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A PROBLEM ON ROUGH PARAMETRIC MARCINKIEWICZ FUNCTIONS

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Abstract

In this note the authors give the $L^2(\mathbb{R}^n)$ boundedness of a class of parametric Marcinkiewicz integral $\mu^\rho_{\Omega,h}$ with kernel function Ω in $L\log^+ L(S^{n-1})$ and radial function $h(|x|) \in l^\infty(L^q)(\mathbb{R}_+)$ for $1 < q \le \infty$.

As its corollary, the $L^p(\mathbb{R}^n)(2 \le p < \infty)$ boundedness of $\mu^{*,\rho}_{\Omega,h,\lambda}$ and $\mu^\rho_{\Omega,h,S}$ with Ω in $L\log^+ L(S^{n-1})$ and $h(|x|) \in l^\infty(L^q)(\mathbb{R}_+)$ are also obtained. Here $\mu^{*,\rho}_{\Omega,h,\lambda}$ and $\mu^\rho_{\Omega,h,S}$ are parametric Marcinkiewicz functions corresponding to the Littlewood-Paley g^*_λ-function and the Lusin area function S, respectively.

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1. Introduction

Suppose that S^{n-1} is the unit sphere of $\mathbb{R}^n (n \ge 2)$ equipped with normalized Lebesgue measure $d\sigma = d\sigma(x')$. Let $\Omega \in L^1(S^{n-1})$ be homogeneous of degree zero on $\mathbb{R}^n$ and

$$\int_{S^{n-1}} \Omega(x')\, d\sigma(x') = 0, \tag{1.1}$$

where $x' = x/|x|$ for any $x \ne 0$.

In 1960, Hörmander [5] defined the parametric Marcinkiewicz function of higher dimension as follows.

$$\mu^\rho_\Omega(f)(x) = \left(\int_0^\infty |F^\rho_t(x)|^2 \frac{dt}{t}\right)^{1/2},$$

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where $\rho > 0$ and

$$F_t^\rho(x) = \frac{1}{t^\rho}\int_{|x-y|\le t} \frac{\Omega(x-y)}{|x-y|^{n-\rho}} f(y)\,dy.$$

When $\rho = 1$, we denote μ_Ω^1 simply by μ_Ω. It is well known that μ_Ω is the usual Marcinkiewicz integral corresponding to the Littlewood-Paley g-function introduced by Stein in [7]. Stein proved that if Ω is continuous and $\Omega \in \mathrm{Lip}_\alpha(S^{n-1})$ $(0 < \alpha \le 1)$, then μ_Ω is of type (p, p) $(1 < p \le 2)$ and of weak type $(1, 1)$. In [1], Benedek, Calderón and Panzone proved that if $\Omega \in C^1(S^{n-1})$, then μ_Ω is of type (p, p) $(1 < p < \infty)$. On the other hand, in 1960, Hörmander [5] proved that if $\Omega \in \mathrm{Lip}_\alpha(S^{n-1})$ $(0 < \alpha \le 1)$, then for $\rho > 0$, μ_Ω^ρ is of type (p, p) $(1 < p < \infty)$. Recently, Sakamoto and Yabuta [6] gave the L^p boundedness of μ_Ω^ρ, $\mu_{\Omega,\lambda}^{*,\rho}$ and and $\mu_{\Omega,S}^\rho$ (see below for the definitions), where ρ is a complex number and $\Omega \in \mathrm{Lip}_\alpha(S^{n-1})$ $(0 < \alpha \le 1)$. It is worth pointing out that Ω was required to satisfy some smoothness conditions in the results mentioned above.

For a long time, an open problem is whether there are some results as above on the L^p boundedness of parametric Marcinkiewicz function μ_Ω^ρ when Ω satisfies only some size condition. The purpose of this note is to give a positive answer. Precisely, we shall consider $L^2(\mathbb{R}^n)$ boundedness of a class of parametric Marcinkiewicz function with kernel functions which lacks smoothness both on the sphere and in radial direction. Let us give some definitions first. The function spaces $l^\infty(L^q)(\mathbb{R}_+)$ are defined as follows. If $1 \le q < \infty$,

$$l^\infty(L^q)(\mathbb{R}_+) = \left\{ h : \|h\|_{l^\infty(L^q)(\mathbb{R}_+)} = \sup_{j\in\mathbb{Z}} \left(\int_{2^{j-1}}^{2^j} |h(r)|^q \frac{dr}{r} \right)^{1/q} < C \right\}. \tag{1.2}$$

If $q = \infty$, $l^\infty(L^\infty)(\mathbb{R}_+) = L^\infty(\mathbb{R}_+)$. By Hölder's inequality, it is easy to check that for $1 < q < r < \infty$

$$l^\infty(L^\infty) \subset l^\infty(L^r) \subset l^\infty(L^q) \subset l^\infty(L^1). \tag{1.3}$$

The parametric Marcinkiewicz function $\mu_{\Omega,h}^\rho$ is defined by

$$\mu_{\Omega,h}^\rho(f)(x) = \left(\int_0^\infty |F_{\Omega,h}^\rho(x,t)|^2 \frac{dt}{t} \right)^{1/2},$$

where ρ is a complex number, $\rho = \alpha + i\tau$ and $h(x)$ is a radial function on $\mathbb{R}^n$ satisfying $h(|x|) \in l^\infty(L^q)(\mathbb{R}_+)$ $(1 \le q \le \infty)$,

$$F_{\Omega,h}^\rho(x,t) = \frac{1}{t^\rho}\int_{|x-y|\le t} \frac{\Omega(x-y)h(|x-y|)}{|x-y|^{n-\rho}} f(y)\,dy.$$

Our main result is the following theorem.

THEOREM 1. *Suppose that* $\Omega \in L\log^+ L(S^{n-1})$ *is a homogeneous function of degree zero on* $\mathbb{R}^n$ *satisfying* (1.1) *and* $h(|x|) \in l^\infty(L^q)(\mathbb{R}_+)$. *If* $1 < q \leq \infty$ *and* $\mathrm{Re}(\rho) = \alpha > 0$, *then* $\|\mu^\rho_{\Omega,h}(f)\|_2 \leq C/\sqrt{\alpha}\|f\|_2$, *where* C *is independent of* ρ *and* f.

As an application of Theorem 1, we obtain also the $L^p(\mathbb{R}^n)(p \geq 2)$ boundedness of the parametric Marcinkiewicz functions $\mu^{*,\rho}_{\Omega,h,\lambda}$ and $\mu^\rho_{\Omega,h,S}$ with the same kernel function Ω and $h(x)$, where $\mu^{*,\rho}_{\Omega,h,\lambda}$ and $\mu^\rho_{\Omega,h,S}$ are respectively defined by

$$\mu^{*,\rho}_{\Omega,h,\lambda}(f)(x) = \left(\iint_{\mathbb{R}^{n+1}_+} \left(\frac{t}{t+|x-y|}\right)^{n\lambda} |F^\rho_{\Omega,h}(y,t)|^2 \frac{dydt}{t^{n+1}}\right)^{1/2}, \quad \lambda > 1,$$

$$\mu^\rho_{\Omega,h,S}(f)(x) = \left(\int_{\Gamma(x)} |F^\rho_{\Omega,h}(y,t)|^2 \frac{dydt}{t^{n+1}}\right)^{1/2},$$

where $\Gamma(x) = \{(y,t) \in \mathbb{R}^{n+1}_+ : |x-y| < t\}$.

THEOREM 2. *If* $2 \leq p < \infty$, *then under the conditions of Theorem* 1 *we have* $\|\mu^{*,\rho}_{\Omega,h,\lambda}(f)\|_p \leq (C/\sqrt{\alpha})\|f\|_p$ *and* $\|\mu^\rho_{\Omega,h,S}(f)\|_p \leq (C/\sqrt{\alpha})\|f\|_p$, *where* $C = C_{\lambda,n,p}$ *is independent of* ρ *and* f.

REMARK 1. Note that

$$\begin{aligned}\mathrm{Lip}_\alpha(S^{n-1})(0 < \alpha \leq 1) \subset L^\infty(S^{n-1}) \subset L^q(S^{n-1})(q > 1) \\ \subset L\log^+ L(S^{n-1}) \subset L^1(S^{n-1}),\end{aligned}$$

and all inclusions are proper. Therefore in Theorem 1 and Theorem 2, the smoothness condition assumed on Ω has been removed and Theorem 1 and Theorem 2 are improvement and extension of the known results mentioned above for $p = 2$ and $2 \leq p < \infty$, respectively.

REMARK 2. After finishing this paper, we were told that in recent work [4], using a method which is quite different from one in this paper, Fan and Sato also gave the L^2-boundedness of Marcinkiewicz integral μ^ρ_Ω when $\Omega \in L\log^+ L(S^{n-1})$ and $h \equiv 1$. From their result, one can deduce that (L^2, L^2) bound of μ^ρ_Ω is only smaller than $C((\mathrm{Re}\,\rho)^{-3/2} + (\mathrm{Re}\,\rho)^{-1/2})$. However, it is smaller than $C(\mathrm{Re}\,\rho)^{-1/2}$ by our method. Hence the conclusion of Theorem 1 in this paper is better than the relevant result in [4].

2. Proofs of Theorem 1 and Theorem 2

Let us begin by recalling a known fact.

LEMMA 1. *Let* $\Omega(x') \in L^\infty(S^{n-1})$. *Then for any* $0 < \theta < 1$ *there is a constant* C *such that for all* $j \in \mathbb{Z}$,

$$\left(\int_{2^j}^{2^{j+1}} \left|\int_{S^{n-1}} \Omega(u') e^{-2\pi i r u'\cdot x}\, d\sigma(u')\right|^2 \frac{dr}{r}\right)^{1/2} \le C\|\Omega\|_{L^\infty(S^{n-1})} |2^j x|^{-\theta/2}.$$

See [3] for the proof.

LEMMA 2. *Let* $\Omega(x') \in L^\infty(S^{n-1})$ *and* $h(r) \in l^\infty(L^q)(\mathbb{R}_+)$, $1 \le q \le 2$. *Then for any* $0 < \theta < 1$ *there is a constant* C *such that for all* $j \in \mathbb{Z}$,

$$\tag{2.1} \begin{aligned} &\int_{2^j}^{2^{j+1}} \left|\int_{S^{n-1}} \Omega(u') e^{-2\pi i r u'\cdot x} h(r)\, d\sigma(u')\right| \frac{dr}{r} \\ &\quad \le C\|h\|_{l^\infty(L^q)(\mathbb{R}_+)} \left(\|\Omega\|_{L^\infty(S^{n-1})} |2^j x|^{-\theta/2}\right)^{2/q'} \left(\|\Omega\|_{L^1(S^{n-1})}\right)^{(q'-2)/q'}. \end{aligned}$$

PROOF. Denote by

$$K(h) = \int_{2^j}^{2^{j+1}} \left|\int_{S^{n-1}} \Omega(u') e^{-2\pi i r u'\cdot x} h(r)\, d\sigma(u')\right| \frac{dr}{r}.$$

First let us consider the case $q = 2$. By Lemma 1 and Hölder's inequality we obtain

$$\tag{2.2} \begin{aligned} K(h) &\le \|h\|_{l^\infty(L^2)(\mathbb{R}_+)} \left(\int_{2^j}^{2^{j+1}} \left|\int_{S^{n-1}} \Omega(u') e^{-2\pi i r u'\cdot x}\, d\sigma(u')\right|^2 \frac{dr}{r}\right)^{1/2} \\ &\le C\|h\|_{l^\infty(L^2)(\mathbb{R}_+)} \|\Omega\|_{L^\infty(S^{n-1})} |2^j x|^{-\theta/2}. \end{aligned}$$

On the other hand, for $q = 1$ we have

$$\tag{2.3} K(h) \le \int_{2^j}^{2^{j+1}} \int_{S^{n-1}} |\Omega(u')|\, d\sigma(u') |h(r)| \frac{dr}{r} \le \|h\|_{l^\infty(L^1)(\mathbb{R}_+)} \|\Omega\|_{L^1(S^{n-1})}.$$

Hence if we see K as a sublinear operator acted on the spaces $l^\infty(L^q)(\mathbb{R}_+)$ for $1 \le q \le 2$, then (2.2) and (2.3) show that K is a bounded operator from $l^\infty(L^2)(\mathbb{R}_+)$ to L^∞ and from $l^\infty(L^1)(\mathbb{R}_+)$ to L^∞, respectively. Using the Riesz-Thorin interpolation theorem for sublinear operator [2] between (2.2) and (2.3), we know there exists an η satisfying $0 < \eta < 1$ and $1/q = (1-\eta) + \eta/2$ such that

$$K(h) \le C\|h\|_{l^\infty(L^q)(\mathbb{R}_+)} \left(\|\Omega\|_{L^\infty(S^{n-1})} |2^j x|^{-\theta/2}\right)^{\eta} \left(\|\Omega\|_{L^1(S^{n-1})}\right)^{1-\eta}.$$

It is easy to see that $\eta = 2/q'$. Thus we finish the proof of Lemma 2. □

Now let us turn to the proof of Theorem 1.

PROOF OF THEOREM 1. By (1.3) we need only consider the case $1 < q \le 2$. First we have

$$\text{(2.4)}\quad \hat{F}^{\rho}_{\Omega,h}(\xi,t) = \int_{\mathbb{R}^n} e^{-2\pi i x\cdot\xi} F^{\rho}_{\Omega,h}(x,t)\,dx = \hat{f}(\xi)\frac{1}{t^{\rho}}\int_{|u|\le t}\frac{\Omega(u)h(|u|)}{|u|^{n-\rho}}e^{-2\pi i u\cdot\xi}\,du.$$

Using Plancherel's theorem and (2.4), the square of $L^2(\mathbb{R}^n)$-norm of $\mu^{\rho}_{\Omega,h}(f)$ is equal to

$$\int_0^\infty\int_{\mathbb{R}^n}|\hat{F}^{\rho}_{\Omega,h}(\xi,t)|^2 d\xi\frac{dt}{t} = \int_{\mathbb{R}^n}|\hat{f}(\xi)|^2\left(\int_0^\infty\left|\int_{|u|\le t}\frac{\Omega(u)h(|u|)}{|u|^{n-\rho}}e^{-2\pi i u\cdot\xi}du\right|^2\frac{dt}{t^{1+2\alpha}}\right)d\xi.$$

Since

$$\text{(2.5)}\quad \left(\int_0^\infty\left|\int_{|u|\le t}\frac{\Omega(u)h(|u|)}{|u|^{n-\rho}}e^{-2\pi i u\cdot\xi}du\right|^2\frac{dt}{t^{1+2\alpha}}\right)^{1/2}$$
$$= \left(\int_0^\infty\left|\int_0^\infty\int_{S^{n-1}}\Omega(u')e^{-2\pi i r u'\cdot\xi}\frac{\chi_{[0,t]}(r)}{r^{1-\rho}}h(r)d\sigma(u')\,dr\right|^2\frac{dt}{t^{1+2\alpha}}\right)^{1/2}$$
$$\le \int_0^\infty\left(\int_0^\infty\left|\int_{S^{n-1}}\Omega(u')e^{-2\pi i r u'\cdot\xi}h(r)\,d\sigma(u')\right|^2\frac{\chi_{[0,t]}(r)}{t^{1+2\alpha}}\,dt\right)^{1/2}\frac{dr}{r^{1-\alpha}}$$
$$= \int_0^\infty\left|\int_{S^{n-1}}\Omega(u')e^{-2\pi i r u'\cdot\xi}h(r)\,d\sigma(u')\right|\left(\int_r^\infty\frac{dt}{t^{1+2\alpha}}\right)^{1/2}\frac{dr}{r^{1-\alpha}}$$
$$= \frac{1}{\sqrt{2\alpha}}\int_0^\infty\left|\int_{S^{n-1}}\Omega(u')e^{-2\pi i r u'\cdot\xi}h(r)\,d\sigma(u')\right|\frac{dr}{r}.$$

On the other hand, note that for any $s > 0$, we have

$$\left(\int_{2^{j-1}}^{2^j}|h(rs)|^q\frac{dr}{r}\right)^{1/q} = \left(\int_{2^{j-1}s}^{2^j s}|h(r)|^q\frac{dr}{r}\right)^{1/q} \le 2\|h\|_{l^\infty(L^q)(\mathbb{R}_+)}.$$

Therefore, by (2.5) to prove Theorem 1 it suffices to show that for $\Omega \in L\log^+ L(S^{n-1})$, there is a constant C such that

$$\text{(2.6)}\qquad \sup_{x'\in S^{n-1}}\int_0^\infty\left|\int_{S^{n-1}}\Omega(u')e^{-2\pi i r u'\cdot x'}h(r)d\sigma(u')\right|\frac{dr}{r} \le C.$$

For any $x' \in S^{n-1}$, we denote $G(x',r) = \int_{S^{n-1}}\Omega(u')e^{-2\pi i r u'\cdot x'}d\sigma(u')$ and write

$$\int_0^\infty|G(x',r)h(r)|\frac{dr}{r} = \int_0^2|G(x',r)h(r)|\frac{dr}{r} + \int_2^\infty|G(x',r)h(r)|\frac{dr}{r} =: \mathrm{I}+\mathrm{II}.$$

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Below we shall show that I and II are uniformly bounded on $x' \in S^{n-1}$. By (1.1), we have

$$\text{(2.7)}\quad \mathrm{I} = \int_0^2 \left| \int_{S^{n-1}} \Omega(u')(e^{-2\pi i r u'\cdot x'} - 1) h(r)\, d\sigma(u') \right| \frac{dr}{r} \le C \|h\|_{l^\infty(L^q)(\mathbb{R}_+)} \|\Omega\|_{L^1(S^{n-1})}.$$

In order to give the estimate of II, we need to use some idea from [8]. Set

$$\begin{aligned} E_0 &= \{u' \in S^{n-1} : |\Omega(u')| \le 2\}, \\ E_l &= \{u' \in S^{n-1} : 2^l < |\Omega(u')| \le 2^{l+1}\} \quad \text{for} \quad l \ge 1, \\ \Omega_l(u') &= \Omega(u')\chi_{E_l}(u') \quad \text{for} \quad l \ge 0, \\ G_l(x', r) &= \int_{S^{n-1}} \Omega_l(u') e^{-2\pi i r u'\cdot x'} d\sigma(u') \quad \text{for} \quad l \ge 0, \end{aligned}$$

where $\chi_{E_l}(u')$ is the characteristic function of E_l. Taking $s \in \mathbb{N}$ such that $s\theta > q'$, where $0 < \theta < 1$ is defined in Lemma 1. Then we have

$$\begin{aligned} \mathrm{II} &\le \sum_{j=1}^{\infty} \int_{2^j}^{2^{j+1}} |G_0(x', r)h(r)| \frac{dr}{r} + \left(\sum_{l>0} \sum_{1\le j \le sl} + \sum_{l>0} \sum_{j>sl} \right) \int_{2^j}^{2^{j+1}} |G_l(x', r)h(r)| \frac{dr}{r} \\ &=: \mathrm{II}_1 + \mathrm{II}_2 + \mathrm{II}_3 . \end{aligned}$$

Now let us give the estimates for II_1, II_2 and II_3, respectively. By Hölder's inequality

$$\text{(2.8)}\qquad \int_{2^j}^{2^{j+1}} |G_0(x', r)h(r)| \frac{dr}{r} \le \|h\|_{l^\infty(L^q)(\mathbb{R}_+)} \left(\int_{2^j}^{2^{j+1}} |G_0(x', r)|^{q'} \frac{dr}{r} \right)^{1/q'} .$$

Since $|G_0(x', r)| \le 2|S^{n-1}|$ and $2 \le q' < \infty$, by (2.8) we have

$$\begin{aligned} \text{(2.9)}\qquad & \int_{2^j}^{2^{j+1}} |G_0(x', r)h(r)| \frac{dr}{r} \\ &\le C\|h\|_{l^\infty(L^q)(\mathbb{R}_+)} \left(\int_{2^j}^{2^{j+1}} |G_0(x', r)|^2 |G_0(x', r)|^{q'-2} \frac{dr}{r} \right)^{1/q'} \\ &\le C\|h\|_{l^\infty(L^q)(\mathbb{R}_+)} \left(\int_{2^j}^{2^{j+1}} |G_0(x', r)|^2 \frac{dr}{r} \right)^{1/q'} . \end{aligned}$$

By Lemma 1 and (2.9) we see that

$$\text{(2.10)}\qquad \mathrm{II}_1 = \sum_{j=1}^{\infty} \int_{2^j}^{2^{j+1}} |G_0(x', r)h(r)| \frac{dr}{r}$$

$$\leq C\|h\|_{l^\infty(L^q)(\mathbb{R}_+)} \sum_{j=1}^{\infty} |2^j x'|^{-\theta/q'} \leq C\|h\|_{l^\infty(L^q)(\mathbb{R}_+)}.$$

For II_2 and $1 < q \leq 2$ we obtain

$$\begin{aligned}
(2.11)\quad \mathrm{II}_2 &\leq \sum_{l>0} \sum_{1\leq j\leq sl} \int_{2^j}^{2^{j+1}} \int_{S^{n-1}} |\Omega_l(u')| d\sigma(u') |h(r)| \frac{dr}{r} \\
&\leq C\|h\|_{l^\infty(L^q)(\mathbb{R}_+)} \sum_{l>0} \sum_{1\leq j\leq sl} (\log 2)^{1/q'} \cdot \|\Omega_l\|_{L^1(S^{n-1})} \\
&\leq C\|h\|_{l^\infty(L^q)(\mathbb{R}_+)} \sum_{l>0} l \log 2 \cdot 2^{l+1} |E_l| \leq C\|h\|_{l^\infty(L^q)(\mathbb{R}_+)} \|\Omega\|_{L \log^+ L(S^{n-1})}.
\end{aligned}$$

Finally, let us estimate II_3. Applying Lemma 2, we have

$$\begin{aligned}
(2.12)\quad \mathrm{II}_3 &= \sum_{l>0} \sum_{j>sl} \int_{2^j}^{2^{j+1}} \left| \int_{S^{n-1}} \Omega_l(u') e^{-2\pi i r u'\cdot x'} h(r) d\sigma(u') \right| \frac{dr}{r} \\
&\leq C\|h\|_{l^\infty(L^q)(\mathbb{R}_+)} \sum_{l>0} \sum_{j>sl} (\|\Omega_l\|_{L^\infty(S^{n-1})} |2^j x'|^{-\theta/2})^{2/q'} (\|\Omega_l\|_{L^1(S^{n-1})})^{(q'-2)/q'} \\
&\leq C\|h\|_{l^\infty(L^q)(\mathbb{R}_+)} \sum_{l>0} \sum_{j>sl} (2^l \cdot 2^{-j\theta/2})^{2/q'} (2^l |S^{n-1}|)^{(q'-2)/q'} \\
&\leq C\|h\|_{l^\infty(L^q)(\mathbb{R}_+)} \sum_{l>0} 2^l \cdot 2^{-sl\theta/q'} \leq C\|h\|_{l^\infty(L^q)(\mathbb{R}_+)}.
\end{aligned}$$

It is easy to see that the constants in (2.7) and (2.10)–(2.12) are independent of x'. Therefore, (2.6) follows from (2.7) and (2.10)–(2.12). Thus we complete the proof of Theorem 1. □

Before giving the proof of Theorem 2, we give a lemma.

LEMMA 3. *Let $\lambda > 1$. Then under the conditions of Theorem 1, there is a constant $C > 0$ such that for any nonnegative and locally integrable function ϕ,*

$$\left(\int_{\mathbb{R}^n} \mu^{*,\rho}_{\Omega,h,\lambda}(f)(x)^2 \phi(x) dx \right)^{1/2} \leq \frac{C_{\lambda,n}}{\sqrt{\alpha}} \left(\int_{\mathbb{R}^n} |f(x)|^2 M\phi(x)\, dx \right)^{1/2},$$

where M denotes the Hardy-Littlewood maximal operator.

The proof of Lemma 3 follows by using the method in [9, pages 241–242] and the conclusion of Theorem 1. We omit the details here. Now let us return to the proof of Theorem 2.

PROOF OF THEOREM 2. For $2 \le p < \infty$, we have

$$\begin{aligned}(2.13)\qquad \|\mu^{*,\rho}_{\Omega,h,\lambda}(f)\|_p &= \left\{\left(\int_{\mathbb{R}^n} [\mu^{*,\rho}_{\Omega,h,\lambda}(f)(x)^2]^{p/2} dx\right)^{2/p}\right\}^{1/2} \\ &= \left\{\sup_{\phi}\left|\int_{\mathbb{R}^n} \mu^{*,\rho}_{\Omega,h,\lambda}(f)(x)^2\phi(x)\,dx\right|\right\}^{1/2},\end{aligned}$$

where the supremum is taken over all $\phi(x)$ satisfying $\|\phi\|_{(p/2)'} \le 1$. Applying Lemma 3, Hölder's inequality and the $L^{(p/2)'}(1 < (p/2)' \le \infty)$ boundedness of Hardy-Littlewood maximal operator M, we get

(2.14)
$$\left(\int_{\mathbb{R}^n} \mu^{*,\rho}_{\Omega,h,\lambda}(f)(x)^2|\phi(x)|\,dx\right)^{1/2} \le \frac{C}{\sqrt{\alpha}}\left(\int_{\mathbb{R}^n} |f(x)|^2 M\phi(x)\,dx\right)^{1/2} \le \frac{C}{\sqrt{\alpha}}\|f\|_p.$$

By (2.13) and (2.14) we have $\|\mu^{*,\rho}_{\Omega,h,\lambda}(f)\|_p \le C/\sqrt{\alpha}\|f\|_p$. On the other hand, using the idea in [9] it is easy to prove that $\mu^{\rho}_{\Omega,h,S}(f)(x) \le 2^{\lambda n}\mu^{*,\rho}_{\Omega,h,\lambda}(f)(x)$. Thus we complete the proof of Theorem 2. □

Finally, we give another direct application of Lemma 3. It is well known that if $\omega \in A_1$, then $M\omega(x) \le C\omega(x)$ a.e. on $\mathbb{R}^n$. Hence by Lemma 3, we get immediately the weighted L^2 boundedness for $\mu^{*,\rho}_{\Omega,h,\lambda}$ and $\mu^{\rho}_{\Omega,h,S}$.

COROLLARY 1. *Under the conditions of Theorem* 1, *if* $\omega \in A_1$, *then*

$$\|\mu^{\rho}_{\Omega,h,S}(f)\|_{2,\omega} \le C_{\lambda,n}\|\mu^{*,\rho}_{\Omega,h,\lambda}(f)\|_{2,\omega} \le \frac{C_{\lambda,n}}{\sqrt{\alpha}}\|f(x)\|_{2,\omega}.$$

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