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A NOTE ON RAMSEY'S THEOREM

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In memory of Leo Moser

In this note we prove some results concerning Ramsey's theorem [5]. If $n \geq 2$ is a positive integer, $\langle n \rangle$ will denote the complete graph on n vertices. We shall formulate our results in terms of the "arrow symbol" introduced by Erdös and Rado [1]. If $u \geq 2$ and $k \geq 1$ are positive integers then

$$n \to (u)_k \tag{1}$$

means that if the edges of an $\langle n \rangle$ are colored arbitrarily in k colors then there results a $\langle u \rangle$ all of whose edges have the same color. It follows from Ramsey's theorem that if u and k are given then $n \to (u)_k$ for all sufficiently large n. $n \not\to (u)_k$ will mean the negation of (1).

It is known ([2] and [3]) that

$$n \to (\log n/2 \log 2)_2$$

and that

$$n \not\to (2 \log n/\log 2)_2. \tag{2}$$

It is also known (see for example [2] or [4]) that ($c_1, c_2, \ldots$ are absolute constants)

$$n \to (c_1 \log n/k \log k)_k \tag{3}$$

and in [6] it is remarked that the arguments used in [3] to prove (2) can be used to prove

$$n \not\to (c_2 \log n/\log k)_k. \tag{4}$$

The object of this note is to narrow somewhat the wide gap between (3) and (4). We shall prove by a fairly simple argument that

$$n \not\to (c_3 \log n/k)_k. \tag{5}$$

LEMMA. *If $a \not\to (u)_b$ and $c \not\to (u)_d$ then*

$$ac \not\to (u)_{b+d}. \tag{6}$$

Proof. Let $\langle a \rangle$ have vertices $p_1, p_2, \ldots, p_a$ and color the edges of $\langle a \rangle$ in b colors in such a way that there does not result a monochromatic $\langle u \rangle$. Similarly, let $\langle c \rangle$ have vertices $p_1', p_2', \ldots, p_c'$ and color the edges of $\langle c \rangle$ in d colors (different from

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those used to color the edges of $\langle a \rangle$) so that there does not result a monochromatic $\langle u \rangle$. Let $\langle ac \rangle$ have vertices p_{ij}, $i=1, 2, \ldots, a$, $j=1, 2, \ldots, c$. Color the edge (p_{ij}, p_{ie}) the same as the edge (p'_j, p'_e) in $\langle c \rangle$ and, if $i \neq k$, color the edge (p_{ij}, p_{ke}) the same as the edge (p_i, p_k) in $\langle a \rangle$. Suppose in $\langle ac \rangle$ there is a monochromatic $\langle u \rangle$ with vertices $p_{i_1 j_1}, p_{i_2 j_2}, \ldots, p_{i_u j_u}$, say. It cannot occur that $i_1 = i_2 = \cdots = i_u$ since this would imply that $\langle c \rangle$ contains a monochromatic $\langle u \rangle$. Also, we cannot have $i_1, i_2, \ldots, i_u$ all distinct since this would imply the existence of a monochromatic $\langle u \rangle$ in $\langle a \rangle$. Hence we must have $i_1 = i_2 \neq i_3$, say. However, this clearly implies that the edges $(p_{i_1 j_1}, p_{i_2 j_2})$ and $(p_{i_1 j_1}, p_{i_3 j_3})$ are colored differently. Hence $\langle ac \rangle$ does not contain a monochromatic $\langle u \rangle$ and (6) is proved.

Now we prove (5). There is no harm in assuming that k is even, say $k=2l$. From (2) we get for all sufficiently large a

$$a \not\to (2 \log a/\log 2)_2.$$

By repeated application of (6) we get

$$a^l \not\to (2 \log a/\log 2)_{2l}.$$

Thus if n satisfies

$$a^{l-1} < n \leq a^l, \tag{7}$$

we have

$$n \not\to (2 \log a/\log 2)_k. \tag{8}$$

It is clear that (5) follows from (7) and (8).

References

1. P. Erdös and R. Rado, *A partition calculus in set theory*, Bull. Amer. Math. Soc. **62** (1956), 427–489.

2. P. Erdös and G. Szekeres, *A combinatorial problem in geometry*, Compositio Math. **2** (1935), 463–470.

3. P. Erdös, *Some remarks on the theory of graphs*, Bull. Amer. Math. Soc. **53** (1947), 292–294.

4. R. E. Greenwood and A. M. Gleason, *Combinatorial relations and chromatic graphs*, Canad. J. Math. **7** (1955), 1–17.

5. F. P. Ramsey, *On a problem in formal logic*, Proc. London Math. Soc. **30** (1930), 264–286.

6. P. Erdös and E. Szemeredi, *On a Ramsey type theorem* (to appear).

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