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TOTALLY INTEGRALLY CLOSED AZUMAYA ALGEBRAS

BY
R. MACOOSH AND R. RAPHAEL

ABSTRACT. Enochs introduced and studied totally integrally closed rings in the class of commutative rings. This article studies the same question for Azumaya algebras, a study made possible by Atterton's notion of integral extensions for non-commutative rings.

The main results are that Azumaya algebras are totally integrally closed precisely when their centres are, and that an Azumaya algebra over a commutative semiprime ring has a tight integral extension that is totally integrally closed. Atterton's integrality differs from that often studied but is very natural in the context of Azumaya algebras. Examples show that the results do not carry over to free normalizing or excellent extensions.

Totally integrally closed rings were introduced by Enochs [4]. We recall that a commutative ring D is totally integrally closed if for any homomorphism of commutative rings $\sigma: B \longrightarrow D$ and any integral extension (in the sense of [10, p. 254]) C of B there is a homomorphism $C \longrightarrow D$ extending σ. In this article we consider the noncommutative case of totally integrally closed Azumaya algebras.

Throughout, all rings are assumed associative with 1 and all homomorphisms are 1-preserving. The centre of a ring R will be written $Z(R)$. If R is a commutative ring, A an R-algebra, we mean by a two-sided A/R-module (see [3, p. 41]) a left and right A-module M such that

i) $(am)a' = a(ma')$, for all m in M, a, a' in A; and

ii) $rm = mr$, for all m in M, r in R.

For any two-sided A/R-module M the set $\{m \in M | am = ma, \text{ for all } a \in A\}$ will be denoted M^A (see [3, p. 42]).

1. **Azumaya algebras.** The following basic results will be needed for our study of Azumaya algebras.

LEMMA 1.1. [3, p. 54] *If A is an Azumaya R-algebra, then for any R-module M, $(M \otimes A)^A \cong M$ as R-modules under the map $m \otimes 1 \leftrightarrow m$ and for any two-sided A/R-module N, $N^A \otimes A \cong N$ as two-sided A/R-modules under the map $\sum n_i \otimes a_i \leftrightarrow \sum n_i a_i$.*

LEMMA 1.2. [3, p. 61] *Let A be an Azumaya R-algebra. Then for any commutative R-algebra S, $A \otimes_R S$ is an Azumaya S-algebra.*

2. **Integral extensions of a noncommutative ring.** If $A \subseteq B$ are commutative rings, an element $b \in B$ is said to be integral over A if (i) $b^n + a_1 b^{n-1} + \cdots + a_n = 0$

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for some $a_1, \cdots, a_n \in A$. This definition is equivalent to (ii) there exists in B a finitely generated A-module M such that $bM \subseteq M$ and zero is the only annihilator of M in $A[b]$ (see [10, p. 254]). Both of these definitions generalize to noncommutative rings but are then no longer equivalent as we shall presently see. Of the two, the following is the more usually encountered notion of integrality (see [6]).

DEFINITION 2.1. Let $A \subseteq B$ be rings. For the purpose of this definition B is called an extension of A if $B = AB^A$, where $B^A = \{b \in B \mid ab = ba, \text{ all } a \in A\}$. An extension B of A is said to be *integral* if each $b \in B$ satisfies a monic polynomial with coefficients in A.

It is pointed out by Procesi [8, p. 130] that if B is an extension of A in the sense of Definition 2.1, the elements of B which satisfy an equation of integral dependence over A do not in general form a ring. As well, the following example shows that an integral extension of A need not be an Azumaya algebra if A is.

EXAMPLE 2.2. Let F be a field and let B be the subring of upper triangular matrices of $M_2(F)$. By the Cayley-Hamilton theorem B is then an integral extension of the Azumaya algebra F. If B were an Azumaya algebra we should have by Lemma 1.1 an isomorphism of two-sided B/F-modules $M_2(F) \cong M_2(F)^B \otimes_F B$. Since $M_2(F)^B = F$ this means that $M_2(F) \cong F \otimes B \cong B$, contradicting the fact that $M_2(F)$ and B do not have the same dimension.

In view of Example 2.2 we now study a notion of integrality due to Atterton [1] which will be seen to differ radically from that defined in Definition 2.1 but is the natural one in the context of Azumaya algebras. Throughout the remainder of this article integral will mean integral in the sense of Atterton.

DEFINITION 2.3. (Atterton) [1, p. 434]. Let B be a ring, A a subring of B containing the identity of B. An element b of B is said to be integral over A if there exists a finitely generated unitary A-module M, all of whose generators belong to $Z(B)$, such that $1 \in M$ and $Mb \subseteq M$. M is then a right and left A-Module and $bM \subseteq M$. One easily sees that a central element of B will be integral over A if it satisfies a monic polynomial equation over A [1. p. 437].

Atterton proved that the integral closure of A in B is a ring containing A [1, Theorem 1, p. 435], and that if B is integral over A then $Z(A) \subseteq Z(B)$ [1, p. 435].

COROLLORY 2.4. *Let $A \subseteq B$ be rings, B integral over A. Then $B = AZ(B)$.*

REMARK 2.5. We conclude from 2.4 that if $A \subseteq B$ are rings it is not true that B is integral over A if each $b \in B$ satisfies a monic polynomial with coefficients in A. This is clear from the Cayley-Hamilton theorem because a noncommutative matrix ring cannot be integral over a commutative field.

LEMMA 2.6. *Let $A \subseteq B$ be rings, B integral over A. Then $Z(B)$ is precisely the set B^A.*

PROOF. Let $b \in B^A$ and let $b' \in B$. Since b' is integral over A there exist $a_1, \cdots, a_n \in A$ and $z_1, \cdots, z_n \in Z(B)$ such that $b' = \sum a_i z_i$. Therefore $bb' = b'b$. The opposite implication is trivial.

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LEMMA 2.7. *If A is an Azumaya algebra and B is an integral extension of A then $B \cong A \otimes_{Z(A)} Z(B)$ as $Z(A)$-algebras.*

PROOF. We have $Z(A) \subseteq Z(B)$ hence B is a two-sided $A/Z(A)$-module and therefore $B^A \otimes A \cong B$ as two-sided $A/Z(A)$-modules by Lemma 1.1.

COROLLARY 2.8. *If A is an Azumaya algebra and B is an integral extension of A then B is an Azumaya algebra.*

PROOF. Lemma 2.7 and Lemma 1.2.

LEMMA 2.9. *Let $A \subseteq B$ be Azumaya algebras, B integral over A. Then $Z(B)$ is integral over* $(Z(A)$.

PROOF. Let $z \in Z(B)$. Then there exists an A-module $N = Az_1 + \cdots + Az_n$, $z_i \in Z(B)$, such that $1 \in N$ and $zN \subseteq N$. To show that z is integral over $Z(A)$ we need, according to definition (ii) above, a finitely generated $Z(A)$-module M contained in $Z(B)$, such that $zM \subseteq M$ and zero is the only annihilator of M in $Z(A)[z]$. Let $M = Z(A)z_1 + \cdots + Z(A)z_n$; then $N = AM$. Furthermore $M \subseteq N^A$, hence $M \otimes_{Z(A)} A \subseteq N^A \otimes_{Z(A)} A \cong N = AM$, A being a flat $Z(A)$-module by [3, Theorem 3.4, p. 52]. Therefore the homomorphism of $M \otimes A$ onto N defined by $\sum m_i \otimes a_i \longrightarrow \sum m_i a_i$ is an isomorphism (of two-sided $A/Z(A)$-modules), and from $M \otimes A \cong N$ it follows that $M \cong (M \otimes A)^A \cong N^A$. Since $z(N^A) \subseteq N^A$, we thus have $zM \subseteq M$. To complete the proof we note that any annihilator of M in $Z(A)[z]$ will annihilate all the generators of N and hence will annihilate N. Since $1 \in N$ this element will equal zero.

COROLLARY 2.10. *If A is a matrix ring over a commutative ring R or a group algebra RG, G a finite group whose order is invertible in R, then the Atterton integral extensions of A arise from integral extensions of $Z(A)$.*

PROOF. If R is a commutative ring it follows from [3, Example II and Example III, p. 41] that the matrix ring $M_n(R)$ is an Azumaya R-algebra, and so is RG whenever G is a finite group whose order is a unit in R. Now by Corollary 2.4, Lemma 2.7, Corollary 2.8 and Lemma 2.9, an integral extension of $M_n(R)$ must be an Azumaya S-algebra of the form $M_n(R) \otimes_R S \cong SM_n(R) = M_n(S)$, where S is an integral extension of R. Similarly an integral extension of RG is $RG \otimes_R S \cong SRG = SG$.

It should be noted that Lemma 2.9, although true for Azumaya algebras, does not hold generally. We are indebted to Walter Burgess for providing the following example of a ring extension which is integral while the inclusion of the centres is not. We shall also see that this is an example of an integral extension in the sense of Definition 2.1 and that furthermore it is an example of a free normalizing extension. Recall that if $A \subseteq B$ are rings, B is a free normalizing extension of A if B is a free right and left A-module with basis $\{b_1, \ldots, b_n\} \subseteq B$ such that $b_1 = 1, B = Ab_1 + \cdots + Ab_n$, $Ab_i = b_iA$ for $i = 1, \ldots, n$ (see [7]).

EXAMPLE 2.11. Let F be a field and let $A = F\langle y, z\rangle$, the free F-algebra in the noncommuting indeterminates y and z. Let $B = (F[x])\langle y, z\rangle / I$, where I is the ideal generated by $x^2 + xy$. Clearly the centre of A is F. As well $\bar{x}$ is in the centre of B, and since it satisfies a monic over $\bar{A}$, it will follow that the ring B is integral over A once it is established that $I \cap A = (0)$. Suppose that $g(y, z)$ lies in I. Then for appropriate polynomials

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$\alpha(x,y,z), \beta(x,y,z)$, one has $g(y,z) = \sum \alpha(x,y,z)(x^2+xy)\beta(x,y,z)$. There is an onto ring homomorphism $F[x]\langle y,z\rangle \longrightarrow F\langle y,z\rangle$ determined by $x \longrightarrow 0$. Substituting 0 for x in the expression for g gives us $g(y,z) = \sum \alpha(0,y,z)(0)\beta(0,y,z)$. Thus $g(y,z) = 0$ and we do have an integral extension.

Now we claim that the element $\bar{x}$ which is central in B is not integral over F. If this were the case, then one would have $\sum f_i x^i = \sum \alpha'(x,y,z)(x^2+xy)\beta'(x,y,z)$ for appropriate polynomials α', β'. There is an onto ring homomorphism $F[x]\langle y,z\rangle \longrightarrow F[x,y]$ determined by letting z go to zero. Substituting zero for z in the equation gives $\sum f_i x^i = \sum \alpha'(x,y,0)(x^2+xy)\beta'(x,y,0)$ in the commutative ring $F[x,y]$. Thus we have $\sum f_i x^i = (x^2+xy)h(x,y)$ for some polynomial h in x and y. This is impossible simply by considering the degree in y of the polynomials on each side of the equality. Thus $\bar{x}$ is not integral over F.

To see that B is a free normalizing extension of A we observe that a typical element b of B has the form

$$b = \sum g(y,z) + \sum f(\bar{x})(y,z) \tag{1}$$

where the summations are over sequences of y and z with coefficients in F and $F[\bar{x}]$ respectively. Since $\bar{x} \in Z(B)$ and $\bar{x}^n = y^n\bar{x}$ for $n > 1$ we can write (1) as $b = \sum g'(y,z) + [\sum f'(y,z)]\bar{x} \in A + A\bar{x}$. Thus $B = A.1 + A\bar{x}$ and furthermore B is free with basis $\{1, \bar{x}\}$ as both a left and right A-module. For if $a_1 = a_2\bar{x}$ then $a_1 - a_2x \in I$ so that a_1 can be written as the sum of a_2x and a multiple of x^2+xy. Therefore a_1 is a multiple of x, say $a_1 = tx$ for some t in $F[x]\langle y,z\rangle$, and this is impossible because x cannot be cancelled in the right hand side. A little work shows that B is not an excellent extension of A in the sense of [7, p. 1].

Finally, since $B = AB^A$ is finitely generated as an A-module, it follows from [6, Theorem 1] that B is also an integral extension of A according to the definition of integrality in Definition 2.1.

In Example 2.11 we have presented an example of a free normalizing extension where Lemma 2.9 fails if the Azumaya condition is dropped. We recall that the Lemma depends on the fact that the centre of a ring is contained in the centre of its integral extensions (see Definition 2.3), The next example shows that excellent extensions do not in general have this property, even when the Azumaya condition is satisfied.

EXAMPLE 2.12. Let A denote the field $Q(i)$, where Q is the field of rational numbers and $i^2 = -1$, and let B denote the division ring $Q(1,i,j,k)$ of quaternions over Q. By [3, p. 50–51] both of A and B are Azumaya algebras and clearly $Q(i)$ is not contained in the centre Q of B. We show that $B = A.1 + Aj$ is an excellent extension of A. Suppose $b \in B$. If $b = p + qi + rj + sk$, $p,q,r,s \in Q$, we can write $b = p + qi + (r + si)j$ and hence $B = A.1 + Aj$ as an A-module. As well, the sum is direct because $\{1,i,j,k\}$ is a basis for B as a Q-vector space. Thus B is a free normalizing extension of A with basis $\{1,j\}$. If now N_B is a submodule of M_B then, since B is a division ring, each basis for N_B can be extended to one for M_B. Therefore N_B is always a direct summand of M_B and so we have an excellent extension. It is interesting to observe that each b in B satisfies a monic quadratic polynomial over A, but B is neither Atterton integral over A nor is it an integral extension of A in the sense of Definition 2.1.

3. **Totally integrally closed Azumaya algebras.** In order to extend the notion of a totally integrally closed ring to Azumaya algebras we need to determine suitable homomorphisms between Azumaya algebras over their centres.

DEFINITION 3.1. Let $\mathfrak{A}$ be the category of Azumaya algebras whose morphisms are the homomorphisms $A \longrightarrow B$ which map $Z(A)$ into $Z(B)$ for each pair A, B in $\mathfrak{A}$. $\mathfrak{A}$ is a generalization of the category of commutative rings in which Enochs worked. We say that an object D in $\mathfrak{A}$ is totally integrally closed if for any $\mathfrak{A}$-morphism $\sigma: B \longrightarrow D$ and any integral extension C of B there is an $\mathfrak{A}$-morphism $C \longrightarrow D$ extending σ. Both of these mappings are then homomorphisms of $Z(B)$-algebras.

In the category of commutative rings Enochs obtained the following results: [4, Theorem 1] (The proof of this Theorem was discussed later by Borho and Weber [2]). A commutative ring R is a subring of a totally integrally closed ring if and only if R is semiprime [4, Theorem 2]. If R is a commutative semiprime ring, there is a totally integrally closed integral extension of R which is also a tight extension of R (a tight extension of R is a ring extension of R each of whose non-zero ideals intersect R in a non-zero ideal).

These results lead us to consider the Azumaya algebras in $\mathfrak{A}$ whose centres are semiprime rings.

THEOREM 3.2. *An Azumaya algebra over a semiprime ring is totally integrally closed if and only if its centre is totally integrally closed.*

PROOF. Let D be a totally integrally closed Azumaya algebra whose centre $Z(D)$ is a semiprime ring and assume $Z(D)$ is not totally integrally closed. Then there exists a totally integrally closed commutative ring $\Omega \underset{\neq}{\supset} Z(D)$ such that the monomorphism $Z(D) \longrightarrow \Omega$ is integral and tight. Put $B = D \otimes_{Z(D)} \Omega$. B is an Azumaya algebra with centre Ω and since D is a flat $Z(D)$-module it follows that $D \subseteq B$. Furthermore B is integral over D. For if ω is an element of Ω we have an equation $\omega^n + z_1\omega^{n-1} + \cdots + z_n = 0$ for some $z_1, \cdots, z_n \in Z(D)$ and therefore the D-module $M = D + D\omega + \cdots + D\omega^{n-1}$ has the property that $1 \in M$ and $\omega M \subseteq M$, showing that ω is integral over D. Because $\Omega \cong Z(D) \otimes_{Z(D)} \Omega$ this means that $1 \otimes \omega$ is integral over D. As well $d \otimes 1 \in D \otimes_{Z(D)} Z(D) \cong D$ is integral over D. Thus B is an integral extension of D.

Now by Definition 3.1 the identity mapping on D can be extended to a homomorphism $\tau: B \longrightarrow D$ onto D. If $\operatorname{Ker} \tau = I \neq (0)$ then by [3, Corollary 3.7, p. 54] I contains the non-zero ideal $I \cap \Omega$ which implies that $I \cap Z(D) \neq (0)$, Ω being a tight extension of $Z(D)$. This contradicts the fact that $\tau | D$ is the identity on D and therefore we have an isomorphism of B onto D, a further contradiction since $Z(B) = \Omega$ is totally integrally closed and $Z(D)$ is not.

Conversely let D be an Azumaya algebra whose centre $Z(D)$ is totally integrally closed, let B be an Azumaya algebra and let $\sigma: B \longrightarrow D$ be a homomorphism of rings taking $Z(B)$ into $Z(D)$. If C is an integral extension of B then $C \cong B \otimes Z(C)$ is an Azumaya algebra by Corollary 2.8, and since $Z(C)$ is integral over $Z(B)$ by Lemma 2.9 there is a homomorphism $h: Z(C) \longrightarrow Z(D)$ extending $\sigma | Z(B): Z(B) \longrightarrow Z(D)$. Now define $\tau: B \otimes Z(C) \longrightarrow D$ by $\tau = \sigma \otimes h$. Then we have $\tau(B) = \sigma(B)$ as required.

THEOREM 3.3. *Let A be an Azumaya algebra over a semiprime ring. Then there exists a totally integrally closed Azumaya algebra that is an integral, tight extension of A.*

PROOF. Since $Z(A)$ is semiprime there exists a totally integrally closed commutative extension $\Omega(Z(A))$ of $Z(A)$ such that the monomorphism $Z(A) \longrightarrow \Omega(Z(A))$ is integral and tight. Put $D = A \otimes_{Z(A)} \Omega(Z(A))$. D is an Azumaya $\Omega(Z(A))$-algebra, hence totally integrally closed, and since A is flat we have $A = Z(A) \otimes_{Z(A)} A \subseteq \Omega(Z(A)) \otimes_{Z(A)} A = D$. As in the proof of Theorem 3.2 D is an integral extension of A. To see that it is also a tight extension of A we recall from [3, Corollary 3.7, p. 54] that if $(0) \neq I$ is any two-sided ideal of D then $I = [I \cap \Omega(Z(A))]D$, therefore $I \cap \Omega(Z(A))$ is a non-zero ideal of $\Omega(Z(A))$ and because $\Omega(Z(A))$ is a tight extension of $Z(A)$ there is in $Z(A)$ a non-zero ideal $(I \cap \Omega(Z(A)) \cap Z(A)) \subseteq I \cap A$.

REMARK 3.4. Let R be a commutative semiprime ring, let A be an Azumaya R-algebra, and let Ω be the integral closure of R in the algebraic closure of its complete ring of quotients (see [9, Corollary 2.10, p. 1143]). Ω is totally integrally closed, therefore the monomorphism $A \longrightarrow A \otimes_R \Omega$ is integral and tight, and $A \otimes_R \Omega$ is totally integrally closed.

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Department of Mathematics
Concordia University, Loyola Campus
7141 Sherbrooke St. W.,
Montreal, Quebec
H4B 1R6