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CONTRACTIVE REPRESENTATION THEORY FOR THE UNITARY GROUP OF $C(X, M_2)$

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1. Introduction. One motivation for studying representation theory for the unitary group $G = U(\mathfrak{A})$ of a unital C^*-algebra $\mathfrak{A}$ arises from Theoretical Physics. (In the latter connection, Segal **[9]** and Arveson **[1]** have developed a representation theory for G. Their approach is in a different direction from ours.) Another motivation for studying the representation theory of G arises out of the desire to unify the theories of amenable von Neumann algebras and amenable locally compact groups.

A serious problem for such a representation theory is the absence of Haar measure on G in general.

In [7], the author introduced the class $\text{Rep}_d\, G$ of contractive unitary representations of G, the strong metric condition involved compensating for the lack of Haar measure. A unitary representation π of G on a Hilbert space $\mathfrak{H}$ is said to be contractive if

$$\|\pi(u) - \pi(v)\| \leqq d(u, v)(=\|u - v\|) \quad \text{for all } u, v \in G$$

(or equivalently, $\|\pi(u) - 1\| \leqq d(u, 1)$ for all $u \in G$). If ϕ is a *-representation of $\mathfrak{A}$ on a Hilbert space, then

$$\phi_{|_G} \in \text{Rep}_d\, G.$$

However, in general, there are many elements of $\text{Rep}_d\, G$ not arising from such a restriction.

An important good property of $\text{Rep}_d\, G$ is that its elements can be "disintegrated" into irreducible contractive representations, so that the study of $\text{Rep}_d\, G$ reduces to that of $\hat{G}_d$, the set of equivalence classes of irreducible elements in $\text{Rep}_d\, G$. A subset of $\hat{G}_d$ is $\hat{G}_{\mathfrak{A}}$, the set of restrictions to G of the elements of $\hat{\mathfrak{A}}$. It is obvious that

$$\hat{G}_d \supset \hat{G}_{\mathfrak{A}} \cup \{1\} \cup (\hat{G}_{\mathfrak{A}})^{\sim},$$

where $\sim$ is the conjugation operation.

A natural question is that of determining $\hat{G}_d$ for various classes of C^*-algebras. In [7], this question is answered for two such classes: the class of commutative C^*-algebras and the class of AW^*-algebras (which, of

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course, contains the class of von Neumann algebras). The answers in the two cases are as follows.

THEOREM A. *Let X be a compact, Hausdorff space and $\mathfrak{A} = C(X)$. Let $\mathscr{S}$ be the family of open and closed subsets of X and $S_{\mathscr{S}}(X)$ be the set of probability measures μ on X such that $\mu(\mathscr{S}) = \{0, 1\}$. Then*

$$(*) \quad \hat{G}_d = (S_{\mathscr{S}}(X) \cup \{0\} \cup -S_{\mathscr{S}}(X)) \times H^1(X, \mathbf{Z})^{\wedge}.$$

The equality (*) is interpreted as follows. We can express G as a direct product $G_e \times K$, where G_e is the identity component of G. Clearly, K can be identified with

$$G/G_e = H^1(X, \mathbf{Z}),$$

and if

$$\mu \in S_{\mathscr{S}}(X) \cup \{0\} \cup -S_{\mathscr{S}}(X) \quad \text{and} \quad \gamma \in H^1(X, \mathbf{Z})^{\wedge},$$

we obtain an element $\alpha_{\mu,\gamma}$ of $\hat{G}_d$ by setting:

$$\alpha_{\mu,\gamma}((e^{ig}, k)) = e^{i\mu(g)}\gamma(k)$$

($g \in C(X, \mathbf{R})$, $k \in K$). The theorem asserts that the map $(\mu, \gamma) \rightarrow \alpha_{\mu,\gamma}$ is a bijection onto $\hat{G}_d$. Note that

$$(G_e)^{\wedge}_d = (S_{\mathscr{S}}(X) \cup \{0\} \cup -S_{\mathscr{S}}(X)).$$

In this case, $\hat{G}_{\mathfrak{A}} \cup \{1\} \cup (\hat{G}_{\mathfrak{A}})^{\sim}$ is identified with $(X \cup \{0\} \cup -X) \times \{1\}$, where if $x \in X$, x is identified with the point mass δ_x and $-x$ with $-\delta_x$. Clearly, $\hat{G}_d$ is much larger than $\hat{G}_{\mathfrak{A}} \cup \{1\} \cup (\hat{G}_{\mathfrak{A}})^{\sim}$ in general.

THEOREM B. *Let $\mathfrak{A}$ be an AW^*-algebra. Then*

$$\hat{G}_d = \hat{G}_{\mathfrak{A}} \cup \{1\} \cup (\hat{G}_{\mathfrak{A}})^{\sim}.$$

This result shows that, in the AW^*-case, contractive representation theory for G is equivalent to representation theory for $\mathfrak{A}$.

With Theorem A in mind, a natural next step is to investigate $\hat{G}_d$ in the case where $\mathfrak{A} = C(X, M_2)$, the algebra of 2×2 matrices with entries in $C(X)$. In this case,

$$G = C(X, U(2)).$$

The determination of $\hat{G}_d$ seems to be substantially more difficult than the corrresponding determination of Theorem A. The difficulties come from two directions: the first is Lie theoretic and the second is algebraic topological.

The theorem of this paper, which we now state, determines $\hat{G}_d$ subject to certain strong topological conditions on X.

THEOREM. *Let X be a connected, compact, CW-complex of dimension $\leqq 2$ and with $H^1(X, \mathbf{Z}) = \{0\}$. Then $\hat{G}_d$ is canonically isomorphic to*

$$X \cup \{1\} \cup -X.$$

The above isomorphism is interpreted as follows. Each $x \in X$ is identified with the representation $f \to f(x)$ of G, while $-x$ is identified with the conjugate representation $f \to \overline{f(x)}$. Of course, 1 is the trivial representation.

The topological conditions of the theorem essentially reduce the proof to the determination of the set of norm-decreasing, irreducible representations of the Banach-Lie algebra $C(X, su(2))$. This set $\hat{\mathfrak{G}}_d$ is determined in Section 2.

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2. Determination of $C(X, su(2))^{\wedge}_d$. Let G be a Banach-Lie group with metric d. The set of irreducible, unitary, contractive representations of G on a Hilbert space is denoted by $\hat{G}_d$. The class of all contractive, unitary representations of G on a Hilbert space is denoted by $\text{Rep}_d\, G$. The Hilbert space of a representation π of G is often denoted by $\mathfrak{H}_\pi$.

Associated with G is its Banach-Lie algebra $\mathfrak{G}$. (A reference for facts about Banach-Lie groups and Banach-Lie algebras is **[2]**.) In this paper, G will be a closed subgroup of $U(\mathfrak{A})$ for some C^*-algebra $\mathfrak{A}$, and the metric d on G will be that inherited from the norm of $\mathfrak{A}$. The Banach-Lie algebra $\mathfrak{G}$ of G can then be identified with the obvious Lie subalgebra of the algebra $\text{Sk}(\mathfrak{A})$ of skew hermitian elements of $\mathfrak{A}$: thus $k \in \mathfrak{G}$ if and only if $e^{tk} \in G$ for all $t \in \mathbf{R}$.

The class of all norm-decreasing, Lie homomorphisms α from $\mathfrak{G}$ into Sk $\mathfrak{H}(= \text{Sk}\, B(\mathfrak{H}))$, where $\mathfrak{H}$ is a Hilbert space, is denoted by $\text{Rep}_d\, \mathfrak{G}$. The set of equivalence classes of irreducible elements of $\text{Rep}_d\, \mathfrak{G}$ is denoted by $\hat{\mathfrak{G}}_d$. If $\pi \in \text{Rep}_d\, G$, then it is easy to see that its differential $d\pi$ at e belongs to $\text{Rep}_d\, \mathfrak{G}$. If G is connected and $\pi \in \hat{G}_d$, then clearly $d\pi \in \hat{\mathfrak{G}}_d$. The trivial representations of G and $\mathfrak{G}$ are denoted by 1 and 0 respectively. Clearly, $1 \in \hat{G}_d$ and $0 \in \hat{\mathfrak{G}}_d$.

Let X be a compact, Hausdorff space. The groups G that we are interested in here are $SU(2)$ and $U(2)$ (where $\mathfrak{A} = M_2$) and $C(X, SU(2))$ and $C(X, U(2))$ (where $\mathfrak{A} = C(X, M_2)$). In these cases, $\mathfrak{G}$ is $su(2)$, $u(2)$, $C(X, su(2))$ and $C(X, u(2))$ respectively.

Our aim in this section is to determine what $C(X, su(2))_d^\wedge$ is. We require the following simple proposition involving $SU(2)$ and $su(2)$. Let

$$\pi_2 : SU(2) \rightarrow U(M_2) \quad \text{and} \quad \alpha_2 : su(2) \rightarrow \text{Sk } M_2$$

be the identity representations. Of course, $d\pi_2 = \alpha_2$.

PROPOSITION 1. (i) *If* $\alpha : su(2) \rightarrow \text{Sk } \mathfrak{H}$ *belongs to* $\text{Rep}_d\, su(2)$, *then there exists a norm-continuous homomorphism*

$$\pi : SU(2) \rightarrow U(B(\mathfrak{H}))$$

such that $d\pi = \alpha$.

(ii) $su(2)_d^\wedge = \{0, \alpha_2\}$ *and* $SU(2)_d^\wedge = \{1, \pi_2\}$.
(iii) *If* $\alpha \in \text{Rep}_d\, su(2)$, *then there exists* $\pi \in \text{Rep}_d SU(2)$ *such that* $d\pi = \alpha$.

Proof. (i) (This is improved in (iii) below.) It is routine that there exists a norm-continuous local homomorphism π' on a neighbourhood of e in $SU(2)$ such that $d\pi' = \alpha$. Now use the simple-connectedness of $SU(2)$ to extend π' to the desired homomorphism π.

(ii) Obviously, $\{0, \alpha_2\} \subset su(2)_d^\wedge$. Conversely, let $\alpha \in su(2)_d^\wedge$, and $\mathfrak{H}$ be the Hilbert space of α. By (i), we can find a unitary representation π of $SU(2)$ on $\mathfrak{H}$ with $d\pi = \alpha$. Clearly, π is irreducible, and since $SU(2)$ is compact, $\mathfrak{H}$ is finite-dimensional. So

$$\alpha \in \{\alpha_{2l+1} : l \geqq 0,\ l \text{ or } 2l \text{ belongs to } \mathbf{Z}\},$$

the standard enumeration of $su(2)^\wedge$ (e.g. [**12**], p. 92). Let $\{Z_1, Z_2, Z_3\}$ be the standard basis for $su(2)$: so

$$(1) \quad Z_1 = \frac{1}{2}\begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}, Z_2 = \frac{1}{2}\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, Z_3 = \frac{1}{2}\begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}.$$

Of course, $[Z_1, Z_2] = Z_3$, $[Z_2, Z_3] = Z_1$ and $[Z_3, Z_1] = Z_2$. A highest-weight norm one vector ξ_l for α_l satisfies the equality:

$$i\alpha_l(Z_3)\xi_l = l\xi_l.$$

If $\alpha = \alpha_l$, then

$$l = \|\alpha(Z_3)\xi_l\| \leqq \|\alpha(Z_3)\| \leqq \|Z_3\| = 1$$

so that $l \leqq 1$, i.e., $l \in \left\{0, \dfrac{1}{2}\right\}$. So $\alpha \in \{0, \alpha_2\}$. So $su(2)_d^\wedge = \{0, \alpha_2\}$, and the corresponding result for $SU(2)_d^\wedge$ immediately follows.

(iii) Let $\alpha \in \text{Rep}_d\, su(2)$ and π be as in (i). Then π is a direct sum of irreducible representations π_δ of $SU(2)$. If α is decomposed into the corresponding direct sum of representations α_δ, then $\alpha_\delta \in su(2)_d^\wedge$, and since $d\pi_\delta = \alpha_\delta$, we have, by (ii), that

$$\pi_\delta \in \{0, \pi_2\} = SU(2)^\wedge_d.$$

Hence $\pi \in \mathrm{Rep}_d\, G$.

Before proceeding with determining $C(X, su(2))^\wedge_d$, it will be helpful, for motivation, to consider the corresponding associative version for $C(X, M_2)^\wedge$. The straight-forward procedure, due to Naimark and Fell ([**6**], p. 337 and [**4**], Theorem 1.1), can be summarised as follows. Let $\mathfrak{A} = C(X, M_2)$ and $\Phi \in \hat{\mathfrak{A}}$. Let E, F be disjoint, compact subsets of X and $\mathfrak{A}_E$, $\mathfrak{A}_F$ be the ideals of functions $f \in \mathfrak{A}$ which vanish off E and F respectively. Now $\Phi(\mathfrak{A}_E)$ and $\Phi(\mathfrak{A}_F)$ are ideals in the irreducible algebra $\Phi(\mathfrak{A})$, and

$$\Phi(\mathfrak{A}_E)\Phi(\mathfrak{A}_F) = \{0\};$$

so one or other of $\Phi(\mathfrak{A}_E)$, $\Phi(\mathfrak{A}_F)$ is $\{0\}$. It then follows that the set

$$\{x \in X : \Phi(\mathfrak{A}_E) \neq \{0\} \text{ for every compact neighbourhood } E \text{ of } x\}$$

is a singleton $\{x_0\}$, and that Φ is of the form (x_0, π) where $\pi \in \hat{M}_2$ and

$$(x_0, \pi)(f) = \pi(f(x_0)).$$

Thus $\hat{\mathfrak{A}} = X \times \hat{M}_2$.

Adapting this argument with $\mathfrak{A}$ replaced by $\mathfrak{G} = C(X, su(2))$, we obtain, in an obvious notation, that $[\mathfrak{G}_E, \mathfrak{G}_F] = \{0\}$ whenever E and F are disjoint, compact subsets of X. Unfortunately, for $\alpha \in \hat{\mathfrak{G}}_d$, we then have

$$[\alpha(\mathfrak{G}_E), \alpha(\mathfrak{G}_F)] = \{0\}$$

rather than $\alpha(\mathfrak{G}_E)\alpha(\mathfrak{G}_F) = \{0\}$, i.e., the elements of $\alpha(\mathfrak{G}_E)$ are only known to commute with those of $\alpha(\mathfrak{G}_F)$, and the above argument breaks down.

To overcome this difficulty, we will extend $\mathfrak{G}$ to a larger Lie algebra $B(X, su(2))$ of $su(2)$-valued functions on X. (This part of the argument is reminiscent of a proof of the spectral theorem.) Using characteristic functions, we can produce contractive representations of $su(2)$ associated with α. Using Proposition 1 and a tensor product argument, we will then show that we do in fact have

$$\alpha(\mathfrak{G}_E)\alpha(\mathfrak{G}_F) = \{0\},$$

and the associative argument can then be pushed through.

Let $B(X)$ be the algebra of bounded, real-valued functions on X which are pointwise limits of sequences in $C(X)$. Thus $f \in B(X)$ if and only if there exists a sequence $\{f_n\}$ in $C(X)$ such that $f_n \to f$ pointwise on X. Clearly, we can always suppose that

$$\|f_n\| \leqq \|f\| \quad \text{for all } n,$$

and that if $f \geqq 0$, then $0 \leqq f_n$ for all n. Suppose that f belongs to the norm closure of $B(X)$. Then we can write

$$f = \sum_{n=1}^{\infty} g_n$$

where $g_n \in B(X)$ and

$$\sum_{n=1}^{\infty} \|g_n\| < \infty,$$

and approximating the g_n by continuous functions then gives that $f \in B(X)$. So $B(X)$ is a commutative real C^*-algebra.

Let

$$\mathscr{B}(X) = \{E \subset X: \chi_E \in B(X)\}.$$

Clearly $\mathscr{B}(X)$ is an algebra of sets, and contains the family of compact, G_δ-subsets of X.

We define $B(X, su(2))$ to be the set of 2×2 matrix-valued functions of the form

$$\begin{bmatrix} if_1 & if_2 - f_3 \\ if_2 + f_3 & -if_1 \end{bmatrix} \quad (f_1, f_2, f_3 \in B(X)).$$

(Note that $C(X, su(2))$ can be defined in the same way: recall that

$$su(2) = \left\{ \begin{bmatrix} ib & id - e \\ id + e & -ib \end{bmatrix} : b, d, e \in \mathbf{R} \right\}.$$

Clearly, $B(X, su(2))$ is a real Banach-Lie algebra under the sup norm, containing $C(X, su(2))$ as a closed subalgebra. It is easy to see that $B(X, su(2))$ is the space of bounded functions $F: X \to su(2)$, where F is the pointwise limit of a sequence of functions in $C(X, su(2))$. Since $B(X)$ is a commutative C^*-algebra, $B(X, su(2))$ is canonically identified with

$$B(X) \check{\otimes} su(2)$$

(injective tensor product norm). Of course, since $su(2)$ is finite-dimensional, $B(X) \check{\otimes} su(2)$ is, as a space, an algebraic tensor product (no completion necessary). If $f \in B(X)$ and $Z \in su(2)$ then $f \otimes Z$ is the function given by:

$$x \to f(x)Z \quad (x \in X).$$

We sometimes write fZ in place of $f \otimes Z$. Let

$$\mathfrak{G} = C(X, su(2)).$$

PROPOSITION 2. *Let $\alpha \in \mathrm{Rep}_d\, \mathfrak{G}$ and $\mathfrak{H}$ be the Hilbert space of α. Then there exists a norm continuous representation*

$$\beta: B(X, su(2)) \to \mathrm{Sk}\, \mathfrak{H}$$

such that:

(i) $\beta_{|\mathfrak{G}} = \alpha$;
(ii) $\|\beta(f \otimes Z)\| \leqq \|f \otimes Z\| (= \|f\|\, \|Z\|)$;

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(iii) *if* $\{f_n\}$ *is a bounded sequence in* $B(X, su(2))$, $f \in B(X, su(2))$ *and* $f_n \to f$ *pointwise on* X, *then* $\beta(f_n) \to \beta(f)$ *in the weak operator topology.*

Proof. Let $M(X)$ be the space of bounded, complex, regular Borel measures on X. For $\xi, \eta \in \mathfrak{H}$ and $Z \in su(2)$ there exists, by the Riesz representation theorem, a unique measure $\mu^Z_{\xi,\eta} \in M(X)$ such that

$$(\alpha(\phi \otimes Z)\xi, \eta) = \int \phi d\mu^Z_{\xi,\eta} \quad (\phi \in C(X)).$$

The map

$$(\xi, \eta) \to \mu^Z_{\xi,\eta}$$

is sesquilinear. For each $f \in B(X)$, define $\beta(f \otimes Z) \in \text{Sk } \mathfrak{H}$ by setting:

$$(2) \quad (\beta(f \otimes Z)\xi, \eta) = \int f d\mu^Z_{\xi,\eta}.$$

(Of course, every function in $B(X)$ is Borel.) Clearly (ii) holds.

The map $(f, Z) \to \beta(f \otimes Z)$ is bilinear, and so extends to a norm-decreasing map, also denoted by β, from the projective tensor product space $B(X) \hat{\otimes} su(2)$ into Sk $\mathfrak{H}$. Since $su(2)$ is finite-dimensional, the projective and injective norms are equivalent on $B(X) \otimes su(2)$. So β is continuous on $B(X, su(2))$.

Let f_n and f be as in (iii). Let $\{Z_1, Z_2, Z_3\}$ be as in (1); we can write

$$f_n = \sum_{i=1}^{3} f_n^i \otimes Z_i, \quad f = \sum_{i=1}^{3} f^i \otimes Z_i.$$

For each i, the sequence $\{f_n^i\}$ is bounded in $B(X)$, and $f_n^i \to f^i$ pointwise on X. The assertion of (iii) now follows using (2) and the dominated convergence theorem.

The fact that β is a Lie homomorphism follows from the corresponding fact for α together with (iii) and the separate continuity of multiplication in $B(\mathfrak{H})$ for the weak operator topology. (We use here the fact that if $g \in B(X, su(2))$, then there exists a bounded sequence $\{f_n\}$ in $C(X, su(2))$ converging to g pointwise.)

As above, let $\mathfrak{G} = C(X, su(2))$. Clearly $0 \in \hat{\mathfrak{G}}_d$. Further, if $x \in X$, then the pair $(x, \alpha_2) \in \hat{\mathfrak{G}}_d$, where we define

$$(x, \alpha_2)f = \alpha_2(f(x)).$$

Our next result shows that these account for the whole of $\hat{\mathfrak{G}}_d$.

PROPOSITION 3. $\hat{\mathfrak{G}}_d = (X \times \{\alpha_2\}) \cup \{0\}$.

Proof. Let $\alpha \in \hat{\mathfrak{G}}_d \sim \{0\}$. We must show that $\alpha \in X \times \{\alpha_2\}$. Let $\mathfrak{H}$ be the Hilbert space of α and let β be the extension to $B(X, su(2))$ given in Proposition 2.

Let $E, F \in \mathscr{B}(X)$ with $E \cap F = \emptyset$. We first show:

(3) $\beta(\chi_E su(2))\beta(\chi_F su(2)) = \{0\}$.

To this end, let $\beta_E, \beta_F \in \text{Rep}_d\, su(2)$ be given by:

$$\beta_E(Z) = \beta(\chi_E Z), \quad \beta_F(Z) = \beta(\chi_F Z).$$

From Proposition 1, β_E is a direct sum of representation β_E^i $(i \in I)$ in $su(2)_d^\wedge$. Further, for each i, $\beta_E^i \in \{0, \alpha_2\}$. Let

$$J = \{i \in I : \beta_E^i = \alpha_2\}$$

and $\mathfrak{H}_i$ be the space of β_E^i. Let

$$\mathfrak{H}_E = \bigoplus_{i \in J} \mathfrak{H}_i \quad \text{and} \quad \mathfrak{H}_0 = \mathfrak{H}_E^{\perp}.$$

Let β_E' be the restriction of β_E to $\mathfrak{H}_E$. Then β_E' is a direct sum of copies of α_2, $\beta_E(\mathfrak{H}_0) = \{0\}$ and $\beta_E = \beta_E' \oplus 0$. Suppose that $\mathfrak{H}_E \neq \{0\}$ (i.e., $\beta_E \neq 0$). Now $\mathfrak{H}_E$ can be identified with a tensor product $\mathbf{C}^2 \otimes \mathfrak{K}$, and with this identification, $\beta_E' = \alpha_2 \otimes 1$. Further, the commutant of $\beta_E'(su(2))$ is $1 \otimes B(\mathfrak{K})$ (c.f. [**8**], p. 187).

Now

$$[\chi_E su(2), \chi_F su(2)] = \{0\}$$

since $\chi_E \chi_F = 0$. Applying β, we see that $\beta_F(su(2))$ is contained in the commutant of $\beta_E(su(2))$. It follows that both $\mathfrak{H}_E$ and $\mathfrak{H}_0$ are $\beta_F(su(2))$ invariant. Let γ_F be the restriction of β_F to $\mathfrak{H}_E$. Since $\gamma_F(su(2))$ is contained in the commutant of $\beta_E(su(2))$, we can define an element $\delta_F \in \text{Rep}_d\, su(2)$ on $\mathfrak{K}$ by setting $\gamma_F = 1 \otimes \delta_F$. So δ_F is a direct sum of representations in $\{0, \alpha_2\}$. Suppose that $\delta_F \neq 0$. Then at least one α_2 occurs in this direct sum, and we can find $\eta \in \mathfrak{K}$, $\|\eta\| = 1$ such that

$$\delta_F(Z_3)\eta = \frac{1}{2} i\eta.$$

Pick $\xi \in \mathbf{C}^2$, $\|\xi\| = 1$ such that

$$\alpha_2(Z_3)\xi = \frac{1}{2} i\xi.$$

Recalling that $E \cap F = \emptyset$ and using Proposition 2, we have

$$\|\beta(\chi_E Z_3 + \chi_F Z_2)\| = \|\beta(\chi_{E \cup F} \otimes Z_3)\| \leqq \|Z_3\| = \frac{1}{2}.$$

So

$$\frac{1}{2} \geqq \|\beta((\chi_E + \chi_F)Z_3)(\xi \otimes \eta)\|$$

$$\geqq |((\beta_E(Z_3) + \beta_F(Z_3))\xi \otimes \eta, \xi \otimes \eta)|$$
$$= |(\alpha_2(Z_3)\xi \otimes \eta + \xi \otimes \delta_F(Z_3)\eta, \xi \otimes \eta)|$$
$$= 1.$$

This is a contradiction. So either $\mathfrak{H}_E = \{0\}$ or $\mathfrak{H}_E \neq \{0\}$ and $\gamma_F = 0$, and (3) immediately follows.

Now suppose that E and F are, in addition, compact, G_δ-subsets of X. For $\phi \in B(X)$, let

$$\phi_E = \phi\chi_E \in B(X) \quad \text{and} \quad B_E(X) = \{\phi_E: \phi \in B(X)\}.$$

Let $\phi \in B(X)$, $\psi \in B_E(X)$ and $\omega \in B_F(X)$. Let $Z, T \in su(2)$. We now claim:

(4) $\beta(\psi Z_1)\beta(\phi Z_2) = \beta(\psi Z_1)\beta(\phi_E Z_2)$,

(5) $\beta(\psi Z_1)\beta(\omega Z_2) = 0$.

To prove this, let $\xi, \eta \in \mathfrak{H}$ and $\eta' = \beta(\chi_E Z_1)^*\eta$. Let

$$\mu = \mu_{\xi,\eta'}^{Z_2},$$

in the notation of the proof of Proposition 2. Let C be a compact, G_δ-subset of $Y = X \sim E$. By (3), with $C = F$, we have $\mu(C) = 0$. It follows that $\mu_{|Y} = 0$, and hence that $\mu(\phi - \phi_E) = 0$. Thus

$$(\beta(\chi_E Z_1)(\beta(\phi Z_2) - \beta(\phi_E Z_2))\xi, \eta) = 0,$$

and so (4) is true in the case $\psi = \chi_E$. A similar argument, using compact G_δ-subsets of E, then establishes (4). The equality (5) follows from (4) by putting $\phi = \omega$.

Let $\mathfrak{A}$ be the C^*-subalgebra of $B(\mathfrak{H})$ generated by $\beta(B(X, su(2)))$, I_E be the closed subalgebra of $\mathfrak{A}$ generated by the set

$$\{B(\psi Z): \psi \in B_E(X), Z \in su(2)\}$$

and I_F be the corresponding subalgebra for F. From (4) and (5), both I_E and I_F are ideals in $\mathfrak{A}$, and $I_E I_F = \{0\}$. Since $\alpha \in \hat{\mathfrak{G}}_d$, $\mathfrak{A}$ is irreducible on $\mathfrak{H}$, and as in the associative version discussed earlier, either $I_E = \{0\}$ or $I_F = \{0\}$. Continuing along the same lines as this version, use a partition of unity argument together with the facts that distinct points x, y of X can be separated by disjoint, compact G_δ-neighbourhoods and that $\alpha \neq 0$ to obtain that there is exactly one point $x_0 \in X$ such that

$$\alpha(\{\phi_C: \phi \in C(X, su(2))\}) \neq \{0\}$$

for every compact G_δ-neighbourhood C of x_0. The continuity of α then gives that for $f \in C(X, su(2))$, $\alpha(f)$ depends only on the value of $f(x_0)$, and that there exists $\beta \in su(2)_d^\wedge$ such that $\alpha = (x_0, \beta)$ as required. Of course, $\beta = \alpha_2$.

3. Proof of the main theorem. Let

$$Q:C(X, SU(2)) \times C(X, \mathbf{T}) \rightarrow G$$

be given by:

$$Q(f, g)(x) = f(x)g(x).$$

Clearly, Q is a continuous homomorphism, and since X is connected,

$$\ker Q = \{ (I, 1), (-I, -1) \}.$$

Let d' be the metric on $C(X, SU(2)) \times C(X, \mathbf{T})$ given by:

$$d'((f, g), (\phi, \psi)) = \|f - \phi\| + \|g - \psi\| \quad (= d(f, \phi) + d(g, \psi)).$$

Then

$$d(Q(f, g), Q(\phi, \psi)) = \|fg - \phi\psi\| \leqq \| (f - \phi)g\| + \|\phi(g - \psi) \| \leqq d'((f, g), (\phi, \psi)),$$

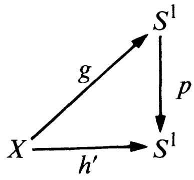
so that Q is contractive.

We now claim that Q is surjective. To this end, let

$$h \in C(X, U(2)) \quad \text{and} \quad h'(x) = \det h(x)(x \in X).$$

Clearly, $h' \in C(X, \mathbf{T}) = C(X, S^1)$.

We claim that h' has a square root $g \in C(X, S^1)$. To this end, let $p:S^1 \rightarrow S^1$ be given by $p(z) = z^2$. From **[5]**, p. 156, there exists $g \in C(X, S^1)$ such that the diagram

(6) 

commutes if and only if

$$p_*(\pi_1(S^1)) \supset (h')_*(\pi_1(X)).$$

We assert that

$$(h')_*(\pi_1(X)) = \{0\}$$

so that the required square root g exists. To prove that

$$(h')_*(\pi_1(X)) = \{0\},$$

first note that since $\pi_1(S^1) = \mathbf{Z}$ is abelian, the homomorphism $(h')_*$ is of the form $\alpha \circ r$ where

$$\alpha:H_1(X, \mathbf{Z}) \rightarrow \mathbf{Z}$$

is a homomorphism and r is the quotient map from $\pi_1(X)$ onto its abelianisation $H_1(X, \mathbf{Z})$. The finitely-generated abelian group $H_1(X, \mathbf{Z})$ is of the form

$$\left(\bigoplus_{i=1}^{k} \mathbf{Z}_{n_i}\right) \oplus \mathbf{Z}^r,$$

and since

$$H^1(X, \mathbf{Z}) = \mathrm{Hom}(H_1(X, \mathbf{Z}), \mathbf{Z}) = \{0\};$$

we must have $r = 0$. So $H_1(X, \mathbf{Z})$ is finite. But then $\alpha(H_1(X, \mathbf{Z}))$ is a finite subgroup of $\mathbf{Z}$ and so is $\{0\}$. Hence

$$(h')_*(\pi_1(X)) = \{0\}$$

as required.

Let g be as in (6). Then $g \in C(X, \mathbf{T})$, $g^2 = h'$,

$$h/g \in C(X, SU(2)) \quad \text{and} \quad Q(h/g, g) = h.$$

So Q is onto. Hence $\hat{G}_d$ can be regarded, using Q, as a subset of

$$(C(X, SU(2)) \times C(X, \mathbf{T}))^\wedge_d,$$

in the obvious way. Using Schur's lemma, the latter set is just

$$C(X, SU(2))^\wedge_d \times C(X, \mathbf{T})^\wedge_d.$$

We know, from Theorem A, what $C(X, \mathbf{T})^\wedge_d$ is. We now have to determine $C(X, SU(2))^\wedge_d$.

We claim that $C(X, SU(2))$ is connected. For let

$$f \in C(X, SU(2)).$$

Then f is homotopic to a cellular map $f':X \to S^3$ and since $\dim X = 2 < 3$, f' cannot be onto. It follows that f' is homotopic to a trivial map. So all functions in $C(X, SU(2))$ are homotopic to one another. Hence $C(X, SU(2))$ is connected.

Let $\pi \in C(X, SU(2))^\wedge_d$. Since $C(X, SU(2))$ is connected, it follows that $\alpha = d\pi$ belongs to

$$C(X, su(2))^\wedge_d = (X \times \{\alpha_2\}) \cup \{0\}$$

by Proposition 3. Hence, in an obvious notation,

$$\pi \in (X \times \{\pi_2\}) \cup \{1\},$$

and one readily checks that

$$C(X, SU(2))^\wedge_d = (X \times \{\pi_2\}) \cup \{1\}.$$

It remains to determine which of the elements of

$$C(X, SU(2))^\wedge_d \times C(X, \mathbf{T})^\wedge_d$$

"pass through" Q to give elements of $C(X, U(2))^{\wedge}_d$.

Let

$$p = (a, b) \in C(X, SU(2))^{\wedge}_d \times (C(X, \mathbf{T})^{\wedge}_d).$$

Since

$$\ker Q = \{(I, 1), (-I, -1)\},$$

p will define a representation of G if $p((I, 1)) = p((-I, -1))$, i.e., if

(7) $a(I)b(1) = a(-I)b(-1)$.

Suppose that $a = (x, \pi_2)$. Let $b \in S_{\mathscr{S}}(X)$ (in the notation of Theorem A). Since X is connected, $S_{\mathscr{S}}(X) = P(X)$, the set of probability measures on X. Then (7) becomes:

$$I \cdot e^{ib(0)} = -I \cdot e^{ib(\pi)},$$

which is always true. Now let $b = 0$. Then (7) becomes: $I = -I$, which is always false. If $b \in -P(X)$ then, as above, (7) is always satisfied.

Suppose, now, that $a = 1$. Then (7) becomes:

$$e^{ib(0)} = e^{ib(\pi)},$$

and for $b \in P(X) \cup \{0\} \cup -P(X)$, this is satisfied only when $b = 0$. So the set of pairs (a, b) satisfying (7) is:

(8) $[(X \times \{\pi_2\}) \times (P(X) \cup -P(X))] \cup \{(1, 0)\}$.

It remains to determine which of these pairs is contractive on G.

Let $a = (x, \pi_2)$ and $b \in P(X)$. Suppose that $b \neq \delta_x$. Then we can find $y \in X \sim \{x\}$ with y in the support of b. Since $y \neq x$, we can find compact neighbourhoods U_x of x and U_y of y in X such that $U_x \cap U_y = \emptyset$, and functions $f, g \in C(X, \mathbf{R})$ such that

$$f(X \sim U_x) = \{0\},\ 0 \leqq f \leqq 1, f(x) = 1, \quad \text{and}$$

$$g(X \sim U_y) = \{0\},\ 0 \leqq g \leqq \frac{1}{2}, g(y) = \frac{1}{2}.$$

Let

$$w = e^{fZ_3}e^{ig} \in C(X, U(2)).$$

Let $\pi = (a, b)$, a representation of $C(X, SU(2)) \times C(X, \mathbf{T})$. Since the eigenvalues of Z_3 are $\pm\frac{1}{2}i$, we have

$$\begin{aligned}\|\pi(w) - \pi(I)\| &= \|e^{f(x)Z_3}e^{ib(g)} - I\| \\ &= \max\{|e^{i((1/2)+b(g))} - 1|, |e^{i(-(1/2)+b(g))} - 1|\}.\end{aligned}$$

Since y is in the support of b, we have $0 < b(g) \leqq 1/2$, and it follows that

$$\|\pi(w) - \pi(I)\| = |e^{i((1/2)+b(g))} - 1| > |e^{(1/2)i} - 1|.$$

But

$$\|w - I\| = \max\{\sup_{z \in U_x} \|e^{f(z)Z_3} - I\|, \sup_{z \in U_y} |e^{ig(z)} - 1|\}$$
$$= \max\{|e^{(1/2)i} - 1|, |e^{(1/2)i} - 1|\} < \|\pi(w) - \pi(I)\|.$$

So for π to be contractive, we must have $b = \delta_x$. Similarly for π to be contractive when $a = (x, \pi_2)$, $b \in -P(X)$, we must have $b = -\delta_x$.

Suppose, now, that $(a, b) = (1, 0)$. Trivially, (a, b) is contractive on

$$C(X, SU(2)) \times C(X, \mathbf{T})_e.$$

So the set of contractive (a, b)'s is:

$$(\{\pi_2\} \times (X \cup -X)) \cup \{(1, 0)\}.$$

The theorem now follows.

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