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ON WEIGHTED GEODESICS IN GROUPS

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ABSTRACT. A word W in a group G is a geodesic (weighted geodesic) if W has minimum length (minimum weight with respect to a generator weight function α) among all words equal to W. For finitely generated groups, the word problem is equivalent to the geodesic problem. We prove: (i) There exists a group G with solvable word problem, but unsolvable geodesic problem. (ii) There exists a group G with a solvable weighted geodesic problem with respect to one weight function α_1, but unsolvable with respect to a second weight function α_2. (iii) The (ordinary) geodesic problem and the free-product geodesic problem are independent.

1. **Introduction, Geodesics**. Let G be a group given in terms of generators $x_1, x_2, \ldots$ and defining relations $R_1, R_2, \ldots$. The group G is said to be *recursively presented* if there exists an effective process which lists the words R_n. A word W in G is called a *geodesic* if the length of W, denoted by $f(W)$, is minimum among the lengths of words equal to W. Note that if W is a geodesic, then W represents a path of minimum length from 1 to W in the graph of the group G.

The *weak geodesic problem* is said to be solvable for the group G if, for any word W in G, we can decide whether or not W is a geodesic. The *strong geodesic problem* is said to be solvable for G if, for any word W in G, we can find a geodesic W^* which is equal to W.

The following two comments are in order.

COMMENT 1. The weak and strong geodesic problems are equivalent.

COMMENT 2. If G is finitely generated, then the word problem is equivalent to the (weak) geodesic problem.

PROOF OF COMMENTS. Let E be a word in G such that $E = 1$. Recall that E is freely equal to a product of conjugates of defining relators:

$$E \approx \prod_{i=1}^{m} T_i^{-1} R_{j_i} T_i$$

Observe that, for a given n, there are only a finite number of such words where $m \leq n$, $j_i \leq n$, $f(T_i) \leq n$, and T_i only involves the generators $x_1, x_2, \ldots, x_n$. Accordingly, we can effectively list all the words $E_1, E_2, \ldots$ which are equal to 1.

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First we prove Comment 1. Suppose we can solve the weak geodesic problem in G, and we know that $W = W_1$ is not a geodesic. Then we freely reduce $W_1E_1, W_1E_2, W_1E_3, \ldots$ until we obtain a word W_2 such that $f(W_2) < f(W_1)$. Such a word W_2 exists since W_1 is not a geodesic. Either W_2 is a geodesic, or we can repeat the process with W_2 to obtain a word W_3. The sequence $W_1, W_2, \ldots$ cannot be infinite, so we must finally obtain a geodesic W^* which is equal to W. Thus Comment 1 is proved.

Next we prove Comment 2. Suppose we can solve the word problem in G. Let W be any word in G. Since G is finitely generated, there exists only a finite number of words W_i such that $f(W_i) < f(W)$. Hence we can decide whether or not there exists W_i such that $W = W_i$. If not, then W is a geodesic; otherwise, W is not a geodesic. On the other hand, suppose the weak geodesic problem is solvable for G. As in Comment 1, we can find in a geodesic W^* such that $W = W^*$. Clearly, $W = 1$ if and only if $f(W^*) = 0$. Thus Comment 2 is proved.

Our first result tells us that the condition of being finitely generated is necessary for Comment 2. That is:

THEOREM 1. *There exists a recursively presented group G with solvable word problem, but unsolvable geodesic problem.*

PROOF. The standard way to prove such theorems is to use an injective semi-computable function $\phi: N \to N$. That is, given a positive integer k, we can compute $\phi(k)$, but we cannot decide if k belongs to the Im ϕ. Such functions are known to exist; c.f. Britten [1, Lemma 2.31]. Now let G be the group with generators

$$x_1, x_2, x_3, \ldots \quad \text{and} \quad y_1, y_2, y_3, \ldots$$

and defining relations

$$y_1 = x^2_{\phi(1)}, y_2 = x^2_{\phi(2)}, y_3 = x^2_{\phi(3)}, \ldots$$

Clearly, G is recursively presented since the function ϕ is recursive. Given any word $W = W(x_i, y_j)$ in G, we can replace each y_j by $x^2_{\phi(j)}$ to obtain a word only involving x's. Note G is freely generated by the x's. Hence we can decide if $W = 1$. In other words, the word problem is solvable for G. On the other hand $V = x_k^2$ is a geodesic if and only if k belongs to Im ϕ. But Im ϕ is nonrecursive. Hence the geodesic problem is unsolvable for G. Thus the theorem is proved.

We close this section with an open problem. Let C(word) and C(geodesic) denote, respectively, the time complexities for solving the word and geodesic problems for a group G. The proof of Comment 2 shows that

$$C(\text{geodesic}) \leq m\,C(\text{word})$$

where m is equal to the number of generators of the group G. On the other hand, for the free group $F = gp(x_1, x_2, \ldots)$, we have

$$C(\text{geodesic}) = C(\text{word})$$

since, in both cases, we simply freely reduce a given word W. This leads to:

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PROBLEM A. *Characterize those groups or find other classes of groups for which* C(geodesic) = C(word).

2. **Weighted Geodesics**. My interest in geodesics actually comes from certain problems in graph theory and computer science. Recall that graph theorists are frequently interested in finding shortest paths where the edges of the graph are weighted. Also, in computer science, one only counts the number of multiplications, not additions, when calculating the determinant of a matrix. These notions naturally lead to the definition of a weighted geodesic in a group G where the generators are viewed as operations.

Let G be a group with generators $x_1, x_2, \ldots$. We say α is a *(generator) weight function* for G if α assigns a nonnegative integer $\alpha(x_i)$ to each generator of G. Then α induces a weight $\alpha(W)$ for each word W in G in the usual way, as the sum of the weights of the generators in W. (Note that $\alpha(1) = 0$ and $\alpha(W) = \alpha(W^{-1})$.)

The definition of a weighted geodesic is analogous to the nonweighted case. Specifically, we define

$$g(W) = \inf\{\alpha(V) : V = W\}.$$

Then W is a *weighted geodesic* iff $g(W) = \alpha(W)$. Note that if $W_1 = W_2$ then $g(W_1) = g(W_2)$.

The *weak weighted geodesic problem* is said to be solvable for the group G if, for any word W in G, we can decide whether or not $g(W) = \alpha(W)$. (This is analogous to the nonweighted case.) The *strong weighted geodesic problem* is solvable for G if, for any word W in G, we can find a geodesic W^* which is equal to W. Observe that Comment 1 still holds for this general situation. Specifically, the sequence $W_1, W_2, \ldots$ in the proof of Comment 1 must still stop since $f(W_k)$ is an integer and $f(W_{k+1}) > f(W_k)$. On the other hand, Comment 2 requires an additional condition.

We say α is a *positive weight function* for a group G if α assigns a positive integer $\alpha(x_i)$ to each generator x_i of G. In such a case, $W = 1$ if and only if $g(W) = 0$. Moreover, if G is finitely generated, then, for any integer N, there exists only a finite number of words W such that $\alpha(W) \leq N$. Using these facts, the proof of the next theorem is essentially the same as the proof of the above comments.

THEOREM 2. *Suppose G is a recursively presented group with a positive weight function α.*

(a) *If G has a solvable weighted geodesic problem, then G has a solvable word problem.*

(b) *If G is finitely generated, then the word problem and the weighted geodesic problem are equivalent for G.*

The condition that α assign positive weights to the generators is necessary. For example, suppose G is a (Boone-Novikov) finitely presented group with an unsolvable word problem. (See [3].) We can then let α assign 0 to every generator in G to obtain a group with a solvable geodesic problem (all words are geodesics) and an unsolvable word problem. The condition that G be finitely generated in part (b) is also necessary as seen by the following theorem.

THEOREM 3. *There exists a recursively presented group G with a solvable weighted geodesic problem with respect to one positive weight function α_1 but with an unsolvable weighted geodesic problem with respect to another positive weight function α_2.*

PROOF. Again we use an injective semi-computable function $\phi : N \to N$ (as in the proof of Theorem 1). Now let G be the group with generators

$$x_1, x_2, x_3 \ldots \quad \text{and} \quad y_1, y_2, y_3, \ldots$$

and defining relations

$$y_1 = x_{\phi(1)}, \; y_2 = x_{\phi(2)}, \; y_3 = x_{\phi(3)}, \ldots$$

Observe that G is recursively presented. Also G is freely generated by the x's. Consider the weight function:

$$\alpha_1(x_1) = 1, \; \alpha_1(x_2) = 1, \ldots, \alpha_1(y_1) = 1, \alpha_1(y_2) = 1, \ldots$$

Then G with the weight function α_1 has solvable geodesic problem. On the other hand, consider the weight function

$$\alpha_2(x_1) = 2, \; \alpha_2(x_2) = 2, \ldots, \alpha_2(y_1) = 1, \alpha_2(y_2) = 1, \ldots$$

Here x_k is a geodesic if and only if k does not belong to $\operatorname{Im} \phi$. This means that the geodesic problem with respect to α_2 is unsolvable for G. Thus the theorem is proved. (Note that G does have a solvable word problem.)

3. **Free product lengths and weights**. This section is motivated by the group G of the Rubik's Cube. Recall that G has six natural generators, $s_1, s_2, \ldots, s_6$, where s_i rotates one of the 6 faces by 90°. The time that it takes to return the Rubik's Cube to the identity position may not be through a geodesic. The reason is that it takes less time to execute s_i^2, rotating a face 180°, than to execute $s_j s_k$, rotating one face 90° and another face 90°. In fact, the time it takes to execute s_i^2 may be approximately the same as the time it takes to execute s_i.

The above discussion leads us to the following definitions. First of all, let $f^*(W)$ denote the free product length of a word W in G. Also, when G has a weight function α, let $\alpha^*(W)$ denote the free product weight of W. For example, suppose a and b are generators of a group G and $\alpha(a) = 1$ and $\alpha(b) = 2$; then

$$f^*(a^3 b^{-4} a b^5 a^{-2}) = 5 \quad \text{and} \quad \alpha^*(a^3 b^{-4} a b^5 a^{-2}) = 7$$

More generally, let β be a function which assigns nonnegative integers to the words of a group G such that:

$$\beta(1) = 0, \quad \beta(W) = \beta(W^{-1}), \quad \beta(W_1 W_2) \leq \beta(W_1) + \beta(W_2).$$

(The functions f, f^*, α and α^* are examples of such a function.) The notion of a *β-geodesic*, that is, a geodesic with respect to the function β, is now clear. Thus we can speak of G having a *solvable β-geodesic problem* (weak or strong), that is, a solvable geodesic problem with respect to the function β.

The main result of this section follows.

THEOREM 4. *There exists a recursively presented group G_1 with solvable f^*-geodesic problem and unsolvable f-geodesic problem, and there exists a recursively presented group G_2 with solvable f-geodesic problem and unsolvable f^*-geodesic problem. That is, the f-geodesic problem and the f^*-geodesic problem are independent.*

PROOF. Let G_1 be the group in Theorem 1 which has an unsolvable f-geodesic problem. Observe that G_1 has a solvable f^*-geodesic problem since replacing $x^2_{\phi(j)}$ by y_j does not decrease the free product length of a word in G_1.

Now let G_2 be the group with generators

$$x_1, y_1, z_1, x_2, y_2, z_2, x_3, y_3, z_3, \ldots$$

and defining relations

$$z_1^3 = x_{\phi(1)}y_{\phi(1)},\ z_2^3 = x_{\phi(2)}y_{\phi(2)},\ z_3^3 = x_{\phi(3)}y_{\phi(3)}, \ldots$$

where $\phi: N \to N$ is an injective semi-computable function as in Theorem 1. The f^*-geodesic problem is unsolvable for G_2 since $x_k y_k$ is an f^*-geodesic if and only if k does not belong to $\operatorname{Im}\phi$. It remains to show that the usual geodesic problem, i.e. the f-geodesic problem, is solvable in G_2.

Consider the following subgroups of G_2:

$$F_s = gp(z_s, x_{\phi(s)}, y_{\phi(s)}) \qquad \text{where } s \in N,$$

$$H_k = gp(x_k, y_k) \qquad \text{where } k \notin \operatorname{Im}\phi$$

Observe that G_2 is the free product of the F_s and the H_k. Note that each H_k is free on x_k, y_k, and that F_s has the single defining relation $z_s^3 = x_{\phi(s)}y_{\phi(s)}$. Accordingly, by the Freiheitsatz [2], F_s has a solvable word problem and $x_{\phi(s)}$ and $y_{\phi(s)}$ are free generators in F_s. Thus the geodesic problem is solvable for each H_k and each F_s.

Now let W be any word in G_2. Clearly, W may be written in a normal form

$$W = W_1 W_2 W_3 \ldots W_m$$

where W_j only involves x_k and y_k or W_j only involves z_s, $x_{\phi(s)}$ and $y_{\phi(s)}$. The solution of the geodesic problem for G_2 follows from the following crucial remark! A freely reduced word $W_j = W_j(x_k, y_k)$ is a geodesic regardless of whether W_j belongs to an F_s or to a H_k. This remark follows from the fact that $x_{\phi(s)}$ and $y_{\phi(s)}$ are free generators in G_s and that replacing $x_{\phi(s)}y_{\phi(s)}$ by z_s^3 in W_j can never lead to a decrease in the length of W_j. Thus the theorem is proved.

We close this paper with two more problems. First observe that, for a given word W in a group G, even when G is finitely generated, there may be an infinite number of words V such that $f^*(V) < f^*(W)$. Therefore, the proof of Comment 2 does not hold for the function f^*.

PROBLEM B. *Find a finitely generated group G which has a solvable word problem (and hence a solvable f-geodesic problem), but an unsolvable f*-geodesic problem.*

PROBLEM C. *Let G be a small-cancellation group which is known to have a solvable word problem (and hence a solvable f-geodesic problem). Show that G also has a solvable f*-geodesic problem.*

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