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ON MODULUS OF NONCOMPACT CONVEXITY AND ITS PROPERTIES

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ABSTRACT. In this paper we prove some properties of the so-called modulus of noncompact convexity. This notion was recently introduced by K. Goebel and T. Sękowski [6] and it appears to be an interesting and useful generalization of the classical Clarkson modulus of convexity. We extend the results obtained in [6] showing that the modulus of noncompact convexity is continuous and has some extra properties in reflexive Banach spaces. The properties applicable in the fixed point theory are also stated.

Introduction. In the geometric theory of Banach spaces the notion of the modulus of convexity plays a very significant role. It allows us to classify Banach spaces from the point of view of their geometrical structure. In this regard the modulus of convexity is a useful tool in the fixed point theory. A lot of facts concerning this notion and its applications may be found in [3, 4, 5, 9], for example.

Recently K. Goebel and T. Sękowski [6] have proposed an interesting generalization of the notion of the modulus of convexity. Namely, with help of the concept of Kuratowski's measure of noncompactness they defined the so-called modulus of noncompact convexity. By means of this modulus they proved a few interesting facts concerning the geometric theory of Banach spaces.

The goal of this paper is to give some further facts concerning properties of the modulus of noncompact convexity.

1. **Notations, Definitions and known results**. Let $(E, \| \ \|)$ be an infinitely dimensional Banach space and let $B(x, r)$, $S(x, r)$ denote the ball and the sphere centred at x and of radius r. For brevity we will write B, S instead of $B(\Theta, 1)$ and $S(\Theta, 1)$. If X is a subset of E, $x \in E$, then $\overline{X}$, $\text{Conv}X$, $\text{dist}(x, X)$ will denote the closure, the closed convex hull of X and the distance from a point x to X, respectively. Similarly, $\text{dist}(X, Y)$ will denote the distance between sets X and Y. By $B(X, r)$ we denote the "ball" centered at a set X and with radius r, i.e. $B(X, r) = \bigcup_{x \in X} B(x, r)$. For a bounded set X, $\alpha(X)$ will denote the Kuratowski's measure of noncompactness:

$$\alpha(X) = \inf[d > 0\colon X \text{ can be covered with a finite number of sets of diameters smaller than } d].$$

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The symbol $\chi(X)$ will denote the Hausdorff's measure of noncompactness:

$$\chi(X) = \inf[\epsilon > 0: X \text{ can be covered with a finite number of balls of radii smaller than } \epsilon].$$

In the sequel we will use first of all the following properties of the function χ:

1° $\chi(X) = 0 \Leftrightarrow \bar{X}$ is compact

2° $X \subset Y \Rightarrow \chi(X) \leq \chi(Y)$

3° $\chi(\bar{X}) = \chi(\text{Conv}X) = \chi(X)$

4° $\chi(\lambda X) = |\lambda|\chi(X),\ \lambda \in R$

5° $\chi(X + Y) \leq \chi(X) + \chi(Y)$

6° $\chi(x + X) = \chi(X)$

7° $\chi(B(x, r)) = \chi(S(x, r)) = r$

Let us notice that the function α has also the properties 1° − 6° and $\alpha(B(x, r)) = \alpha(S(x, r)) = 2r$. For further properties of these measures we refer to [1].

Recall that the classical Clarkson's *modulus of convexity* of the space E [2] is the function δ: $\langle 0, 2\rangle \rightarrow \langle 0, 1\rangle$ defined by

$$\delta_E(\epsilon) = \inf\left[1 - \frac{\|x + y\|}{2}: x, y \in \bar{B}, \|x - y\| \geq \epsilon\right].$$

The coefficient of convexity of E is understood as

$$\epsilon_0(E) = \sup[\epsilon: \delta_E(\epsilon) = 0].$$

The space is called uniformly convex if $\epsilon_0 = 0$. The notion of *the modulus of noncompact convexity* was defined in [6] in the following way

$$\bar{\Delta}_E(\epsilon) = \inf[1 - \text{dist}(\Theta, X): X \subset \bar{B}, X = \text{Conv}X, \alpha(X) \geq \epsilon].$$

Actually $\bar{\Delta}$: $\langle 0, 2\rangle \rightarrow \langle 0, 1\rangle$ and is a nondecreasing function. Moreover, $\delta_E(\epsilon) \leq \bar{\Delta}_E(\epsilon)$ for any Banach space E. It was shown in [6] that this inequality may be sharp for some spaces. Analogously the number $\bar{\epsilon}_1(E) = \sup[\epsilon: \bar{\Delta}_E(\epsilon) = 0]$ was called *the coefficient of noncompact convexity* and spaces with $\bar{\epsilon}_1 = 0$, $\bar{\Delta}$-*uniformly convex*. Of course, $\bar{\epsilon}_1(E) \leq \epsilon_0(E)$ and in the case of Day's space D, $\bar{\epsilon}_1(D) = 0$ and $\epsilon_0(D) = 2$ [6]. The main result proven in [6] may be summarized in the below given theorem.

THEOREM 1. *If* $\bar{\epsilon}_1(E) < 1$ *then* E *is reflexive and has normal structure.*

In what follows we shall use the notion of the modulus of noncompact convexity defined with help of the Hausdorff's measure of noncompactness:

$$\Delta: \langle 0, 1\rangle \rightarrow \langle 0, 1\rangle,\ \Delta_E(\epsilon) = \inf[1 - \text{dist}(\Theta, X): X \subset \bar{B}, X = \text{Conv}X, \chi(X) \geq \epsilon].$$

In the similar way by $\epsilon_1(E)$ we denote the coefficient of noncompact convexity of E

(with respect to the modulus Δ). We say that E is Δ-*uniformly convex* if $\epsilon_1 = 0$. Let us notice that the well-known dependence $\chi(X) \leq \alpha(X) \leq 2\chi(X)$ (cf. [1]) yields

$$\bar{\Delta}_E(\epsilon) \leq \Delta_E(\epsilon) \leq \bar{\Delta}_E(2\epsilon),\ \epsilon \in \langle 0,\ 1\rangle$$

for any Banach space E. Hence $\epsilon_1(E) \leq \bar{\epsilon}_1(E) \leq 2\epsilon_1(E)$. This inequality permits to formulate the following

THEOREM 2. *If $\epsilon_1(E) \leq 1/2$ then the space E is reflexive and has normal structure.*

2. **Continuity of modulus of noncompact convexity**. This section is devoted to showing that the modulus of noncompact convexity $\Delta_E(\epsilon)$ is continuous on the interval $\langle 0,\ 1)$. We will need the following result due to Radström [10]:

LEMMA 1. *Let X, Y, Z be nonempty subsets of a Banach space E such that $Y =$ Conv$Y$ and Z is bounded. Then $X + Z \subset Y + Z$ implies that $X \subset Y$.*

We will also use the following result.

LEMMA 2. $\chi(B(X, r)) = \chi(X) + r$.

PROOF. The properties of the function χ and the equality $B(X, r) = X + r\cdot B$ imply $\chi(B(X, r)) \leq \chi(X) + r$. In order to prove the converse inequality let us remark at first that

$$\chi(X + rB) \geq \chi(x + rB) = r, \tag{1}$$

where x is an arbitrary point from X. Further notice that in view of the definition of χ there exist a finite set H and a number $r_1 > \chi(X + rB)$ such that $X + rB \subset \mathrm{H} + r_1B$. Hence

$$X + rB \subset \mathrm{Conv}H + (r_1 - r)B + rB.$$

Because the set Conv$H + (r_1 - r)\bar{B}$ is closed and convex, by virtue of Lemma 1 we get $X \subset \mathrm{Conv}H + (r_1 - r)\bar{B}$, what implies

$$\chi(X) \leq \chi(\mathrm{Conv}H) + r_1 - r = r_1 - r$$

and finally

$$\chi(X) + r \leq r_1.$$

The above inequality together with (1) completes the proof.

Now we can prove our main result.

THEOREM 3. *The function Δ is continuous on the interval* $\langle 0,\ 1)$.

PROOF. First note that the function Δ is nondecreasing on the interval $\langle 0,\ 1\rangle$. Further, let us fix $\epsilon_1 \in \langle 0,\ 1)$ and take an arbitrary $\epsilon_2 \in (\epsilon_1,\ 1)$. For $\eta > 0$ arbitrarily small we may choose a set X_1 contained in $\bar{B}$ such that Conv$X_1 = X_1$, $\chi(X_1) \geq \epsilon_1$ and

$$1 - \mathrm{dist}(\Theta, X_1) \leq \Delta(\epsilon_1) + \eta. \tag{2}$$

Next, putting $k = (1 - \epsilon_2)/(1 - \epsilon_1)$ we see that $k \in (0, 1)$. Consider the set $Y = kX_1$. Obviously $\chi(Y) = k\chi(X_1)$ and $\mathrm{dist}(\Theta, Y) = k\mathrm{dist}(\Theta, X_1)$, $\mathrm{dist}(Y, S) \geq 1 - k$, so that if we take the set $X_2 = B(X_1, 1 - k)$ we may easily verify that $X_2 \subset B$, $\mathrm{Conv}X_2 = X_2$ and

$$\mathrm{dist}\,(\Theta, X_2) = k\mathrm{dist}(\Theta, X_1) - 1 + k. \tag{3}$$

Moreover, in view of Lemma 2 we obtain

$$\chi(X_2) = k\chi(X_1) + 1 - k \geq k\epsilon_1 + 1 - k = \epsilon_2.$$

Now by (2) and (3) we infer

$$\begin{aligned} 1 - \mathrm{dist}(0, X_2) &= 1 - k\mathrm{dist}(\Theta, X_1) + 1 - k = \\ &= k(1 - \mathrm{dist}(\Theta, X_1)) + 2(1 - k) \leq k(\Delta(\epsilon_1) + \eta) + 2(1 - k). \end{aligned}$$

Hence

$$\Delta(\epsilon_2) \leq k(\Delta(\epsilon_1) + \eta) + 2(1 - k).$$

Finally, keeping in mind that η was chosen arbitrarily we have

$$\Delta(\epsilon_2) \leq k\Delta(\epsilon_1) + 2(1 - k)$$

which implies

$$\begin{aligned} \Delta(\epsilon_2) - \Delta(\epsilon_1) \leq k\Delta(\epsilon_1) - \Delta(\epsilon_1) + 2(1 - k) &= (1 - k)(2 - \Delta(\epsilon_1)) \leq 2(1 - k) \\ &= 2(\epsilon_2 - \epsilon_1)/(1 - \epsilon_1). \end{aligned}$$

Thus the proof is completed.

It is worth while to mention that our method of proving allows us to show continuity of the function $\bar{\Delta}$ on the interval $\langle 0, 1)$ but not on the whole interval $\langle 0, 2)$. This is caused by the fact that for Kuratowski's measure α the equality $\alpha(B(X, t)) = \alpha(X) + 2t$ is no longer true; we have only $\alpha(X) + t \leq \alpha(B(X, t)) \leq \alpha(X) + 2t$.

3. **The case of reflexive space**. Throughout this section we will assume that E is a reflexive Banach space. This assumption permits us to deduce that for a nonempty, closed and convex subset X of E, for any $y \in E$, there is at least one $x \in X$ with the property $\mathrm{dist}(y, X) = \|y - x\|$ [8]. We show below that this fact has some significance in order to obtain additional properties of a modulus of noncompact convexity.

Let us assume that a number $\epsilon \in (0, 1)$ is fixed. Let us take an arbitrary $\eta > 0$ and a set $X \subset B$, $X = \mathrm{Conv}X$, $\chi(X) \geq \epsilon$, such that

$$1 - \mathrm{dist}(\Theta, X) \leq \Delta(\epsilon) + \eta. \tag{4}$$

Next, let k be an arbitrary number in the interval $(0, 1)$. Choose $x \in X$ with the property $\mathrm{dist}(\Theta, X) = \|x\|$ and consider the set $X_1 = kX + ((1 - k)/\|x\|)x$. Then $\chi(X_1) \geq k\epsilon$ and $\mathrm{dist}(\Theta X_1) = k\mathrm{dist}(\Theta, X) + 1 - k$. Moreover $X_1 \subset B$. Further we have

$$\mathrm{dist}(\Theta, X) = (1/k)(\mathrm{dist}(\Theta, X_1) + k - 1),$$

and (4) gives

$$1 - \Delta(\epsilon) \leq \operatorname{dist}(\Theta, X) + \eta = (1/k)(\operatorname{dist}(\Theta, X_1) + k - 1) + \eta.$$

This inequality implies

$$1 - \Delta(\epsilon) \leq (1/k)(1 - \Delta(k\epsilon) + k - 1) + \eta$$

and finally

$$\Delta(k\epsilon) \leq k\Delta(\epsilon).$$

Thus we can formulate our next result.

THEOREM 4. *If E is a reflexive Banach space then $\Delta_E(\epsilon)$ is a subhomogeneous function, i.e.*

$$\Delta(k\epsilon) \leq k\Delta(\epsilon)$$

for any e, $k \in \langle 0, 1\rangle$.

From the above theorem we may deduce some simple corollaries.

COROLLARY 1. $\Delta(\epsilon) \leq \epsilon$ *for any* ϵ $\langle 0, 1\rangle$.

COROLLARY 2. *The function Δ is strictly increasing on the interval $\langle \epsilon_1(E), 1\rangle$.*

Indeed, for $t_1 < t_2 \leq 1$, $\epsilon_1(E)$, if we put in Theorem 4 $\epsilon = t_2$, $k = t_1/t_2$, we have $\Delta(t_1) \leq (t_1/t_2)\Delta(t_2)$ what implies

$$\Delta(t_2)/\Delta(t_1) \geq t_2/t_1 > 1.$$

Thus $\Delta(t_1) < \Delta(t_2)$.

COROLLARY 3. $\Delta(t_2) - \Delta(t_1) \geq (t_2 - t_1)/\Delta(t_1)$ *for any* $t_1, t_2 \in (\epsilon_1(E), 1\rangle$, $t_1 \leq t_2$.

COROLLARY 4. *The function $\epsilon \to \Delta(\epsilon)/\epsilon$ is nondecreasing on the interval $\langle 0, 1\rangle$ and $\Delta(\epsilon_1 + \epsilon_2) \geq \Delta(\epsilon_1) + \Delta(\epsilon_2)$ provided $\epsilon_1 + \epsilon_2 \leq 1$.*

We omit the simple proofs of the last two corollaries.

4. **Stability**. As we have established in Theorem 2 every Banach space for which $\epsilon_1 < 1/2$ has normal structure. Thus, according to the well-known Kirk's fixed point theorem such a space has the fixed point property with respect to nonexpansive self-mappings of a nonempty, bounded, closed and convex set [7, 5]. We show now that this property is stable with regard to the slight change of the norm.

Assume that $(E, \|\ \|_1)$ is a Banach space for which $\epsilon_1 < 1/2$. Let $\|\ \|_2$ be the equivalent norm on the space E i.e. there exist positive constants m and M such that

$$m\|x\|_1 \leq \|x\|_2 \leq M\|x\|_1$$

for every $x \in E$. Let χ_1 and χ_2 denote the Hausdorff's measures of noncompactness in the spaces $(E, \|\ \|_1)$, $(E, \|\ \|_2)$, respectively. Then we can easily show that

$$m\chi_1(X) \leq \chi_2(X) \leq M\chi_1(X)$$

for any bounded subset X of the space E.
Further let Δ_1, Δ_2 be moduli of noncompact convexity with respect to the suitable norms. Let us fix $\epsilon > 0$ and $\eta \in (0, 1)$. Next, let us take $X \subseteq \bar{B}_2$, $X = \text{Conv}X$, $\chi_2(X) \geq \epsilon$ and such that

$$\text{dist}_2(\Theta, X) \geq 1 - \Delta_2(\epsilon) - \eta$$

(here the indices denote that we consider the ball or the distance with respect to the suitable norm). Then we have $\chi_1(X) \geq \epsilon/M$ and

$$\text{dist}_1(\Theta, X) \geq (1/M)\, \text{dist}_2(\Theta, X).$$

Moreover, $X \subset \bar{B}_1(0, 1/m)$. Hence we get

$$(1/M)\, \text{dist}_2(\Theta, X) \leq \text{dist}_1(\Theta, X) \leq (1 - \Delta_1(m\epsilon/M))(1/m)$$

what implies

$$1 - \Delta_2(\epsilon) - \eta \leq (M/m)(1 - \Delta_1(m\epsilon/M)).$$

Finally the last inequality yields

$$\Delta_2(\epsilon) \geq 1 - k(1 - \Delta_1(\epsilon/k)) \tag{5}$$

where $k = M/m \geq 1$.
Let $B > 1$ be a unique solution of the equation

$$1 - (1/B) = \Delta_1(1/2B), \tag{6}$$

which exists in view of continuity of the function Δ_1 (Theorem 3). Now, if $1 \leq k < B$ then $k(1 - \Delta_1(1/2k)) < 1$ so that (5) allows us to infer that $\Delta_2(1/2) > 0$. This assertion means that the coefficient of noncompact convexity for the norm $\| \ \|_2$ is smaller than $1/2$ and in view of Theorem 2 the space $(E, \| \ \|_2)$ has normal structure. Thus we have

THEOREM 5. *Let E be a Banach space with $\epsilon_1 < 1/2$ and let $B > 1$ satisfy (6). If F is another Banach space having the Banach-Mazur distance from E smaller than B then its coefficient of noncompact convexity is also smaller than $1/2$.*

Let us remark that similar result for the coefficient of convexity was obtained in [5].

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