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COEFFICIENT ESTIMATES FOR A CLASS OF STAR-LIKE FUNCTIONS

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1. Introduction. In this note we continue the study, initiated in [**1**], of the class $S^*(\alpha)$ of functions

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k \tag{1.1}$$

that are analytic and univalent in the unit disc U and satisfy the condition

$$-\alpha\frac{\pi}{2} < \arg\frac{zf'(z)}{f(z)} < \alpha\frac{\pi}{2} \qquad (0 < \alpha \leqq 1). \tag{1.2}$$

$S^*(1)$ is the frequently studied class of univalent star-like functions. For each α, $S^*(\alpha)$ is a subclass of the class $K(\alpha)$ of close-to-convex functions of order α introduced by Pommerenke [**4**]. Properties of the class $S^*(\alpha)$ proved useful in studying the coefficient behaviour of bounded univalent functions that are analytic and map U onto a convex domain [**1**]. In this note we investigate the problem of determining

$$A_n(\alpha) = \max_{f\in S^*(\alpha)} |a_n| \tag{1.3}$$

but we are able to give only a partial solution.

In § 3 we introduce the related class $\Sigma^*(\alpha)$ of functions

$$F(z) = \frac{1}{z} + \sum_{k=0}^{\infty} A_k z^k$$

that are analytic and univalent in the punctured disc and satisfy the condition

$$\left(1 - \frac{\alpha}{2}\right)\pi < \arg\frac{zF'(z)}{F(z)} < \left(1 + \frac{\alpha}{2}\right)\pi \qquad (0 < \alpha \leqq 1). \tag{1.4}$$

$\Sigma^*(1)$ is the class of univalent meromorphic star-like functions studied in [**3**; **5**]. For the class of functions $\Sigma^*(\alpha)$, we show that

$$|A_n| \leqq \frac{2\alpha}{n+1}$$

with equality for a fixed integer n if and only if

$$\frac{zF'(z)}{F(z)} = -\left(\frac{1+\epsilon z^{n+1}}{1-\epsilon z^{n+1}}\right)^{\alpha} \qquad (|\epsilon| = 1).$$

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It is convenient to denote by $\mathscr{P}_\alpha$ $(0 < \alpha \leqq 1)$ the class of functions

$$P(z) = 1 + \sum_{k=1}^{\infty} p_k z^k$$

that are analytic in U and subordinate to the function $((1 + z)/(1 - z))^\alpha$. We note that $P(z) \in \mathscr{P}_\alpha$ if and only if $P(z) = [Q(z)]^\alpha$, where $Q(z) \in \mathscr{P}_1$.

For future reference we observe that (1.2) and (1.4) are equivalent to

$$\frac{zf'(z)}{f(z)} = [P(z)]^\alpha \tag{1.5}$$

and

$$\frac{zF'(z)}{F(z)} = -[P(z)]^\alpha, \tag{1.6}$$

respectively, where $P(z)$ belongs to $\mathscr{P}_1$.

2. We begin by determining $A_n(\alpha)$ in the case that $n = 2$ and $n = 3$.

THEOREM 2.1. *Let*

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k$$

belong to $S^*(\alpha)$ $(0 < \alpha \leqq 1)$. *Then* $|a_2| \leqq 2\alpha$, *with equality if and only if*

$$\frac{zf'(z)}{f(z)} = \left[\frac{1 + \epsilon z}{1 - \epsilon z}\right]^\alpha \qquad (|\epsilon| = 1).$$

If $0 < \alpha < \frac{1}{3}$, *then* $|a_3| \leqq \alpha$ *with equality if and only if*

$$\frac{zf'(z)}{f(z)} = \left[\frac{1 + \epsilon z^2}{1 - \epsilon z^2}\right]^\alpha \qquad (|\epsilon| = 1);$$

if $\frac{1}{3} < \alpha \leqq 1$, *then* $|a_3| \leqq 3\alpha^2$ *with equality if and only if*

$$\frac{zf'(z)}{f(z)} = \left[\frac{1 + \epsilon z}{1 - \epsilon z}\right]^\alpha \qquad (|\epsilon| = 1);$$

and if $\alpha = \frac{1}{3}$, *then* $|a_3| \leqq \frac{1}{3}$, *with equality if and only if*

$$\frac{zf'(z)}{f(z)} = \left[\lambda\left(\frac{1 + \epsilon z}{1 - \epsilon z}\right) + (1 - \lambda)\left(\frac{1 + \epsilon^2 z^2}{1 - \epsilon^2 z^2}\right)\right]^{1/3}, \tag{2.1}$$

where $|\epsilon| = 1$ *and* $0 \leqq \lambda \leqq 1$.

Proof. If $f(z) \in S^*(\alpha)$, then by (1.5),

$$\frac{zf'(z)}{f(z)} = [P(z)]^\alpha = \left[1 + \sum_{k=1}^{\infty} p_k z^k\right]^\alpha, \tag{2.2}$$

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where $P(z) \in \mathscr{P}_1$. From (2.2) it follows that $a_2 = \alpha p_1$ and

$$2a_3 = \alpha\left[p_2 + \frac{3\alpha - 1}{2} p_1^2\right]. \tag{2.3}$$

By a well-known theorem due to Carathéodory [**2**], $|p_n| \leqq 2$ and $|p_1| = 2$ if and only if

$$P(z) = \frac{1 + \epsilon z}{1 - \epsilon z},$$

where $|\epsilon| = 1$. This completes the proof of the first part of the theorem.

If $\frac{1}{3} < \alpha \leqq 1$, then since $|p_2| \leqq 2$, (2.3) implies that $|a_3| \leqq 3\alpha^2$ and again equality holds if and only if

$$\frac{zf'(z)}{f(z)} = \left[\frac{1 + \epsilon z}{1 - \epsilon z}\right]^\alpha.$$

If $\alpha = \frac{1}{3}$, then by (2.3) $|a_3| \leqq \frac{1}{3}$ with equality if and only if $|p_2| = 2$. It follows from Carathéodory's theorem that if $|p_2| = 2$, then

$$P(z) = \lambda \frac{1 + \epsilon z}{1 - \epsilon z} + (1 - \lambda) \frac{1 + \epsilon^2 z^2}{1 - \epsilon^2 z^2},$$

where $|\epsilon| = 1$ and $0 \leqq \lambda \leqq 1$; consequently, $zf'(z)/f(z)$ satisfies (2.1).

It remains to consider the case $0 < \alpha < \frac{1}{3}$. By (2.3) we have

$$2 \operatorname{Re} a_3 = \alpha \operatorname{Re}\left\{p_2 - \frac{1 - 3\alpha}{2} p_1^2\right\}. \tag{2.4}$$

Since $P(z) \in \mathscr{P}_1$, the Herglotz representation formula (see [**7**, p. 232]) states that

$$P(z) = \int_0^{2\pi} \frac{1 + ze^{-it}}{1 - ze^{-it}} \, d\mu(t),$$

where $\mu(t)$ is increasing on $[0, 2\pi]$, and $\mu(2\pi) - \mu(0) = 1$. It follows that

$$p_n = 2\int_0^{2\pi} e^{-int} \, d\mu(t) \qquad (n = 1, 2, \ldots).$$

Substituting in (2.4) we have

$$\begin{aligned} 2 \operatorname{Re} a_3 &= 2\alpha \int_0^{2\pi} \cos 2t \, d\mu(t) \\ &\quad - 2\alpha(1 - 3\alpha)\left\{\left[\int_0^{2\pi} \cos t \, d\mu(t)\right]^2 - \left[\int_0^{2\pi} \sin t \, d\mu(t)\right]^2\right\} \\ &\leqq 2\alpha \int_0^{2\pi} \cos 2t \, d\mu(t) + 2\alpha(1 - 3\alpha)\left[\int_0^{2\pi} \sin t \, d\mu(t)\right]^2 \\ &= 2\alpha\left\{1 + (1 - 3\alpha)\left[\int_0^{2\pi} \sin t \, d\mu(t)\right]^2 - 2\int_0^{2\pi} \sin^2 t \, d\mu(t)\right\}. \end{aligned} \tag{2.5}$$

By Jensen's inequality [7, p. 61],

$$\left[\int_0^{2\pi} |\sin t|\, d\mu(t)\right]^2 \leqq \int_0^{2\pi} \sin^2 t\, d\mu(t),$$

and thus it follows from (2.5) that $2 \operatorname{Re} a_3 \leqq 2\alpha$.

If $2 \operatorname{Re} a_3 = 2\alpha$, $\mu(t)$ must satisfy

$$\int_0^{2\pi} \sin^2 t\, d\mu(t) = 0 \tag{2.6}$$

and

$$\int_0^{2\pi} \cos t\, d\mu(t) = 0. \tag{2.7}$$

(2.6) is possible only if $\mu(t)$ is constant on $(0, \pi)$ and on $(\pi, 2\pi)$. For such a $\mu(t)$, (2.7) is possible only if the jump of $\mu(t)$ at $t = \pi$ equals the sum of the jumps at $t = 0$ and $t = 2\pi$. It follows that $\operatorname{Re} a_3 = \alpha$ if and only if

$$\frac{zf'(z)}{f(z)} = \left[\frac{1}{2}\frac{1+z}{1-z} + \frac{1}{2}\frac{1-z}{1+z}\right]^\alpha = \left[\frac{1+z^2}{1-z^2}\right]^\alpha,$$

and therefore $|a_3| = \alpha$ if and only if

$$\frac{zf'(z)}{f(z)} = \left[\frac{1+\epsilon z^2}{1-\epsilon z^2}\right]^\alpha \qquad (|\epsilon| = 1).$$

This completes the proof of the theorem.

The "logical" choice for an extremal function for the problem of determining $A_n(\alpha)$ would be the function $f_\alpha(z)$ defined by

$$\frac{zf_\alpha'(z)}{f_\alpha(z)} = \left(\frac{1+z}{1-z}\right)^\alpha. \tag{2.8}$$

As seen in the previous theorem, if $n = 3$ and $0 < \alpha < \frac{1}{3}$, $f_\alpha(z)$ is not an extremal function for this problem. The next theorem shows however that for each n, $f_\alpha(z)$ is an extremal function provided α is sufficiently near 1.

THEOREM 2.2. *Let* $f(z) = z + \sum_{k=2}^\infty a_k z^k$ *belong to* $S^*(\alpha)$, $0 < \alpha \leqq 1$, *and let* $n > 1$ *be a fixed integer. There exists a number* β_n $(0 < \beta_n < 1)$ *such that if* $\beta_n < \alpha \leqq 1$, $|a_n| = A_n(\alpha)$ *if and only if* $f(z) = \bar{\epsilon} f_\alpha(\epsilon z)$, *where* $f_\alpha(z)$ *is defined by* (2.8) *and* $|\epsilon| = 1$.

Proof. By (1.5),

$$\frac{zf'(z)}{f(z)} = [P(z)]^\alpha = 1 + \sum_{k=1}^\infty \alpha_k z^k, \tag{2.9}$$

where $P(z) \in \mathscr{P}_1$. If $P(z) = 1 + \sum_{k=1}^\infty p_k z^k$, then it follows from (2.9) that

$$\alpha_k = \sum \Psi(\alpha; m_1, \ldots, m_j) p_{m_1} \ldots p_{m_j} \tag{2.10}$$

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(where $\Psi(\alpha; m_1, \ldots, m_j)$ is a polynomial of degree at most k in α) is independent of $P(z)$; and the summation is taken over all j-tuples $(m_1, \ldots, m_j)$ of positive integers which satisfy

$$m_1 \leqq \ldots \leqq m_j \quad \text{and} \quad m_1 + \ldots + m_j = k.$$

Also from (2.9) we have

$$(2.11) \qquad (k-1)a_k = \alpha_1 a_{k-1} + \ldots + \alpha_{k-2}a_2 + \alpha_{k-1}.$$

Using (2.10) and induction we deduce

$$(2.12) \qquad a_n = a_n(\alpha) = \sum \phi(\alpha; m_1, \ldots, m_j)p_{m_1} \ldots p_{m_j}$$

(where $\phi(\alpha; m_1, \ldots, m_j)$ is a polynomial of degree at most $n-1$ in α) is independent of $f(z)$; and the range of summation is as defined in (2.10) with $k = n - 1$. If $\alpha = 1$, then $\alpha_k = p_k$. An induction argument using (2.11) and (2.12) shows that $\phi(1; m_1, \ldots, m_j) > 0$ for all $m_1 \leqq \ldots \leqq m_j$ with $m_1 + \ldots + m_j = n - 1$. It follows that there is a constant β_n, $0 < \beta_n < 1$, such that each $\phi(\alpha; m_1, \ldots, m_j)$ is positive in the interval $(\beta_n, 1]$. Thus by (2.12), $a_n(\alpha) = A_n(\alpha)$ $(\beta_n < \alpha \leqq 1)$ if and only if $|p_j| = 2$ for $1 \leqq j \leqq n-1$; i.e.,

$$P(z) = \frac{1+\epsilon z}{1-\epsilon z}, \qquad |\epsilon| = 1.$$

It follows that for this range of α, the only extremal functions for this problem are functions of the form $\bar{\epsilon}f_\alpha(\epsilon z)$, where $|\epsilon| = 1$.

The previous theorem determines $A_n(\alpha)$ for a given n if α is near 1. We now give a theorem which determines $A_n(\alpha)$ for a given n when α is near 0. This theorem requires the following result.

THEOREM 2.3 (Rogosinski [**6**, p. 70]). *Let $f(z) = a + \sum_{k=1}^\infty a_k z^k$ be subordinate to $F(z) = a + \sum_{k=1}^\infty A_k z^k$ in U. If $F(z)$ is univalent in U and $F(U)$ is convex, then $|a_n| \leqq |A_1|$. If $F(U)$ is not a half plane, then equality can hold for a given n only if $f(z) = F(\epsilon z^n)$ $(|\epsilon| = 1)$.*

If $P(z) \in \mathscr{P}_\alpha$ $(0 < \alpha < 1)$, then $P(z)$ is subordinate to $((1+z)/(1-z))^\alpha$. It follows from Theorem 2.3 that if

$$P(z) = 1 + \sum_{k=1}^\infty p_k z^k,$$

then $|p_n| \leqq 2\alpha$. Moreover, $|p_n| = 2\alpha$ if and only if

$$P(z) = \left(\frac{1+\epsilon z}{1-\epsilon z}\right)^\alpha \qquad (|\epsilon| = 1).$$

We shall also need the following lemma.

LEMMA. *Let* $f(z) = z + \sum_{k=2}^{\infty} a_k(\alpha)z^k$ *be a function in* $S^*(\alpha)$ *for which* $a_n(\alpha) = A_n(\alpha)$. *If*

$$\frac{zf'(z)}{f(z)} = \left[\frac{1 + W_\alpha(z)}{1 - W_\alpha(z)}\right]^\alpha, \tag{2.13}$$

where $|W_\alpha(z)| < 1$ *and* $W_\alpha(0) = 0$, *then*

$$\lim_{\alpha \to 0} W_\alpha(z) = z^{n-1}.$$

Proof. Let

$$P_\alpha(z) = \left[\frac{1 + W_\alpha(z)}{1 - W_\alpha(z)}\right]^\alpha = 1 + \sum_{k=1}^{\infty} p_k(\alpha)z^k. \tag{2.14}$$

It follows from (2.13) and (2.14) that

$$(k - 1)a_k(\alpha) = p_{k-1}(\alpha) + p_{k-2}(\alpha)a_2(\alpha) + \ldots + p_1(\alpha)a_{k-1}(\alpha). \tag{2.15}$$

By Theorem 2.3 and induction we deduce that

$$(k - 1)a_k(\alpha) = p_{k-1}(\alpha) + O(\alpha^2) \qquad (\alpha \to 0).$$

In particular,

$$(n - 1)a_n(\alpha) = \operatorname{Re} p_{n-1}(\alpha) + O(\alpha^2) \leqq 2\alpha + O(\alpha^2) \qquad (\alpha \to 0).$$

If $g(z)$ is the function in $S^*(\alpha)$ defined by

$$\frac{zg'(z)}{g(z)} = \left(\frac{1 + z^{n-1}}{1 - z^{n-1}}\right)^\alpha, \tag{2.16}$$

then

$$g(z) = z + \frac{2\alpha}{n - 1} z^n + \ldots. \tag{2.17}$$

Since $a_n(\alpha) = A_n(\alpha)$ and $\operatorname{Re} p_{n-1}(\alpha) \leqq 2\alpha$,

$$2\alpha \leqq (n - 1)a_n(\alpha) \leqq \operatorname{Re} p_{n-1}(\alpha) + O(\alpha^2) \leqq 2\alpha + O(\alpha^2).$$

It follows that

$$\lim_{\alpha \to 0} \frac{1}{2\alpha} \operatorname{Re} p_{n-1}(\alpha) = 1.$$

The function $[P_\alpha(z)]^{1/2\alpha} \in \mathscr{P}_{1/2}$. If

$$[P_\alpha(z)]^{1/2\alpha} = 1 + \sum_{k=1}^{\infty} q_k(\alpha)z^k,$$

then using the fact that $|p_k(\alpha)| \leqq 2\alpha$ we obtain

$$\operatorname{Re} q_{n-1}(\alpha) = \operatorname{Re} \frac{p_{n-1}(\alpha)}{2\alpha} + o(\alpha) \qquad (\alpha \to 0).$$

Thus $\lim_{\alpha \to 0} \operatorname{Re} q_{n-1}(\alpha) = 1$. $\mathscr{P}_{1/2}$ is a normal compact family of functions,

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and thus it follows from the theory of normal families and the comments following Theorem 2.3 that

$$\left[\frac{1+z^{n-1}}{1-z^{n-1}}\right]^{1/2} = \lim_{\alpha\to 0} [P_\alpha(z)]^{1/2\alpha} = \lim_{\alpha\to 0}\left[\frac{1+W_\alpha(z)}{1-W_\alpha(z)}\right]^{1/2}.$$

This completes the proof of the lemma.

THEOREM 2.4. *For each integer $n > 1$, there exists a number γ_n, $0 < \gamma_n < 1$, such that if $0 < \alpha < \gamma_n$, then $A_n(\alpha) = 2\alpha/(n-1)$. Moreover, if*

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k$$

is a function in $S^*(\alpha)$ for which $|a_n| = 2\alpha/(n-1)$, then

$$\frac{zf'(z)}{f(z)} = \left[\frac{1+\epsilon z^{n-1}}{1-\epsilon z^{n-1}}\right]^\alpha,$$

where $|\epsilon| = 1$.

Proof. Let $f(z) = z + \sum_{k=2}^{\infty} a_k(\alpha)z^k$ be a function in $S^*(\alpha)$ for which $a_n(\alpha) = A_n(\alpha)$. Using the notation of the lemma we have

$$\frac{zf'(z)}{f(z)} = P_\alpha(z) = \left[\frac{1+W_\alpha(z)}{1-W_\alpha(z)}\right]^\alpha \tag{2.18}$$

and $\lim_{\alpha\to 0} W_\alpha(z) = z^{n-1}$. We show that there exists a number γ_n, $0 < \gamma_n < 1$, such that

$$W_\alpha(z) = z^{n-1} \tag{2.19}$$

for $0 < \alpha < \gamma_n$. In view of (2.18), (2.16), and (2.17), this will complete the proof.

Let $W_\alpha(z) = \sum_{k=1}^{\infty} w_k(\alpha)z^k$. If we can show that there exists a $\gamma_n > 0$ such that

$$w_{n-1}(\alpha) = 1 \qquad (0 < \alpha < \gamma_n), \tag{2.20}$$

then (2.19) will follow.

Suppose that (2.20) does not hold. Then there exists a set S which contains arbitrarily small values of $\alpha > 0$ such that

$$|w_{n-1}(\alpha)| = 1 - \lambda(\alpha) \qquad (\alpha \in S)$$

and $0 < \lambda(\alpha) < 1$. By the lemma, $\lim_{\alpha\to 0} \lambda(\alpha) = 0$.

Since $|W_\alpha(z)| < 1$ in $|z| < 1$, Parseval's identity implies that

$$|w_1(\alpha)|^2 + \ldots + |w_{n-1}(\alpha)|^2 \leqq 1.$$

Thus if $\alpha \in S$,

$$|w_k(\alpha)|^2 \leqq 2\lambda(\alpha) \qquad (1 \leqq k \leqq n-2). \tag{2.21}$$

It follows from (2.18) that

$$(2.22)\quad P_\alpha(z) = 1 + 2\alpha\sum_{j=1}^{\infty} [W_\alpha(z)]^j + 2\alpha(\alpha-1)\left\{\sum_{j=1}^{\infty} [W_\alpha(z)]^j\right\}^2 + \dots$$
$$= 1 + 2\alpha W_\alpha(z) + 2\alpha^2 W_\alpha^{\;2}(z) + \alpha h(z),$$

where $h(z)$ is a sum of powers of $W_\alpha(z)$ of degree at least 3.

If $\alpha \in S$, then (2.21) and (2.22) imply that

$$(2.23)\qquad p_k(\alpha) = 2\alpha w_k(\alpha) + \alpha^2 O(\lambda(\alpha)) + \alpha O([\lambda(\alpha)]^{3/2})$$

for $1 \leqq k \leqq n-1$. Substituting (2.23) in (2.15), applying (2.21), and using induction, we obtain

$$\begin{aligned}(n-1)a_n(\alpha) &= 2\alpha w_{n-1}(\alpha) + \alpha^2 O(\lambda(\alpha)) + \alpha O([\lambda(\alpha)]^{3/2})\\ &\leqq 2\alpha[1 - \lambda(\alpha) + \alpha O(\lambda(\alpha)) + O([\lambda(\alpha)]^{3/2})]\\ &< 2\alpha\end{aligned}$$

for sufficiently small α in S. This is a contradiction since (2.17) implies that $(n-1)A_n(\alpha) \geqq 2\alpha$ for $0 < \alpha \leqq 1$. Thus no such set S can exist which implies the existence of a number γ_n with the desired properties.

3. The coefficient problem for $\Sigma^*(\alpha)$. Let

$$F(z) = \frac{1}{z} + \sum_{k=0}^{\infty} A_k z^k$$

belong to $\Sigma^*(1)$. It was shown in [**3**] that for $n \geqq 1$, $|A_n| \leqq 2/(n+1)$ with equality if and only if

$$\frac{zF'(z)}{F(z)} = -\frac{1+\epsilon z^{n+1}}{1-\epsilon z^{n+1}},$$

where $|\epsilon| = 1$. Using this result, we prove the following theorem.

THEOREM 3.1. *Let*

$$F(z) = \frac{1}{z} + \sum_{k=0}^{\infty} A_k z^k$$

belong to $\Sigma^*(\alpha)$ $(0 < \alpha \leqq 1)$. *Then for* $n \geqq 1$,

$$(3.1)\qquad |A_n| \leqq \frac{2\alpha}{n+1}$$

with equality if and only if

$$\frac{zF'(z)}{F(z)} = -\left(\frac{1+\epsilon z^{n+1}}{1-\epsilon z^{n+1}}\right)^\alpha,$$

where $|\epsilon| = 1$.

Proof. Since $F(z) \in \Sigma^*(\alpha)$,

$$\frac{zF'(z)}{F(z)} = -P(z), \tag{3.2}$$

where $P(z) \in \mathscr{P}_\alpha$. Let $G(z)$ be the function in $\Sigma^*(1)$ defined by

$$\frac{zG'(z)}{G(z)} = -P(z)\left[\frac{1 + dz^{n+1}}{1 - dz^{n+1}}\right]^{1-\alpha} \qquad (|d| = 1). \tag{3.3}$$

If $G(z) = 1/z + \sum_{k=0}^{\infty} B_k z^k$, then it follows from (3.2) and (3.3) that $A_k = B_k$ for $1 \leqq k \leqq n - 1$ and

$$(n + 1)B_n = (n + 1)A_n - 2d(1 - \alpha). \tag{3.4}$$

Since $G(z) \in \Sigma^*(1)$, $|(n + 1)B_n| \leqq 2$, i.e.,

$$|(n + 1)A_n - 2d(1 - \alpha)| \leqq 2. \tag{3.5}$$

arg d is arbitrary and thus if we choose

$$\arg d = \arg A_n + \pi, \tag{3.6}$$

(3.5) implies that

$$(n + 1)|A_n| + 2(1 - \alpha) \leqq 2 \qquad \text{or} \qquad (n + 1)|A_n| \leqq 2\alpha.$$

This establishes (3.1). If equality holds; i.e., $(n + 1)|A_n| = 2\alpha$, then

$$(n + 1)|B_n| = (n + 1)|A_n| + 2(1 - \alpha) = 2.$$

It follows from the result for $\Sigma^*(1)$ quoted above that

$$\frac{zG'(z)}{G(z)} = -P(z)\left[\frac{1 + dz^{n+1}}{1 - dz^{n+1}}\right]^{1-\alpha} = -\frac{1 + \epsilon z^{n+1}}{1 - \epsilon z^{n+1}}, \tag{3.7}$$

where $|\epsilon| = 1$ and $\arg \epsilon = \pi + \arg B_n$. In view of (3.4) and (3.6),

$$\arg \epsilon = \pi + \arg B_n = \arg d \pmod{2\pi}. \tag{3.8}$$

Substituting (3.8) in (3.7) we obtain

$$\frac{zF'(z)}{F(z)} = -P(z) = -\left[\frac{1 + \epsilon z^{n+1}}{1 - \epsilon z^{n+1}}\right]^\alpha.$$

This completes the proof of the theorem.

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