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# THE CONE = HYPERSPACE PROPERTY

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The author has recently shown [**11**] that the hyperspace of subcontinua of a solenoid is homeomorphic to the cone over that solenoid. This is an interesting result, for it is the first time that the hyperspace of subcontinua of a complicated space has been recognized. This homeomorphism, moreover, is the expected map; it maps the singletons onto the base of the cone and the point corresponding to the whole space onto the vertex of the cone. We say that spaces for which such natural homeomorphisms exist have the cone = hyperspace property. In the first section we prove the following theorem.

THEOREM. *If a finite-dimensional continuum $X$ has the cone = hyperspace property, then $X$ is an arc, a circle, or an indecomposable continuum each proper subcontinuum of which is an arc.*

In the second section we investigate some spaces for which a homeomorphism exists between the cone and the hyperspace, even though the space does not have the cone = hyperspace property. One consequence of this is a proof that the hyperspace of a circle with a spiral does not have the fixed point property.

In the third section, we consider a hereditarily indecomposable continuum $X$ and show that there exists a monotone, open map of the cone over $X$ onto the hyperspace of $X$.

A continuum is a compact, connected, nonvoid, metric space. $C(X)$, the hyperspace of subcontinua of a continuum $X$, is the space of all subcontinua of $X$ metricized by the Hausdorff metric $\rho$ (that is, $\rho(A, B) = \text{glb}\,\{\epsilon\}$ for all $\epsilon$ such that $A \subset V_\epsilon(B)$ and $B \subset V_\epsilon(A)$, where $V_\epsilon(A)$ is the $\epsilon$-neighborhood of $A$). We write $X'$ for the point in $C(X)$ which corresponds to the continuum $X$. The subspace of $C(X)$ consisting of the degenerate subcontinua of $X$ is isometric to $X$, so we identify $X$ with this subspace. For more information about $C(X)$, see [**5**].

The cone over a continuum $X$ is obtained from the space $X \times [0, 1]$ by identifying $X \times \{1\}$ to a point $v$, called the vertex of the cone. Let $h$ denote this identification map, and let $\pi$ be the projection of $X \times [0, 1]$ onto $X$. The base of the cone, $B(X)$, is the space $X \times \{0\}$. When no confusion will arise, we will write $X$ for $B(X)$, since $B(X)$ is naturally isometric to $X$. We denote the cone over $X$ by $K(X)$.

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**1. Continua with the cone = hyperspace property.** A continuum is said to have the cone = hyperspace property if the triples $(K(X), X, v)$ and $(C(X), X, X')$ are homeomorphic. Although it was not explicitly stated in [**11**], the homeomorphism constructed there implies that the solenoids have the cone = hyperspace property. Furthermore, we know that the simple closed curve and the arc have this property; moreover, these are the only finite-dimensional, locally connected continua which have the cone = hyperspace property, for such a continuum must be an atriodic linear graph, by [**5**, 5.4 and 5.5]. We can show by the methods of [**11**] that the familiar "buckethandle" continuum constructed with semi-circles from the Cantor ternary set and described by Kuratowski [**7**, p. 205] has the cone = hyperspace property.

We prove in this section that each subcontinuum of a finite-dimensional continuum with the cone = hyperspace property is an arc. We restrict ourselves to finite dimensions because if $H$ is the Hilbert cube, then $K(H)$ is homeomorphic to $H$, so the existence of a homeomorphism between $K(H)$ and $C(H)$ is equivalent to the topological equivalence of $H$ and $C(H)$.

THEOREM 1. *If the continuum $X$ has the cone = hyperspace property, then each composant of $X$ is arcwise connected.*

*Proof*. Let $p$ and $q$ be points of $X$ which belong to the same composant, and let $Y$ be a proper subcontinuum of $X$ which contains both $p$ and $q$. Let $g: (K(X), X, v) \to (C(X), X, X')$ be a homeomorphism of triples. If $Y$ is an arc, then we are done. Suppose $Y$ is not an arc. $C(g(Y))$ is arcwise-connected, by [**5**, Theorem 2.7]. Furthermore, $C(g(Y))$ is a subcontinuum of $C(X)$ which does not contain $X'$. Since $X$ has the cone = hyperspace property, there exists an arc $A$ in $K(X) - v$ which contains both $p$ and $q$. The continuum $\pi h^{-1}(A)$ is thus an arcwise-connected subcontinuum of $X$ containing $p$ and $q$. Therefore, there is an arc $B$ in $X$ which contains both $p$ and $q$. $B$ is a proper subcontinuum of $X$, since it does not contain all the points of $Y$. Hence $B$ is a subset of the composant of $X$ which contains $p$ and $q$.

COROLLARY 1. *If the decomposable continuum $X$ has the cone = hyperspace property, then $X$ is arcwise-connected.*

The next theorem is [**10**, Theorem 1].

THEOREM 2. *Suppose $Y, X_1, X_2, \ldots, X_n$ are proper subcontinua of the continuum $X$ such that $Y \cap X_i \neq \emptyset$, $X_i \not\subset Y$, and $(X_i - Y) \cap (X_j - Y) = \emptyset$ when $j \neq i$. Then $C(X)$ contains an $n$-cell.*

COROLLARY 2. *If $C(X)$ has dimension $\leqq 2$, then $X$ is atriodic.*

THEOREM 3. *Suppose $X$ is a continuum with arcwise-connected composants, and suppose $C(X)$ is 2-dimensional. Then $X$ is an arc, an arcwise-connected circle-like continuum, or an indecomposable continuum that contains only arcs as subcontinua.*

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*Proof.* We note the following properties of any proper subcontinuum $Y$ of $X$:

($A$) *$Y$ is atriodic.* $X$ is atriodic, by Corollary 2.

($B$) *$Y$ is decomposable.* Suppose $Y$ is indecomposable. Consider 3 points in different composants of $Y$, and let $Z$ be an arc which contains these points. Then there exist subarcs $X_1$, $X_2$, $X_3$ of $Z$ such that the hypotheses of Theorem 2 are satisfied. $C(X)$, however, is 2-dimensional.

($C$) *$Y$ is unicoherent.* Suppose $Y$ can be expressed as the union of 2 continua $A$ and $B$ such that $A \cap B$ is not connected. Let $C$ be an arc from the complement of $Y$ to a point in $A \cap B$. Then continua $X_1$, $X_2$, $X_3$, and $Y$ can be chosen to satisfy the hypotheses of Theorem 2.

($D$) *If $X$ is indecomposable, then $Y$ is arcwise connected.* If $Y$ is not arcwise-connected between 2 of its points, let $Z$ be an arc between these 2 points. Then $Y \cup Z$ is a proper multicoherent subcontinuum.

Therefore, if $X$ is indecomposable, each proper subcontinuum is a hereditarily unicoherent, hereditarily decomposable, atriodic, arcwise-connected continuum. Such continua must be arcs [**2**]. If $X$ is decomposable and unicoherent, then $X$ is an arc, for the same reason. If $X$ is decomposable and not unicoherent, then $X$ is a circle-like continuum [**4**].

COROLLARY 3. *If $X$ is arcwise-connected, and $C(X)$ is 2-dimensional, then $X$ is an arc or a circle-like continuum.*

The next theorem is proved in [**10**].

THEOREM 4. *If $X$ is a continuum with arcwise connected composants of dimension at least* 2, *then* dim $C(X) = \infty$.

THEOREM 5. *Suppose $X$ is a finite-dimensional continuum such that each composant is arcwise-connected and such that $K(X)$ is homeomorphic to $C(X)$. Then $X$ is an arc, an arcwise-connected circle-like continuum, or an indecomposable continuum that contains only arcs as subcontinua.*

*Proof.* $K(X)$, and hence $C(X)$, is finite-dimensional, so $X$ is 1-dimensional, by Theorem 4. Thus $C(X)$ is 2-dimensional, and the conclusion follows from Theorem 3.

The next theorem is a generalization of [**5**, Theorem 4.4].

THEOREM 6. *Suppose the subcontinuum $Y$ of $X$ has open, connected neighborhoods $V_\epsilon(Y)$ in $X$, where $\epsilon$ is an arbitrarily small positive number. Then $C(X)$ is connected im kleinen at $Y$.*

*Proof.* Let $V = V_\epsilon(Y)$ be an open, connected neighborhood (in $X$) of $Y$. If $\rho(Y, Z) < \epsilon$, then $\rho(Y, \bar{V}) < \epsilon$ and $\rho(Z, \bar{V}) < 2\epsilon$. Let $A_t$ be a segment from $Y$ to $\bar{V}$ and $B_t$ a segment from $Z$ to $\bar{V}$. (For the definition of segment, see [**5**, p. 24].) The continuum

$$W = \{A_t\} \cup \{B_t\}, 0 \leqq t \leqq 1,$$

is of diameter less than $3\epsilon$. Hence any two points of $C(X)$ less than $\epsilon$ apart can be joined by a continuum of diameter less than $3\epsilon$.

COROLLARY 3. *If $X$ is locally connected, then $C(X)$ is locally connected.*

Since the converse of this corollary is true [**5**, Theorem 4.4], we might hope that the converse of Theorem 6 is true. This is not the case. Let $X$ be the curve in the $xy$-plane defined by

$$\begin{aligned} y &= \sin(1/x) && \text{for } 0 < x \leqq 1 \\ -1 \leqq y &\leqq 2 && \text{for } x = 0. \end{aligned}$$

Then the interval $\{(x, y): 0 \leqq y \leqq 1\frac{1}{2}, x = 0\}$ has small open neighborhoods homeomorphic to an open 2-cell.

THEOREM 7. *Suppose the finite-dimensional continuum $X$ has the cone = hyperspace property. Then each proper subcontinuum of $X$ is an arc.*

*Proof.* It suffices to show that the simple closed curve is the only circle-like continuum with the cone = hyperspace property. Consider $W$, the $\sin(1/x)$-circle (or Warsaw circle), obtained from the standard $\sin(1/x)$-curve by identifying a pair of opposite endpoints. $W$ is an arcwise-connected, circle-like continuum.

In $K(W)$ there are points arbitrarily close to $v$ at which $K(W)$ is not connected im kleinen. In $C(W)$, however, $W'$ has a neighborhood at each point of which $C(W)$ is locally connected (in fact, $W'$ has small neighborhoods homeomorphic to 2-cells). Nadler [**8**, Theorem 6] has shown that an arcwise-connected, non-Peanian, circle-like continuum $X$ resembles $W$ in that $X$ is the union of an arc $A$ and a chainable metric continuum $C$ with exactly 2 arc components, and $A \cap C$ is 2 points which are opposite endpoints of both $A$ and $C$. So the proof that $W$ does not have the cone = hyperspace property can be applied to any non-Peanian, arcwise-connected, circle-like continuum.

Let $S$ be the continuum obtained from identifying the two noncut-points of two $\sin(1/x)$-curves. Then an argument similar to that of Theorem 7 may be used to show that, for $S$, and continua resembling $S$, no homeomorphism between the cone and hyperspace is possible.

## 2. Homeomorphisms between $C(X)$ and $K(X)$.

In this section we investigate some spaces $X$ such that $C(X)$ and $K(X)$ are homeomorphic. In particular, we consider the space $Z$ consisting of a ray spiraling down on a circle and show that $C(Z)$ and $K(Z)$ are homeomorphic. In view of the result of Ronald Knill [**6**] (see also [**1**, p. 129]) that $K(Z)$ does not have the fixed point property, we have an example of a space $X$ such that $C(X)$ does not have the fixed point property.*

*I am indebted to Knill for a helpful conversation about this space.

Notice that if a space $X$ is locally connected, then $C(X)$ is an $AR$ [**5**, Theorem 4.4] and hence has the fixed point property. We observe in the next section that if $X$ is a continuum at the other end of the spectrum, namely, a hereditarily indecomposable continuum, then $C(X)$ has the fixed point property. This leads us to raise the following question:

*Question*. When does $C(X)$ have the fixed point property?

$Z$, the circle with a spiral, is defined in polar coordinates as follows: the circle $B$ is defined as $[(r, \theta):r = 1]$ and the spiral $S$ as $[(r, \theta):r = 1 + 1/(1 + \theta), \theta \geqq 0]$. Following Bing [**1**, p. 129], we call the cone over the spiral the *skirt of the cone.*

THEOREM 8. *$C(Z)$ is homeomorphic to $K(Z)$.*

*Proof*. Write subarcs of $Z$ as $[s, t]$ with a counterclockwise orientation. Let $l[s, t]$ denote the length of the arc $[s, t]$, and let $[s, B]$ denote the continuum with complement $[(2, 0), s]$.

Consider the subset of the plane

$$\{(x, y):0 \leqq x \leqq 1, y \geqq 0\}.$$

Compactify this set by adding an arc $[0, 1] \times \{\infty\}$ and delete the set

$$\{(x, y):x = 1, 0 \leqq y < 2\pi\}.$$

Call the resulting space $X$ and let $Y$ be the decomposition space obtained from $X$ be shrinking

$$A = \{(x, y):x = 1, 2\pi \leqq y \leqq \infty\}$$

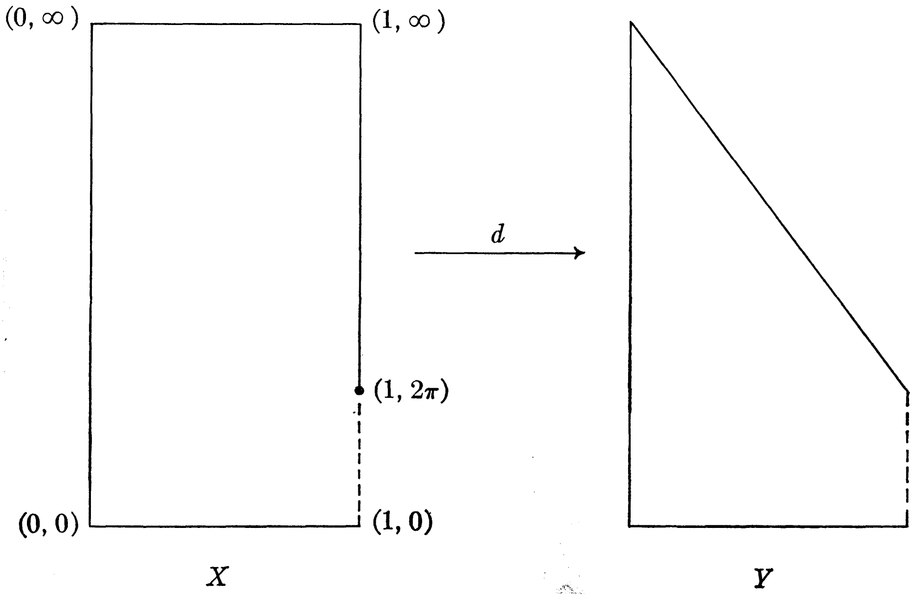


Figure 1.

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to a point. Let $d:X \to Y$ be the map associated with this decomposition. $X$ is pictured in Figure 1. We think of $Y$ as the skirt of the cone.

Let the cone $K(B)$ over $B$ have its vertex $v$ at $(0, 0, 2\pi)$. If $[a, b]$ is a subcontinuum of $B$ and if $R$ is the line segment from $a$ to $v$, define $f([a, b])$ to be the point on $R$ which has $z$-coordinate $l[a, b]$. Then $F$ is a homeomorphism of $C(B)$ onto $K(B)$.

Let $g$ be a homeomorphism from $S$ to $[1, 0)$. Extend $f$ to a multi-valued function from $C(Z)$ to the free union of $X$ and $K(B)$ by defining

$$\begin{aligned} f[s, t] &= (g(s), l[s, t]) \\ f[s, B] &= (g(s), \infty) \\ f(B) &= A. \end{aligned}$$

If we follow $f$ by the collapse $d$ and then wrap the skirt back around the cone in the obvious way, then we have a homeomorphism between $C(Z)$ and $K(Z)$.

Theorem 8 is true for several other continua, such as the sin $(1/x)$-continuum; the proof is simply a modification of the above proof.

**3. Hyperspaces of hereditarily indecomposable continua.** A continuum is indecomposable if it is not the union of 2 of its proper subcontinua. A continuum is hereditarily indecomposable if all of its subcontinua are indecomposable. If $X$ is a hereditarily indecomposable continuum, then $C(X)$ is not homeomorphic to $K(X)$; nevertheless, a strong mapping relation exists between the two spaces.

THEOREM 9. *If $X$ is a hereditarily indecomposable, nondegenerate continuum, then there is a monotone, open, arc-preserving mapping of $K(X)$ onto $C(X)$ which is the identity map on $X$.*

*Proof.* It suffices to construct a suitable map from $X \times I$ onto $C(X)$, by the Transgression Theorem [**3**, p. 123]. Rhee [**9**] has constructed the following continuous map in proving that $C(X)$ is contractible: $\Phi$ is the map from $X \times I$ onto $C(X)$ defined by

$$\Phi(x, t) = A, \text{ if } x \in A \in C(X) \text{ and } \mu(A) = t.$$

$\Phi$ is monotone since $\Phi^{-1}(\{A\}) = A$. $\Phi$ is clearly arc-preserving.

To see that $\Phi$ is open, let $O \times P$ be a standard basic open set in $X \times I$. Let $A$ be a point in $\Phi(0 \times P)$. There exists a point $x$ in $0$ such that $x \in A$. Let $Q$ be an $\epsilon$-neighborhood of $x$ such that $Q \subset 0$. Consider the open set $V$ in $C(X)$ which is the intersection of $\mu^{-1}(P)$ and the $(\epsilon/2)$-neighborhood of $A$. If $B \in V$, then $\mu(B) \in P$ and $A$ is contained in the $(\epsilon/2)$-neighborhood of $B$. Hence there is a point $y$ in $B$ such that $d(x, y) < \epsilon/2$. Hence $y \in 0$ and $B \in \Phi(0 \times P)$.

*Remark.* The hyperspace of subcontinua of a hereditarily indecomposable continuum $X$ has the fixed-point property. For $C(X)$ is contractible and uniquely arcwise-connected [**5**, Theorem 8.4]; Gail Young [**12**] has shown such a continuum must have the fixed-point property. This same reasoning shows that the cone over a hereditarily indecomposable continuum has the fixed-point property–this partially answers a question of Knill [**6**, p. 36].

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