(x) = -\frac{(-2)^{-k-\frac{3}{4}}\,\Gamma(r+\frac{5}{4})}{\pi\, i\,\Gamma(k+r+\frac{1}{2})}\, e^{-x}\, x^{-k+\frac{1}{4}}$$

$$\int_c e^{-\frac{1}{2}s}\, s^{-r-\frac{3}{4}}\,(x+s)^{k+r-\frac{1}{2}}\, D_{2k-\frac{1}{2}}(\sqrt{2s})\, ds.$$

$$(6) \quad \omega_{r,\frac{1}{2}}(x) = \frac{1}{\pi}(r+\tfrac{1}{4})\, e^{-x}\, x^{\frac{1}{2}} \int_c s^{-r-\frac{5}{4}}\,(x+s)^{r-\frac{3}{4}}\ \text{Erfc.}\,(\sqrt{s})\, ds.$$

$$(7) \quad T^p_r\,(x) = \frac{(-1)^{p+r}\,\Gamma(q-\frac{1}{2}r+1)}{2\pi i\,\Gamma(p+q+\frac{1}{2}r+1)}\, x^{-q-\frac{1}{2}r}$$

$$\int_c e^{-\frac{1}{2}s}\, s^{-p-1}\,(x+s)^{p+q+\frac{1}{2}r}\,\omega_{q,r}(s)\, ds.$$

$$(8) \quad T^r_{\frac{1}{2}}\,(x) = \frac{(-1)^{r-k+\frac{1}{4}}\, 2^{-k-\frac{3}{4}}}{\pi\,\Gamma(k+r+\frac{3}{4})}\, x^{-k+\frac{1}{4}}$$

$$\int_c e^{-\frac{1}{2}s}\, s^{-r-\frac{1}{2}}\,(x+s)^{k+r-\frac{1}{4}}\, D_{2k-\frac{1}{2}}(\sqrt{2s})\, ds.$$

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