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ARTICLE

Cushion option on CPPI strategy for defined-contribution pension plans

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Abstract
This paper investigates a well-known downside protection strategy called the constant proportion portfolio insurance (CPPI) in defined contribution (DC) pension fund modeling. Under discrete time trading CPPI, an investor faces the risk of portfolio value hitting the floor which denotes the process of guaranteed portfolio values. In this paper, we question how to deal with so-called 'gap risk' which may appear due to uncontrollable events resulting in a sudden drop in the market. In the market model considered, the risky asset price and the labor income are assumed to be continuous-time stochastic processes, whereas trading is restricted to discrete-time. In this setting, an exotic option (namely, the 'cushion option') is proposed with the aim of reducing the risk that the portfolio value falls below the defined floor. We analyze the effectiveness of the proposed exotic option for a DC plan CPPI strategy through Monte Carlo simulations and sensitivity analyses with respect to the parameters reflecting different setups.

Keywords: CPPI strategy; defined contribution pension plan; exotic (cushion) option; risk management; portfolio allocation

JEL Codes: G11; G22

1. Introduction

Pension planning is one of the most important financial decisions, as it impacts a significant portion of a plan participant's life. Contributions and investment decisions during the accumulation phase are key factors for determining the guarantee that individuals expect to maintain their standard of living after retirement. This is where portfolio insurance strategies come into play by protecting participants' retirement savings from significant market downturns while allowing growth potential. Using techniques such as dynamic asset allocation or option-based approaches, one can ensure that a minimum portfolio value is maintained and the risk of substantial losses near retirement is reduced. This balance between risk mitigation and growth is essential for providing retirees with more predictable and secure financial benefits in the face of long-term market volatility.

In the present paper, we focus on a specific type of portfolio insurance, namely, the *constant proportion portfolio insurance (CPPI)* tailored for a DC plan within a discrete-time trading framework. In such settings, investors always face the risk that the portfolio value may reach the floor due to the inability to trade instantly in response to market changes when trading discretely. If the portfolio value drops below the floor value, the entire wealth is invested in the riskless asset creating a special case of a *cash-locked* position. As a result, the potential gains from the risky asset are missed during an upward market movement, and the portfolio value may fail to meet the guarantee at the termination date. On the other hand, an interruption of the CPPI strategy before the termination of the policy (such as withdrawal and retirement) also poses a gap risk. To reduce such a risk, the investor may consider a financial tool, such as a put option, to improve the course of the asset growth. To cope with such unwanted situations, this paper introduces an exotic option called the 'cushion option' which is designed to generate an additional gain in case of a *gap* that is a sudden drop in the portfolio value. Hence, the option is expected to benefit when the cushion becomes negative between discrete trading times. Since this is a theoretical derivative instrument, our primary assumption is that there are parties in the market who sell these options. Then, based on the position of the pension fund, the plan participant can buy it.

The option is priced as an expectation of the discounted claims with its boundary under the equivalent martingale measure in our complete market. In an incomplete market, on the other hand, there exists a class of equivalent martingale measures to price the exotic option. Applying Schweizer's variance-optimal criterion (Schweizer, 1995), one can then obtain the explicit price of the exotic option under the minimal martingale measure which is not in the scope of this paper. The market we consider exhibits perfect correlation between the labor income and the risky asset processes following the approach of Bodie *et al.* (1992) and Benzoni *et al.* (2007). This assumption is needed to find a replicating portfolio of the labor income. Through sensitivity analyses, we examine the effectiveness of cushion option on CPPI strategy in a DC pension plan to deal with gap risk. We also analyze the optimal strike price interval of the cushion option under predetermined market parameters and question if this option can be proposed as a financial tool to reduce the impact of early withdrawals resulting in lapse risk.

In the related literature, Blake *et al.* (2000) examine the optimal dynamic asset allocation strategy for DC plan taking into account stochastic process for labor income including a non-hedgeable risk component. Considering stochastic behavior of labor income and stochastic inflation rate, a closed form solution is given by Battochio and Menoncin (2004) for the optimal portfolio in a complete financial market. Similarly, Baltas *et al.* (2018) study the optimal management of defined-contribution pension schemes by considering both the portfolio's exposure to various market risks and the model uncertainty regarding the evolution of several unknown market parameters that influence its behavior. The authors delve into the same problem, considering the effects of inflation, mortality, and model uncertainty in Baltas *et al.* (2022).

In the discrete-time trading framework, Haberman and Vigna (2001, 2002) and Temocin *et al.* (2017) investigate an optimal investment strategy in DC plan by using dynamic programming techniques. Among many others, the main risk in a DC pension plan for participants is the investment risk during accumulation phase, in which their pension wealth has been built up so that an appropriately sized annuity (or other investment) can be purchased to provide income during retirement. To moderate investment risk, a minimum guarantee is introduced as a lower bound for pension wealth that will be paid out to the participants upon retirement. With respect to this, Boulier *et al.* (2001) study the optimal management of a defined contribution (DC) plan with deterministic contribution and the guarantee in retirement depending on the level of stochastic interest rate which follows the Vasicek (1977) model. In their setup, the guarantee has a form that is annuity paid out from retirement time until the date of death which is also taken as deterministic. Deelstra *et al.* (2003) investigate the optimal investment problem for DC plan that allows for minimum guarantee with stochastic contribution and assumes that interest rate dynamics are given as in Duffie and Kahn (1996) in its one-dimensional version. Although authors make various assumptions, their aim is to provide the minimum guarantee, which are so-called 'portfolio insurance strategies'.

The concept of CPPI strategy is firstly introduced by Perold (1986) for fixed-income securities and by Black and Jones (1987) for equity instruments. The properties of CPPI strategies in continuous time are studied by Black and Perold (1992). Assuming HARA utiliy function, they show that the CPPI strategy can maximize expected utility. This study is extended by Horsky (2012) with the interest rate that follows the Vasiček model and stock modeled by the Heston process. Temocin *et al.* (2017) study the CPPI strategy in DC pension funds with different floor assumptions under continuous-time and discrete-time trading. The *gap risk* which is the probability of portfolio falling below the floor and the risk of portfolio being fully invested in the risk-free asset without a recovery, called the *cash-lock risk*, are analyzed for discrete time trading as well in Temocin *et al.* 2018.

Bertrand and Prigent (2003) compare the performances of CPPI and Option based portfolio insurance (OBPI) strategies when the volatility of the risky asset is stochastic. They also examine both strategies under first-order stochastic dominance criteria that are related to increasing utility functions (Bertrand and Prigent, 2005). Considering various stochastic dominance criteria up to third order, Zagst and Kraus (2011) compare the two portfolio insurance strategies. They conclude that CPPI is likely to dominate the OBPI strategy at third order. Pézier and Scheller (2013) show that CPPI is superior to the OBPI strategy under discrete trading and asset prices having jump.

In the Black and Scholes framework, which is the basis for most academic studies on CPPI, there is no gap risk, because the probability of the portfolio value being above the floor at any time equals to one. However, addressing the gap risk, Balder *et al.* (2009), Cont and Tankov (2009), and Lacroze and Paulot (2011) show that portfolio value may crash through the floor in incomplete market in which asset price jumps may occur or when the portfolio may only be re-balanced on a finite number of trading days. Balder *et al.* (2009) investigate the CPPI strategy under trading restrictions, where the multiplier, a certain proportion of excess value of current wealth over the floor, cannot be changed between instant trading dates. So that, investor will not have an opportunity to re-balance the portfolio, which then may crash through the floor between trading dates on a downward market move. Cont and Tankov (2009) quantify the gap risk that results from instantaneous price jumps analytically for the CPPI strategy under continuous-time trading. Lacroze and Paulot (2011) show that if the underlying has independent increments, the dynamics of the portfolio at trading dates is described by a discrete-time Markov process in a single variable. Extreme value approach is also used by Bertrand and Prigent (2002) to estimate gap risk of the CPPI strategy. Moreover, Tankov introduces an exotic derivative called gap option in Tankov (2010) and shows that to price and manage the option, jumps must necessarily be included into the model. The author then presents explicit pricing and hedging formulas in the single asset and multi-asset case.

The main difference of our CPPI setting from the literature listed above is that we focus on a CPPI with random floor that grows not only with interest rate but also with the portions of each contribution, unlike the deterministic floors in the classical CPPI strategies. Hence, in this setup, floor becomes a stochastic process due to the dynamics of the labor income as described in Temocin *et al.* (2017).

The plan of this paper is as follows: In Section 2, the classical CPPI strategy is presented. Section 3 gives the CPPI strategy for a DC pension plan, the cushion option is also proposed, and the CPPI strategy for DC pension scheme under the cushion option is derived. In Section 4, the evolution of a wealth in pension fund with and without the cushion option until certain withdrawal time is presented, and the effectiveness of cushion option for CPPI strategy is also discussed. We conclude with a brief discussion of the observations and suggestions for further research in Section 5.

2. CPPI strategy

The CPPI strategy is a self-financing strategy whose aim is to take potential gain on upward market move while guaranteeing at least an specified fixed amount of money at maturity time T. In the CPPI strategy, the investor initially sets a floor which is the lowest acceptable portfolio value. The cushion is calculated as the excess value of current wealth over the floor. The cushion multiplied by a pre-determined multiplier, defined as exposure, is allocated to the risky asset. Remaining funds are invested in the riskless asset. In the classic Black-Scholes market, where two basic assets are traded continuously during time horizon $[0, T]$, and the fund manager invests into the two assets: a riskless

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asset (money market account), B_t, and a risky asset (stock or stock index), S_t, the price dynamics under $\mathbb{P}$ are given by

$$\frac{dB_t}{B_t} = rdt, \quad B_0 = b, \tag{1}$$

$$\frac{dS_t}{S_t} = \mu_s dt + \sigma_s dW_t, \quad S_0 = s, \tag{2}$$

where r, b, s, μ_s, and σ_s are constant and W_t is a Brownian process defined on complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Here, $\mathbb{P}$ is real-world probability measure and the filtration $\mathcal{F} = (\mathcal{F}_t)_{[t \geq 0]}$ represents the history of the Brownian motion up to time t. The basic idea of the CPPI approach (Zagst and Kraus, 2011) is that the terminal portfolio value, V_T, at the end of the investment horizon T stays above an investor-defined floor given as a percentage, $\varphi \geq 0$, of initial value, V_0, i.e.

$$F_T = \varphi_T V_0.$$

Since the market is arbitrage-free, it is impossible to find an investment that returns more than the risk-free rate of return, r, with no risk. Hence, the maximum guaranteed portfolio value at maturity time, T, is limited by

$$\varphi_T \leq e^{rT}.$$

The floor F_t, $0 \leq t \leq T$, denotes the present value of guarantee. By discounting with respect to deterministic interest rate, r, φ_t becomes

$$F_t = \varphi_t V_0, \qquad \varphi_t = \varphi_T e^{-r(T-t)}.$$

Accordingly, the floor follows the dynamics

$$dF_t = rF_t dt.$$

The surplus of current portfolio value, V_t, above the floor F_t is called cushion. The price of cushion, C_t, at any time $t \in [0, T]$ is given as

$$C_t := V_t - F_t. \tag{3}$$

At any time $t \in [0, T]$, the wealth invested into the risky asset, called exposure, is given by

$$e_t = mC_t,$$

where m is a constant multiplier reflecting the risk attitude of the investor. The remaining funds are then invested in the risk-free asset leading to a portfolio value process

$$\frac{dV_t}{V_t} = e_t \frac{dS_t}{S_t} + (V_t - e_t)\frac{dB_t}{B_t}. \tag{4}$$

By Equations (3) and (4), the cushion C_t satisfies

$$dC_t = C_t \left((m(\mu_S - r) + r) dt + m\sigma_S dW_t \right). \tag{5}$$

3. CPPI strategy in DC pension plan under discrete time trading setting

Let τ be set of $n+1$ equidistant trading dates along time period $[0, T]$, such as $\tau = \{t_0 < t_1 < \ldots < t_{n-1} < t_n\}$, where $t_0 = 0$ is the inception date of DC pension plan, $t_n = T$ is the retirement date, and $\triangle t := t_{k+1} - t_k = T/n$ for $k = 0, 1, .., n-1$.

As mentioned earlier, we consider two assets in the financial market for investing pension contributions: risky asset (stock or stock index) and riskless asset (money market account). Money market account, B_t, grows with constant interest rate, r, and its price dynamic with initial value $B_0 = b$ given in Equation (1). Additionally, we assume that the price dynamics of risky asset, S_t, is expressed as in Equation (2). Since trading takes place at equidistant trading dates, the observed stock price is as follows:

$$S_{t_{k+1}} = S_{t_k} e^{(\mu_S - \frac{\sigma_S^2}{2})\triangle t + \sigma_S \triangle W_{t_k}}, \qquad k = 0, \ldots, n-1,$$

where $S_{t_0} = s$ and $\triangle W_{t_k} = W_{t_{k+1}} - W_{t_k}$ for $k = 0, \ldots, n-1$.

The DC fund modeling has two key aspects:

i. There is no consumption before the retirement date.
ii. Labor income plays a central role in the wealth-accumulated phase.
iii. The inheritance gains are not considered which avoids the consideration of longevity and mortality risks.

Labor income is hard to model realistically due to its complicated stochastic components such as financial and political crisis, disability, and mortality. Here, labor income, L_t, is assumed to have a stochastic process reflecting the risk of the financial market. If the participants contribute to the pension fund at each equidistant trading date as a certain proportion, γ, of their labor income, the labor income dynamics are defined as a stochastic differential equation as follows:

$$dL_t = L_t(\mu_L dt + \sigma_L dW_t^*), \quad L_0 = l, \tag{6}$$

with l, μ_L, and σ_L representing starting labor constant, average labor, and standard deviation, respectively, and W_t^* is a Brownian process. In what follows, we restrict the stochastic behavior of the labor income process for the tractability of analytic results. The case we consider is a labor income process that perfectly correlates with risky asset ($dW_t = dW_t^*$) following the approach of Bodie *et al.* (1992).

Based on this setup, the observed labor income becomes

$$L_{t_{k+1}} = L_{t_k} e^{\left(\mu_L - \frac{\sigma_L^2}{2}\right)\triangle t + \sigma_L \triangle W_{t_k}}, \qquad k = 0, \ldots, n-1, \tag{7}$$

where $L_{t_0} = l$ and $\triangle W_{t_k} = W_{t_{k+1}} - W_{t_k}$ for $k = 0, \ldots, n-1$. Therefore, the DC with respect to labor process and the value of proportion, γ, is defined as

$$\gamma_{t_k} = \gamma L_{t_k}, \tag{8}$$

for all $k = 0, 1, .., n$ with dynamics

$$d\gamma_t = \gamma dL_t. \tag{9}$$

We assume that there is a portfolio which has the same dynamics as labor income. For this reason, the labor income can be replicated perfectly between equidistant trading dates using assets in the market. In order to prove this, we need to show the relevant replicating portfolio. Assuming the relation which expresses the equivalence of risk premium between the labor income and the risky asset

$$\mu_L = r + \sigma_L \frac{\mu_S - r}{\sigma_S}, \tag{10}$$

we consider a self-financing replicating strategy $\pi = (\pi_B, \pi_S, \pi_L)$ and choose $\pi_L = -1$ as given in Temocin *et al.* (2017). Then, we have the value of the fund as

$$\begin{aligned} dV_t^\pi &= \pi_B dB_t + \pi_S dS_t - dL_t \\ &= \pi_B(rB_t dt) + \pi_S(S_t\mu_s dt + S_t\sigma_s dW_t) - L_t(\mu_L dt + \sigma_L dW_t) \\ &= (\pi_B rB_t + \pi_S S_t \mu_s - L_t\mu_L)dt + (\pi_S S_t\sigma_s - L_t\sigma_L)dW_t. \end{aligned}$$

Equating the diffusion terms and the drift terms to zero, we obtain the number of assets required as

$$\pi_S(t) = \frac{L\sigma_L}{S\sigma_S} \qquad \text{and} \qquad \pi_B(t) = \frac{L}{rB\sigma_S}(\sigma_S\mu_L - \sigma_L\mu_S).$$

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Unlike the floor in the classical CPPI strategy, we consider a random floor which evolves not only with interest rate but also with the portion of each contribution. In this setup, floor, F_t becomes a stochastic process due to the dynamics of the labor income as described in (Temocin *et al.*, 2017). Hence, the floor at time t is defined as

$$F_t = \begin{cases} \sum_{i=0}^{k} e^{r(t-t_i)} c\gamma_{t_i}, & t \in (t_k, t_{k+1}) \\ F^{-}_{t_{k+1}} + c\gamma_{t_{k+1}}, & t = t_{k+1} \end{cases} \tag{11}$$

for $k = 0, 1, ..., n-1$, where $0 < c < 1$ is a constant that denotes the proportion of the amount of contribution to be added to the floor with $F_0 = c\gamma_{t_0}$. The portfolio value is then given as

$$V_{t_{k+1}} = \begin{cases} (V_{t_k} - mC_{t_k})e^{r\triangle t} + mC_{t_k}\frac{S_{t_{k+1}}}{S_{t_k}} + \gamma_{t_{k+1}}, & C_{t_k} > 0 \\ V_{t_k}e^{r\triangle t} + \gamma_{t_{k+1}}, & C_{t_k} \le 0 \end{cases} \tag{12}$$

for $t \in (t_k, t_{k+1})$ and

$$V_t = \begin{cases} (V_{t_k} - mC_{t_k})\frac{B_t}{B_{t_k}} + mC_{t_k}\frac{S_t}{S_{t_k}}, & C_{t_k} > 0 \\ V_{t_k}\frac{B_t}{B_{t_k}}, & C_{t_k} \le 0 \end{cases} \tag{13}$$

for $t = t_k$ for all $k = 0, 1, .., n-1$.

Then by its definition, the cushion process becomes

$$C_{t_{k+1}} = \begin{cases} C_{t_k}(m\frac{S_{t_{k+1}}}{S_{t_k}} + (1-m)e^{r\triangle t}) + (1-c)\gamma_{t_{k+1}}, & C_{t_k} > 0 \\ C_{t_k}e^{r\triangle t} + (1-c)\gamma_{t_{k+1}}, & C_{t_k} \le 0 \end{cases} \tag{14}$$

for $t \in (t_k, t_{k+1})$ and

$$C_t = \begin{cases} C_{t_k}\left(m\frac{S_t}{S_{t_k}} + (1-m)\frac{B_t}{B_{t_k}}\right), & C_{t_k} > 0 \\ C_{t_k}\frac{B_t}{B_{t_k}}, & C_{t_k} \le 0 \end{cases}$$

for $t = t_k$ for all $k = 0, 1, .., n-1$.

Now that we have described the CPPI-DC design, we next introduce the implementation of the cushion insurance by employing a special exotic option, hereby called the *cushion option*.

3.1. Cushion option in the frame of a CPPI strategy

Due to our trading structure, the risk of cushion becoming negative between trading dates is always present as the re-balancements are done only at trading dates. Considering the cushion dynamics, we can define probability of gap risk, P_{gap} just before contribution payment comes into the system as follows:

$$\begin{aligned} P_{gap} &= P\left(C^{-}_{t_{k+1}} < 0 |\, C_{t_k} > 0\right) \\ &= P\left(C_{t_k}\left(m\frac{S_t}{S_{t_k}} + (1-m)e^{r\triangle t}\right) < 0 |\, C_{t_k} > 0\right) \\ &= P\left(\frac{S_t}{S_{t_k}} < \frac{(m-1)}{m}e^{r\triangle t}\right). \end{aligned}$$

To minimize P_{gap}, we propose the exotic cushion option with payoff function

$$(K - (1-c)\gamma L_{t_{k+1}})^{+}\mathbb{1}_{\left\{\frac{S_{t_{k+1}}}{S_{t_k}} < (1-\frac{1}{m})e^{r\triangle t}\right\}}, \tag{15}$$

where $K = C_{t_k}$. The main aim of this custom-made option is to generate an additional gain in case of a sudden decrease in the portfolio value. The proposed option is expected to benefit when the cushion becomes negative between times t_k and t_{k+1}. Hence, we compare the cushion amounts at these times with the purpose of activating the payoff only when there is a gap phenomenon in the given time period. Here, the strike price, K, represents the cushion with the aim of making sure that there is always sufficient capital for buying the option, so the option price will always be smaller than the cushion due to non-negative labor income. Since this is a theoretical derivative, our primary assumption is that there are parties in the market who sell these options. Then, based on the position of the pension fund, the beneficiary can buy the relevant option. By Girsanov's Theorem (Shreve, 2004), we have the risk-neutral dynamics of the stock price as

$$S_{t_{k+1}} = S_{t_k} exp\left(\left(r - \frac{\sigma_S^2}{2}\right)\triangle t + \sigma_S (W^{\mathbb{Q}}_{t_{k+1}} - W^{\mathbb{Q}}_{t_k})\right),$$

where $W^{\mathbb{Q}}$ is the Brownian motion under the unique risk-neutral measure $\mathbb{Q}$. Thus, we have

$$W^{\mathbb{Q}}_{t_{k+1}} - W^{\mathbb{Q}}_{t_k} = \frac{ln\frac{S_{t_{k+1}}}{S_{t_k}} - \left(r - \frac{\sigma_S^2}{2}\right)\triangle t}{\sigma_S}. \tag{16}$$

Substituting $W^{\mathbb{Q}}_{t_{k+1}} - W^{\mathbb{Q}}_{t_k}$ into the $\mathbb{Q}$-dynamics of $L_{t_{k+1}}$ yields

$$L_{t_{k+1}} = L_{t_k} exp\left(\left(\mu_L - \frac{\sigma_L^2}{2}\right)\triangle t + \sigma_L \frac{ln\frac{S_{t_{k+1}}}{S_{t_k}} - \left(r - \frac{\sigma_S^2}{2}\right)\triangle t}{\sigma_S}\right) \tag{17}$$

$$= L_{t_k}\left(\frac{S_{t_{k+1}}}{S_{t_k}}\right)^{\frac{\sigma_L}{\sigma_S}} exp\left(\left(\mu_L - \frac{\sigma_L^2}{2} - \frac{r\sigma_L}{\sigma_S} + \frac{\sigma_L\sigma_S}{2}\right)\triangle t\right). \tag{18}$$

Hence, the payoff of the cushion option becomes

$$\begin{aligned}(C_{t_k} - (1-c)\gamma L_{t_{k+1}})^+ \mathbb{1}_{\left\{\frac{S_{t_{k+1}}}{S_{t_k}} < (1-\frac{1}{m})e^{r\triangle t}\right\}} &= \left(C_{t_k} - (1-c)\gamma L_{t_k}\left(\frac{S_{t_{k+1}}}{S_{t_k}}\right)^{\frac{\sigma_L}{\sigma_S}}\right. \\ &\left. exp\left(\left(\mu_L - \frac{\sigma_L^2}{2} - \frac{r\sigma_L}{\sigma_S} + \frac{\sigma_L\sigma_S}{2}\right)\triangle t\right)\right)^+ \\ &\times \mathbb{1}_{\left\{\frac{S_{t_{k+1}}}{S_{t_k}} < (1-\frac{1}{m})e^{r\triangle t}\right\}} \\ &= \zeta(K^* - S_{t_{k+1}}^{\frac{\sigma_L}{\sigma_S}})^+ \mathbb{1}_{\left\{\frac{S_{t_{k+1}}}{S_{t_k}} < (1-\frac{1}{m})e^{r\triangle t}\right\}},\end{aligned}$$

where

$$\zeta = \frac{(1-c)\gamma L_{t_k}}{S_{t_k}^{\frac{\sigma_L}{\sigma_S}}} exp\left(\left(\mu_L - \frac{\sigma_L^2}{2} - \frac{r\sigma_L}{\sigma_S} + \frac{\sigma_L\sigma_S}{2}\right)\triangle t\right) \quad \text{and} \quad K^* = \frac{C_{t_k}}{\zeta}.$$

The next proposition gives the price of the cushion with this payoff.

Proposition 1. *The cushion option under CPPI-DC design with payoff*

$$(C_{t_k} - (1-c)\gamma L_{t_{k+1}})^+ \mathbb{1}_{\left\{\frac{S_{t_{k+1}}}{S_{t_k}} < (1-\frac{1}{m})e^{r\triangle t}\right\}},$$

has a price of P_{t_k} given as

$$P_{t_k} = \zeta^* K^* \Phi\left(\min(A_{t_k}, B_{t_k})\right) - \zeta^* S_{t_k}^{\frac{\sigma_L}{\sigma_S}} e^{\left[\frac{\sigma_L}{\sigma_S}(r-\sigma_S^2/2)+\sigma_L^2/2\right]\triangle t} \Phi\left(\min(A_{t_k}, B_{t_k}) - \sigma_L \triangle t\right), \tag{19}$$

where

$$\zeta^* = e^{-r\triangle t}\zeta,$$

with

$$A_{t_k} = \frac{ln\left(\frac{K^*}{S_{t_k}^{\frac{\sigma_L}{\sigma_S}}}\right) - \frac{\sigma_L}{\sigma_S}\left(r - \frac{\sigma_S^2}{2}\right)\triangle t}{\sigma_L\sqrt{\triangle t}} \quad and \quad B_{t_k} = \frac{ln(\frac{m-1}{m}) + (\sigma_S^2/2)\triangle t}{\sigma_S\sqrt{\triangle t}}.$$

Proof. *The details of the proof is given in the Appendix.* □

The cushion option is assumed to be purchased at time t_k for all $k = 0, 1, ..., n-1$. Since $C_{t_k} > P_{t_k}$, new portfolio value, $V_{t_k}^*$, and cushion value, $C_{t_k}^*$, at time t_k are given by

$$V_{t_k}^* = V_{t_k} - P_{t_k}$$

and

$$C_{t_k}^* = max\{V_{t_k}^* - F_{t_k}, 0\},$$

respectively. Then, for $C_{t_k} > 0$ for all $k = 0, 1, .., n-1$, the portfolio value is expressed as

$$V_{t_{k+1}} = \left[(V_{t_k}^* - mC_{t_k}^*)e^{r\triangle t} + mC_{t_k}^* \frac{S_{t_{k+1}}}{S_{t_k}} + \gamma L_{t_{k+1}} + (C_{t_k} - (1-c)\gamma L_{t_{k+1}})^+\right] \mathbb{1}_{\left\{\frac{S_{t_{k+1}}}{S_{t_k}} \leq (1-\frac{1}{m})e^{r\triangle t}\right\}}$$
$$+ \left[(V_{t_k}^* - mC_{t_k}^*)e^{r\triangle t} + mC_{t_k}^* \frac{S_{t_{k+1}}}{S_{t_k}} + \gamma L_{t_{k+1}}\right] \mathbb{1}_{\left\{\frac{S_{t_{k+1}}}{S_{t_k}} > (1-\frac{1}{m})e^{r\triangle t}\right\}}.$$

For $C_{t_k} \leq 0$ for all $k = 0, 1, .., n-1$, the portfolio value, $V_{t_{k+1}} = V_{t_k} e^{r\triangle t} + \gamma L_{t_{k+1}}$, yields the value of the pension fund at t_{k+1}. Henceforth, when $C_{t_k} > 0$, the wealth process between equidistant trading dates is given as

$$V_t = \left[(V_{t_k}^* - mC_{t_k}^*)\frac{B_t}{B_{t_k}} + mC_{t_k}^* \frac{S_t}{S_{t_k}}\right],$$

whereas for $C_{t_k} \leq 0$,

$$V_t = V_{t_k}\frac{B_t}{B_{t_k}},$$

for $t \in (t_k, t_{k+1})$, $k = 0, 1, .., n-1$.

4. Implementation and numerical results

In order to indicate the influence of the cushion option on CPPI-DC plans under certain assumptions, we perform simulations because the realization of real data requires long years and a wide range of market conditions, DC plan, and country-specific characteristics may also change over the years. By means of 10000 Monte Carlo simulations, the CPPI strategy with cushion option and without cushion option is generated and the performance levels of both strategies are investigated under various market and DC plan assumptions. All analyses and runs are coded using MATLAB.

The simulation scenarios based on many factors are expected to expose sensitivity with respect to varying parameter values. As exotic options can be utilized for short-term leverage in financial markets, their contributions are evaluated in two basic time frames:

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Table 1. Parameter values under discrete-time trading setting

Parameter	Value
Interest rate, r	0.03
Stock parameters	
Drift, μ_S	0.12
Volatility, σ_S	0.3
Labor income parameters	
Drift, μ_L	0.06
Volatility, σ_L	0.09
Contribution rate, γ	0.1
Guarantee rate, c	0.8
Multiplier, m	8
Time horizon , T (years)	3, 20

Table 2. The final wealth of CPPI strategies for long and short terms

Cushion option	$T = 20$ years		$T = 3$ years	
With	74.424	169.211	5.227	2.619
Without	39.370	40.503	4.195	1.431

(i) short-term which corresponds to the generally assumed earliest time to terminate the pension plan; $T = 3$ years

(ii) long-term at which the participants fulfill the retirement conditions; $T = 20$ years.

The comparisons with respect to time frames and the parameters are performed based on the value of the fund with and without cushion options. The steps of the proposed approach start with setting up arbitrary fixed values for the contributing parameters. The algorithm then calculates the price of the cushion option for the strike price at each fixed trading date. The next step considers a CPPI scheme for each fixed trading date with and without cushion options for strike price and finds the wealth for each case. The process ends with a comparison of the setup with respect to the cushion option and different values of the contributing parameters.

Portfolio strategies are simulated using the set of parameters summarized in Table 1. The parameter values are chosen to be the same as in Temocin *et al.* (2017) in order to have a benchmark except for some adjustments in the interest rate, r, and market volatility, σ_S, to expose the influence of the cushion-option setup on fund's development. The choice of 3 years for short-term is made with respect to the most commonly official withdrawal duration set by the life and pension insurance companies.

Table 2 shows the mean and standard deviation of the final wealth for the CPPI strategy without and with the cushion option for a period of 3 years and a period of 20 years. According to the results, we see that cushion option is a profitable choice for both periods. Mean, $E(V_T)$, and standard deviation, $\sigma(V_T)$, of final wealth for the CPPI strategy under the cushion option are higher than the mean and standard deviation of final wealth for the CPPI strategy. Considering that downside protection is provided by the CPPI strategy, the higher standard deviation indicates the longer right tail of terminal wealth distribution, and therefore it is considered a positive performance indicator, despite its more usual role as a negative performance indicator.

In Table 3, we illustrate the sensitivities with respect to the parameters for the time frames $T = 20$ years and $T = 3$ years. We see from the results of the simulations that the higher market volatility, σ_S, has a positive effect on moments of terminal wealth for without cushion option CPPI strategy in the 3-year time period. However, we cannot observe accurate information about its effect on the terminal wealth for the CPPI strategy under cushion option for the 20-year time period. An increase in the market drift, μ_S, raises the moment of the terminal wealth for CPPI strategies in both time periods.

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Table 3. CPPI portfolio with and without cushion option under certain parameter values (T=20 years and T=3 years)

		CPPI ($T=20$)				CPPI ($T=3$)			
		With cushion option		Without cushion option		With cushion option		Without cushion option	
Parameter	Value	$\mathrm{E}(\mathrm{V}_\mathrm{T}^\mathrm{c})$	$\sigma(\mathrm{V}_\mathrm{T}^\mathrm{c})$	$\mathrm{E}(\mathrm{V}_\mathrm{T})$	$\sigma(\mathrm{V}_\mathrm{T})$	$\mathrm{E}(\mathrm{V}_\mathrm{T}^\mathrm{c})$	$\sigma(\mathrm{V}_\mathrm{T}^\mathrm{c})$	$\mathrm{E}(\mathrm{V}_\mathrm{T})$	$\sigma(\mathrm{V}_\mathrm{T})$
	0.2	38.394	29.768	34.356	19.755	4.110	1.019	4.124	1.029
σ_S	0.3	74.423	169.210	39.370	40.502	5.227	2.619	4.195	1.431
	0.4	45.398	83.246	48.508	92.320	5.301	3.616	4.293	1.908
	0.06	71.330	160.797	38.697	39.296	5.024	2.463	4.096	1.348
μ_S	0.12	74.423	169.210	39.370	40.502	5.227	2.619	4.195	1.431
	0.18	77.421	178.493	40.212	42.648	5.448	2.782	4.303	1.516
	0.045	72.147	155.243	38.378	35.997	5.212	2.445	4.181	1.291
σ_L	0.09	74.423	169.210	39.370	40.502	5.227	2.619	4.195	1.431
	0.18	82.888	218.635	44.001	57.527	5.284	3.036	4.246	1.772
	0.03	73.343	167.222	38.857	40.053	5.154	2.591	4.134	1.414
μ_L	0.06	74.423	169.210	39.370	40.502	5.227	2.619	4.195	1.431
	0.12	76.613	173.788	40.462	41.397	5.376	2.675	4.320	1.466
	0.7	91.030	193.875	40.877	46.865	5.717	2.868	4.230	1.543
c	0.8	74.423	169.210	39.370	40.502	5.227	2.619	4.195	1.431
	0.9	55.394	119.906	37.640	31.330	4.682	2.059	4.131	1.150
	0.05	37.212	84.605	19.685	20.251	2.613	1.309	2.097	0.716
γ	0.1	74.423	169.210	39.370	40.502	5.227	2.619	4.195	1.431
	0.2	148.848	338.421	78.740	81.006	10.454	5.238	8.389	2.862
	2	38.512	32.223	38.512	32.223	4.109	0.893	4.109	0.893
m	6	74.423	169.210	39.370	40.502	5.227	2.619	4.195	1.431
	10	38.216	41.701	39.475	41.780	4.841	2.357	4.207	1.479

A rise on labor income drift, μ_L, as well as labor income volatility, σ_L, increases the moments of terminal wealth for CPPI strategies in both time horizon. Increasing the guarantee rate, c, resulting in the higher floor, has a negative effect on the moments of both strategies. That is, when the smaller amount of wealth is invested in the risky asset, we take fewer potential gains on the upward market move. CPPI with cushion option yields higher average return compared to no cushion when the contribution rate, γ, increases. When more amount of money is invested in the risky asset, investor can take more gains from increasing market. Higher multiplier, m, yields the moments of CPPI portfolio to be slightly higher in both time periods, but on the other hand higher m decreases moments of CPPI strategy with cushion option in both time horizon. The gap risk increases in the case of the high multiplier, m, and therefore the value of cushion option depending on gap risk raises. As expected, the high price of cushion option lowers the profits. Inversely, smaller multiplier, m, decreases the gap risk in line with the price of cushion option. For a small multiplier such as $m=2$, the cushion option has no significant effect on the CPPI strategy. Note that the initial value for the portfolio is taken to be 0.1 in Tables 2 and 3.

Figure 1 illustrates the portfolio trajectories of both strategies, highlighting their evolution over time. The figure compares the performance across two selected investment horizons. Taking a glance at the graphs, one can see that the portfolio value under the cushion option initially stands below the portfolio value of the CPPI strategy due to hedging cost for both horizons. When a sudden decrease occurs, portfolio value recovers quickly by means of cushion option and makes benefits from increasing in the market. However, portfolio value for the CPPI strategy needs time to recover and cannot make use of an advantage of potential gains on the upward move in the market.

Figures 2 and 3 illustrate the tail properties and the distributional behavior of terminal portfolio value for both strategies with respect to different time frames. As clearly seen, the CPPI strategy under cushion option has a longer right tail in the value of terminal wealth compared to the other case. This illustrative distributional behavior of both strategies supports the results depicted in Table 2.

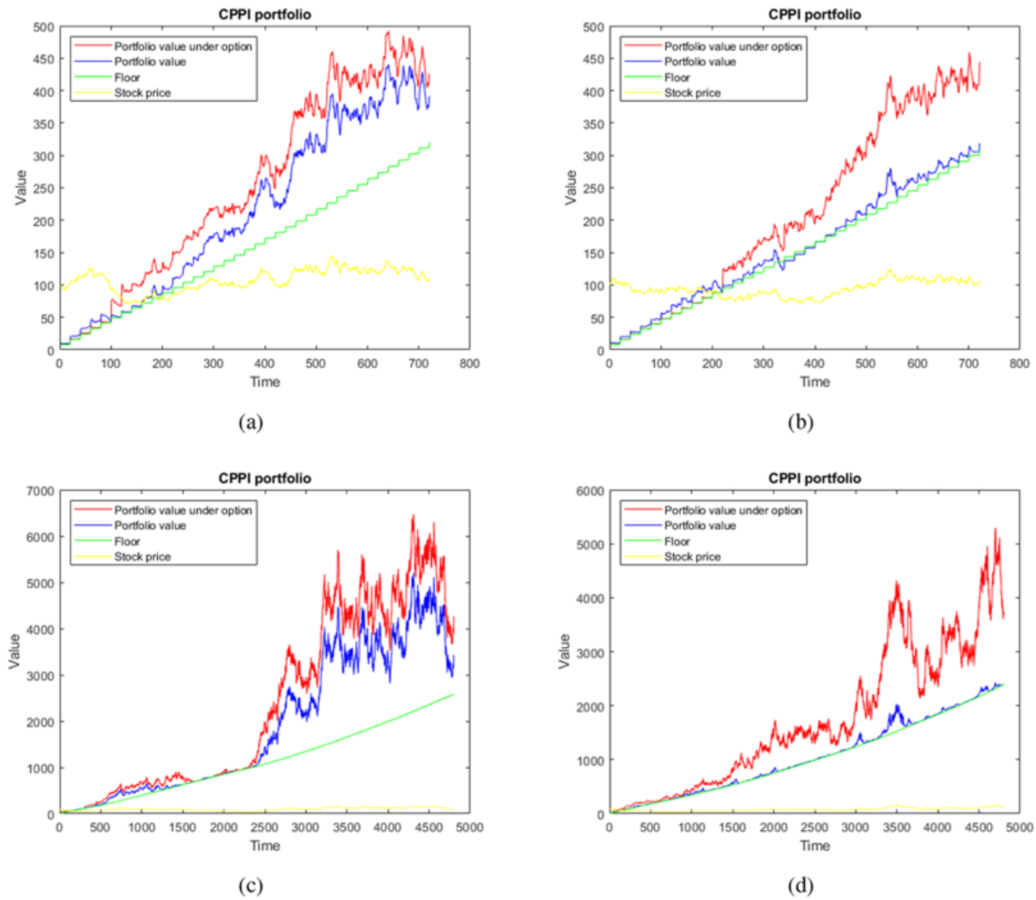

Figure 1. Illustrative trajectories of CPPI portfolio-scheme based on the parameters chosen in Table 1 in the 3-year time period (a and b) and 20-year time period (c and d).

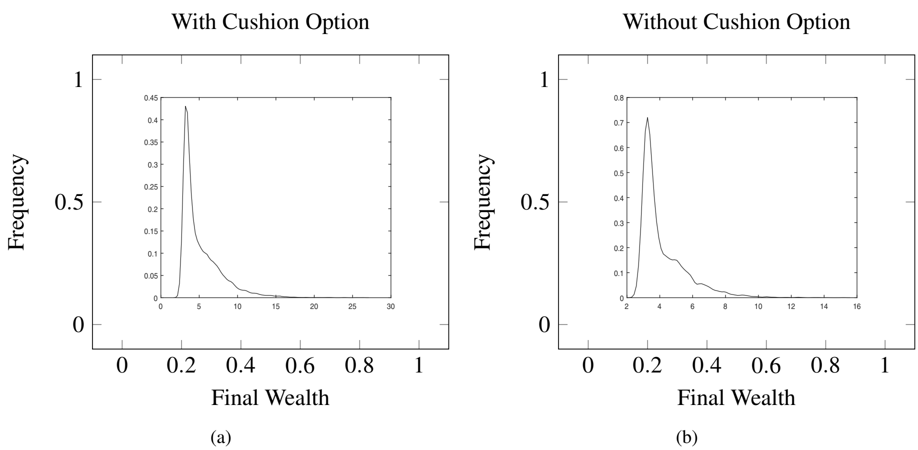

Figure 2. Short-term estimated kernel densities of final wealth for CPPI.

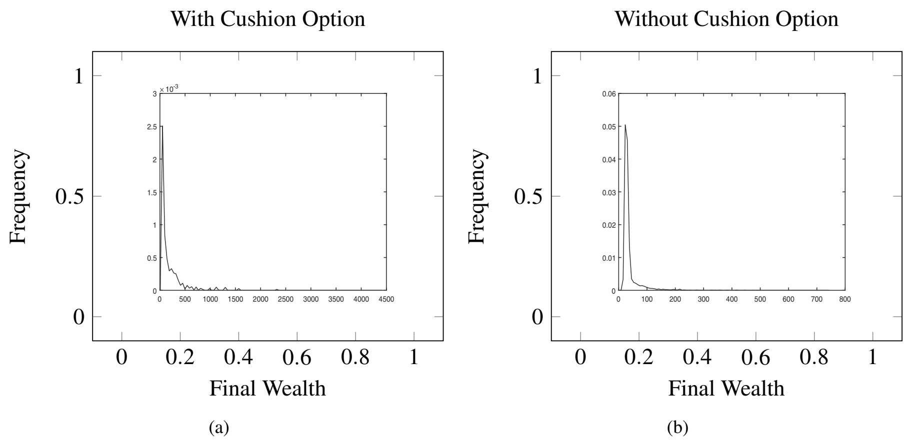

Figure 3. Long-term estimated kernel densities of final wealth for CPPI.

5. Concluding comments

Fund management for DC pension plans is critical for both individuals and social security systems. In this context, the literature explores portfolio insurance as a means to provide downside protection for portfolio values. The CPPI strategy with the random floor applied to DC pension plan under discrete-time trading setting allows us to foresee the development of a guaranteed fund. Considering trading only takes place at predefined dates, the portfolio value can drop below the floor between trading dates. Additionally, contribution at trading dates may not be enough to push up the floor. Therefore, in this paper, we propose a cushion option to reduce the gap risk and derive the price of option as well as the wealth process accordingly. By means of Monte Carlo simulations, the effectiveness of the cushion option on the CPPI strategy is tested by comparing portfolio performance for the CPPI strategy with and without cushion option. To support the effectiveness of cushion option, kernel densities are estimated for terminal wealth of both strategies.

This paper contributes to the valuation of DC plan funds by explicitly pricing a cushion option and demonstrating its utilization through numerical analysis. Monte Carlo simulation results indicate that the CPPI strategy incorporating a cushion option outperforms the standard CPPI strategy over both short-term (e.g., 3-year) and long-term (e.g., 20-year) horizons, as evidenced by higher moments of terminal wealth.

Even though the CPPI strategy provides downside protection, a sudden decline in the market may cause the portfolio value to drop below the floor. If the remaining time before termination date is not enough to recover the portfolio value, the investor faces the risk (gap risk) that the portfolio value stays under the floor representing the acceptable minimum portfolio value. The cushion option which is introduced in this paper is shown to be a profitable choice for a DC plan participant who considers a risk of early withdrawal.

The findings show that for both short- and long-term pension periods, CPPI with cushion option increases the expected value of the wealth and reduces the standard deviation especially for the long-term period. This proposed approach can be utilized to depict the plausible investment strategies having certain guarantees in pension fund system. It should be noted that the market and income are assumed to have similar dynamics due to their close correlation in real life. The influence of market risk through inflation will have an real reduction on the value of the income. As a future avenue of research, we plan to generalize the salary model by accounting for imperfect correlation between labor income and financial markets. Another possible extension is to incorporate regime-switching dynamics to better capture the changing behavior of the economy over the long term. In addition to the generalizations in financial modeling, the stochastic modeling approach presented in this paper has potential applications beyond the domain of finance. Stochastic models, pricing approaches, and risk management techniques can be applied in various fields, such as management (resource allocation and decision-making), biology (population dynamics and genetic processes), environmental networks (sustainable resource management), and financial networks (systemic risk in interconnected institutions). Exploring these interdisciplinary connections could provide new avenues for future research.

Furthermore, our setup can be applied to the hybrid pension scheme for reducing funding shortfall which is prominent risk on calculation of reserve with respect to Solvency II framework by incorporating inflation and longevity risks.

References

Balder S, Brandl M and Mahayni A (2009) Effectiveness of CPPI strategies under discrete-time trading. *Journal of Economic Dynamics and Control* **33**, 204–220.

Baltas I, Dopierala L, Kolodziejczyk K, Szczepański M, Weber GW and Yannacopoulos AN (2022) Optimal management of defined contribution pension funds under the effect of inflation, mortality and uncertainty. *European Journal of Operational Research* **298**, 1162–1174.

Baltas I, Szczepański M, Dopierala L, Kolodziejczyk K, Weber GW and Yannacopoulos AN (2018) Optimal pension fund management under risk and uncertainty: The case study of Poland. *Modeling, Dynamics, Optimization and Bioeconomics IV: DGS VI JOLATE, Madrid, Spain, May 2018, and ICABR, Berkeley, USA, May–June 2017–Selected Contributions*, Springer International Publishing, pp. 31–64.

Battochio P and Menoncin F (2004) Optimal pension management in a stochastic framework. *Insurance: Mathematics and Economics* **34**, 79–95.

Benzoni L, Collin–Dufresne P and Goldstein RS (2007) Portfolio choice over the life–cycle when the stock and labor markets are cointegrated. *The Journal of Finance* **62**, 2123–2167.

Bertrand P and Prigent J-L (2002) Portfolio insurance: the extreme value approach to the CPPI. *Finance* **23**, 68–86.

Bertrand P and Prigent J-L (2003) Portfolio insurance strategies: A comparison of standard methods when the volatility of the stock is stochastic. *International Journal of Business* **8**, 15–31.

Bertrand P and Prigent JL (2005) Portfolio insurance strategies: OBPI versus CPPI. *Finance* **26**, 5–32.

Black F and Jones R (1987) Simplifying portfolio insurance. *The Journal of Portfolio Management* **14**, 48–51.

Black F and Perold A (1992) Theory of constant proportion portfolio insurance. *Journal of Economic Dynamics and Control* **16**, 403–426.

Blake D, Cairns AJG and Dowd K (2003) Optimal dynamic asset allocation for defined-contribution pension plans. The Pension Institute, London, Discussion Paper PI.

Bodie Z, Merton RC and Samuelson WF (1992) Labor supply flexibility and portfolio choice in a life cycle model. *Journal of Economic Dynamics and Control* **16**, 427–449.

Boulier JF, Huang SJ and Taillard G (2001) Optimal management under stochastic interest rates: The case of a protected defined contribution pension fund. *Insurance: Mathematics and Economics* **28**, 173–189.

Cont R and Tankov P (2009) Constant proportion portfolio insurance in the presence of jumps in asset prices. *Mathematical Finance* **19**, 379–401.

Deelstra G, Grasselli M and Koehl PF (2003) Optimal investment strategies in the presence of a minimum guarantee. *Insurance: Mathematics and Economics* **33**, 189–207.

Duffie D and Kahn R (1996) A yield-factor model of interest rates. *Mathematical Finance* **6**, 379–406.

Gulveren A (2016) CPPI Strategy on Defined Contribution Pension Scheme under Cushion Option and Discrete Time Trading Setting. Ankara, Turkey: *Unpublished M.Sc. Thesis, Institute of Applied Mathematics, METU.*

Haberman S and Vigna E (2001) Optimal investment strategy for defined contribution pension schemes. *Insurance: Mathematics and Economics* **28**, 233–262.

Haberman S and Vigna E (2002) Optimal investment strategies and risk measures in defined contribution pension schemes. *Insurance: Mathematics and Economics* **31**, 35–69.

Horsky R (2012), Barrier Option Pricing and CPPI-Optimization, *Unpublished PhD Thesis*, TU Kaiserslautern, Germany.

Lacroze X and Paulot L (2011) One-dimensional pricing of CPPI. *Applied Mathematical Finance* **18**, 207–225.

Perold A (1986) *Constant Portfolio Insurance, Boston, MA: Manuscript, Harvard Business School.*

Pézier J and Scheller J (2013) Best portfolio insurance for long-term investment strategies in realistic conditions. *Insurance: Mathematics and Economics* **52**, 263–274.

Schweizer M (1995) Variance-optimal hedging in discrete time. *Mathematics of Operation Research* **20**, 1–32.

Shreve S (2004) *Stochastic Calculus for Finance II: Continuous-Time Models*, New York: Springer.

Tankov P (2010) Pricing and hedging gap risk. *Journal of Computational Finance* **13**, 33–59.
Temocin BZ, Korn R and Selcuk-Kestel AS (2017) Constant proportion portfolio insurance in defined contribution pension plan management. *Annals of Operations Research* **266**(1), 329–348.
Temocin BZ, Korn R and Selcuk-Kestel AS (2018) Constant proportion portfolio insurance in defined contribution pension plan management under discrete-time trading. *Annals of Operations Research* **260**, 515–544.
Vasiček OA (1977) An equilibrium characterization of the term structure. *Journal of Financial Economics* **5**, 177–188.
Zagst R and Kraus J (2011) Stochastic dominance of portfolio insurance strategies OBPI versus CPPI. *Annals of Operations Research* **185**, 75–103.

Sketch of the proof of Proposition 1 (Gulveren, 2016)

$$P_{t_k} = E^Q \left[e^{-r\triangle t} \zeta \left(K^* - S_{t_{k+1}}^{\frac{\sigma_L}{\sigma_S}} \right)^+ \mathbb{1}_{\left\{ \frac{S_{t_{k+1}}}{S_{t_k}} < (1-\frac{1}{m}) e^{r\triangle t} \right\}} \right],$$

where Q is a risk neutral measure, under which the discounted stock price $e^{-rt}S_t$ is a martingale. Therefore,

$$P_{t_k} = \int_{-\infty}^{\infty} e^{-r\triangle t} \zeta \left(K^* - S_{t_{k+1}}^{\frac{\sigma_L}{\sigma_S}} \right)^+ \mathbb{1}_{\left\{ \frac{S_{t_{k+1}}}{S_{t_k}} < (1-\frac{1}{m}) e^{r\triangle t} \right\}} \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} dz. \tag{A1}$$

In order to get payoff, two conditions (A and B) need to be satisfied:

Condition A:

$$K^* - S_{t_{k+1}}^{\frac{\sigma_L}{\sigma_S}} > 0$$

$$K^* > S_{t_k}^{\frac{\sigma_L}{\sigma_S}} e^{\frac{\sigma_L}{\sigma_S}(r-\sigma_S^2/2)\triangle t + \sigma_L (W_{t_{k+1}}^Q - W_{t_k}^Q)}$$

$$ln \left(\frac{K^*}{S_{t_k}^{\frac{\sigma_L}{\sigma_S}}} \right) > \frac{\sigma_L}{\sigma_S} \left(r - \frac{\sigma_S^2}{2} \right) \triangle t + \sigma_L (W_{t_{k+1}}^Q - W_{t_k}^Q)$$

$$\frac{Z_{\triangle t}}{\sqrt{\triangle t}} < \frac{ln \left(\frac{K^*}{S_{t_k}^{\frac{\sigma_L}{\sigma_S}}} \right) - \frac{\sigma_L}{\sigma_S}(r - \frac{\sigma_S^2}{2})\triangle t}{\sigma_L \sqrt{\triangle t}} := A, \tag{A2}$$

where $Z_{\triangle t} = W_{t_{k+1}}^Q - W_{t_k}^Q$ and $\frac{Z_{\triangle t}}{\sqrt{(\triangle t)}} = z \sim N(0, 1)$.

Condition B:

$$\frac{S_{t_{k+1}}}{S_{t_k}} < (1 - \frac{1}{m}) e^{r\triangle t}$$

$$\frac{S_{t_k} e^{(r-\sigma_S^2/2)\triangle t + \sigma_S (W_{t_{k+1}}^Q - W_{t_k}^Q)}}{S_{t_k}} < (1 - \frac{1}{m}) e^{r\triangle t}$$

$$(r - \sigma_S^2/2)\triangle t + \sigma_S (W_{t_{k+1}}^Q - W_{t_k}^Q) < ln \left(\frac{m-1}{m} \right) + r\triangle t$$

$$\frac{Z_{\triangle t}}{\sqrt{\triangle t}} < \frac{ln \left(\frac{m-1}{m} \right) + (\sigma_S^2/2)\triangle t}{\sigma_S \sqrt{\triangle t}} := B, \tag{A3}$$

where $Z_{\triangle t} = W_{t_{k+1}}^Q - W_{t_k}^Q$ and $\frac{Z_{\triangle t}}{\sqrt{(\triangle t)}} = z \sim N(0, 1)$.

Under these conditions:

$$P_{t_k} = \int_{-\infty}^{B} e^{-r\triangle t}\zeta \left(K^* - S_{t_{k+1}}^{\frac{\sigma_L}{\sigma_S}}\right)^+ \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} dz$$

$$= \int_{-\infty}^{min(A,B)} \zeta^* \left(K^* - S_{t_{k+1}}^{\frac{\sigma_L}{\sigma_S}}\right) \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} dz,$$

where

$$\zeta^* = e^{-r\triangle t}\zeta$$

$$= \int_{-\infty}^{min(A,B)} \zeta^* \left(K^* - S_{t_k}^{\frac{\sigma_L}{\sigma_S}} e^{\frac{\sigma_L}{\sigma_S}(r-\sigma_S^2/2)\triangle t + \sigma_L(W_{t_{k+1}}^Q - W_{t_k}^Q)}\right) \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} dz$$

$$P_{t_k} = \zeta^* K^* \Phi\left(\min(A,B)\right) - \zeta^* S_{t_k}^{\frac{\sigma_L}{\sigma_S}} e^{\frac{\sigma_L}{\sigma_S}(r-\sigma_S^2/2)\triangle t} \int_{-\infty}^{min(A,B)} \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2 - 2\sigma_L z\sqrt{\triangle t}}{2}} dz$$

adding and substracting $\frac{\sigma_L^2 \triangle t}{2}$, we deduce

$$P_{t_k} = \zeta^* K^* \Phi\left(\min(A,B)\right) - \zeta^* S_{t_k}^{\frac{\sigma_L}{\sigma_S}} e^{\left[\frac{\sigma_L}{\sigma_S}(r-\sigma_S^2/2)+\sigma_L^2/2\right]\triangle t} \int_{-\infty}^{min(A,B)} \frac{1}{\sqrt{2\pi}} e^{-\frac{(z-\sigma_L\sqrt{\triangle t})^2}{2}} dz.$$

By change of variable $y = z - \sigma_L^2\sqrt{\triangle t}$,

$$P_{t_k} = \zeta^* K^* \Phi\left(\min(A,B)\right) - \zeta^* S_{t_k}^{\frac{\sigma_L}{\sigma_S}} e^{\left[\frac{\sigma_L}{\sigma_S}(r-\sigma_S^2/2)+\sigma_L^2/2\right]\triangle t} \Phi\left(\min(A,B) - \sigma_L \triangle t\right).$$

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