A NOTE ON EMBEDDING

P.A. Pittas

Let $Q_t = [0,1]$ be equipped with the topology consisting of Q_t, the empty set and all subsets of Q_t of the form $[0,x)$, $0 < x \leqq 1$. R. Nielsen and C. Sloyer [1, p. 514] proved that every T_0-space can be embedded in Q_t^F for a suitable F. The purpose of this note is to generalize this result.

THEOREM 1. *Let X be any T_0-space which is not a T_1-space; let Y be any T_0-space and let F be the family of all continuous functions $f: Y \to X$. Then the evaluation map $e: Y \to X^F$, where X^F is taken in the product topology defined by $e(y)_f = f(y)$, is a homeomorphism of Y onto $e[Y] \subset X^F$.*

Proof. Since X is not a T_1-space, there is an x_1 in X such that $\overline{\{x_1\}} \neq \{x_1\}$; since X is a T_0-space there is a neighborhood U of x_1 such that for x_2 in $\overline{\{x_1\}} \sim \{x_1\}$, $U \cap \{x_2\} = \phi$. Given y_1 and y_2 distinct elements of Y, and V a neighborhood of y_1 not containing y_2, the function $f: Y \to X$, defined by $f(y) = x_1$ if y is in V and $f(y) = x_2$ if y is not in V, is an element of F, and $f(y_1) = x_1 \neq x_2 = f(y_2)$. Also, if C is a closed set in Y and y_0 is not in C, then $g: Y \to X$, defined by $g(y) = x_1$ if y is not in C and $g(y) = x_2$ if y is in C, is an element of F, and $g(y_0) = x_1$ is not in $\overline{g[C]}$, for x_2 is not in U. In other words F distinguishes points and also distinguishes points and closed sets; thus by the Embedding Lemma [2, p. 116], e is a homeomorphism of Y onto $e[Y] \subset X^F$.

Q_t is a T_0-space which is not a T_1-space, so the main theorem in [1] is a particular case of the above theorem. A particularly simple space which may serve as the X of the above theorem is the space $\{a, b\}$ equipped with the topology consisting of $\{a, b\}$, ϕ, and $\{a\}$. (For an example showing that e is not onto X^F see [3].)

The following example shows that if we replace T_0 by T_1 and T_1 by T_2 in Theorem 1, then the theorem is false without additional conditions.

https://doi.org/10.4153/CMB-1969-045-8 Published online by Cambridge University Press

Let the set of natural numbers N be equipped with the finite complement topology (i.e. the topology whose closed sets are the whole space, the empty set and all finite sets). N is a T_1-space which is not T_2. Let A be an uncountable set equipped with the finite complement topology. Let F be the family of all continuous functions $f : A \to N$. Clearly f is in F if and only if $f[A]$ is a singleton. Therefore F does not distinguish points and thus e is not a homeomorphism. However we can prove the following result.

PROPOSITION 1. *Let X be an infinite set equipped with the finite complement topology, let Y be a T_1-space and F be the family of all continuous functions $f : Y \to X$. Then a sufficient condition for Y to be homeomorphic to a subspace of X^F is that for each y in Y there exists an open neighborhood U_y of y with card $U_y \leqq$ card X.*

Proof. F distinguishes points and closed sets, for if C is closed in Y and y_0 is not in C, then $N = U_{y_0} \cap (Y \sim C)$ is an open neighborhood of y_0 with card $N \leqq$ card X. Thus $g : Y \to X$, defined by $g[Y \sim N] = x_1$ and the restriction of g to N is a one-to-one mapping into $X \sim \{x_1\}$, is an element of F and $g(y_0)$ is not in $\{x_1\} = \overline{g[C]}$. Since Y is a T_1-space, F also distinguishes points. Thus by the Embedding Lemma [2, p. 116], e is a homeomorphism of Y onto $e[Y] \subset X^F$.

The following example shows that the condition of Proposition 1 is not necessary. Let X be the set of natural numbers equipped with the finite complement topology. X is a T_1-space which is not T_2. Let $Y = [0, \Omega]$ be the set of ordinal numbers which are less or equal to the first uncountable ordinal Ω. Let Y be equipped with the order topology i.e., the topology generated by all sets of the form $\{x : x < \alpha\}$ and $\{x : x > \beta\}$, where α and β are members of Y. Y is a T_1-space and each neighborhood of Ω has cardinality greater than $\aleph_0$ = card X. Let F be the family of all continuous functions from Y to X. It is easy to show that the evaluation map is a homeomorphism of Y onto $e[Y] \subset X^F$.

Remark. If card Y $\leqq$ card X then the space Y is the required neighborhood for each y in Y.

PROPOSITION 2. *Let X, Y and F be as in Proposition 1 and card Y > card X. Then a necessary condition for Y to be homeomorphic to a subspace of X^F is that there be at least one proper closed subset C of Y with card C = card Y.*

Proof. Suppose card C < card Y for each proper closed subset C of Y. Then clearly f is in F if and only if $f[Y]$ is a singleton (because card X < card Y). Suppose for a contradiction that there is a homeomorphism h of Y into X^F. If y_1 and y_2 are distinct elements of Y, then $h(y_1) \neq h(y_2)$ and, for some f in F, $h(y_1)_f \neq h(y_2)_f$ so that the composition $P_f \circ h : Y \to X$ of the projection P_f and h is a continuous mapping with $P_f \circ h[Y]$ not a singleton, which is a contradiction.

The following example shows that the condition of Proposition 2 is not a sufficient one. Let X be a countable set equipped with the finite complement topology and A any set such that card A > c, where c is the power of the continuum. Let $Y = \{A\} \cup A$ be equipped with the topology consisting of $\{A\}$ and all the sets in the finite complement topology on Y. Let F be the family of all continuous $f : Y \to X$. Since $f \in F$ if and only if f[A] is a singleton, we have card $F = \aleph_0$, so that card $X^F = \aleph_0^{\aleph_0} = c <$ card A and Y cannot be homeomorphic to any subspace of X^F.

This research was supported by NRC Grant No. A-3999. The author wishes to thank Professor M. Edelstein for his most helpful suggestions.

REFERENCES

1. R. Nielsen and C. Sloyer, On embedding in quasi-cubes. Amer. Math. Monthly 75 (1968) 514-515.

2. J. L. Kelley, General topology. (Van Nostrand, Princeton, N.J., 1955).

3. Problem No. 5566. Amer. Math. Monthly 75 (1968) 198.

Dalhousie University